Preface

This collection is intended for use with a Co-requisite course, pairing a concurrent MAT 1023 - College Algebra (TCCN MAT 1314) with an NCBO (Non-Course Based Option) support course. This book is designed to supplement the MAT materials and provide a free, opensource option for students to reference in the NCB support course.

The arrangement of materials within the modules is structured to support the Lessons covered in the MAT course. As a result the sequence may appear odd compared to a traditional developmental math textbook.

This book utilizes topics from modules in OpenStax PreAlgebra, Intermediate Algebra, and College Algebra collections. This collection is intended for developmental students taking a Co-requisite course in math intended for a stem degree pathway.

Coverage and Scope

By design, the support course in our Co-requisite model emphasizes "Just-in-time" support of topics discussed in the related math course. It is not meant to be a complete review of all topics from earlier developmental math courses like Elementary or

Intermediate Algebra. As a result, each chapter is arranged to review only those topics needed to understand concepts discussed in the specific math course chapter it supports. The material is presented as a sequence of modules tied to Lesson chapters of the MAT 1023 course. I.e. Module 1 supports section 1.1 of MAT 1023. The order of topics was carefully planned to support progression throughout the course (by providing foundational concepts related to the Lesson Topics, and/or previewing material for Lessons), and to facilitate a thorough understanding of each concept.

Chapter 1: Pre-knowledge Concepts
Chapter 1 reviews arithmetic operations with fractions, decimals and real numbers, and concepts of exponents, radicals, polynomials, and rational expresssions to give the student a solid base that will support their study of algebra.

Chapter 2: Equations and Inequalities
In Chapter 2, students will review topics related to graphing basics, linear equations, complex numbers and other types of equations.

Chapter 3: Functions and Graphs
Chapter 3 covers the basics of Functions, slope, and transformations/composition of functions.

Chapter 4: Polynomial Functions
Chapter 4 covers Quadratic and Polynomial functions, as well as introduces polynomial division.

Chapter 5: Exponential and Logarithmic Functions
In Chapter 5, students learn concepts related to working with logs and exponential functions.
Chapter 6: Systems of Equations
In Chapter 6, students learn about the basics of systems of linear equations and inequalities.

All chapters incorporate multiple topics, the titles of which can be viewed in the **Topics Covered** box at the start of each module.

About the Authors

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Foundations

This module provides a review of those topics needed prior to the start of the semester.

Topics Covered in this Module

In case you missed something in class, or just want to review a specific topic covered in this Module, here is a list of topics covered:

1. Identify Counting Numbers and Whole Numbers [link]
2. Use Variables and Algebraic Symbols [link]
3. Use Negatives and Opposites [link]
4. Add and Subtract Integers [link]
5. Multiply and Divide Integers [link]
6. Identify Multiples [link]
7. Use Common Divisibility Tests [link]
8. Find all the Factors of the Given Number [link]
9. Fraction Basics [link]
10. Multiply and Divide Fractions[link]
11. Add, Subtract Fraction [link]
12. Decimal Basics [link]
13. Solve Equations, Subtraction and Addition [link]
14. Solve Equations, Multiplication and Division [link]
15. Percents Basics [link]

16. Key Concepts [link]

Identify Counting Numbers and Whole Numbers

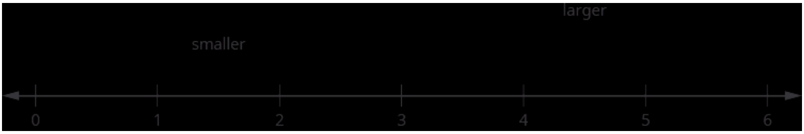
Learning algebra is similar to learning a language. You start with a basic vocabulary and then add to it as you go along. You need to practice often until the vocabulary becomes easy to you. The more you use the vocabulary, the more familiar it becomes.

Algebra uses numbers and symbols to represent words and ideas. Let's look at the numbers first. The most basic numbers used in algebra are those we use to count objects: $1,2,3,4,5,\ldots$ and so on. These are called the **counting numbers**. The notation "…" is called an ellipsis, which is another way to show "and so on", or that the pattern continues endlessly. Counting numbers are also called natural numbers.

Doing the Manipulative Mathematics activity Number Line-Part 1 will help you develop a better understanding of the counting numbers and the whole numbers.

The counting numbers start with 1 and continue.
1,2,3,4,5…

Counting numbers and whole numbers can be visualized on a **number line** as shown in [link].



The point labeled 0 is called the **origin**. The points are equally spaced to the right of 0 and labeled with the counting numbers. When a number is paired with a point, it is called the **coordinate** of the point.

The discovery of the number zero was a big step in the history of mathematics. Including zero with the counting numbers gives a new set of numbers called the **whole numbers**.

Whole Numbers
The whole numbers are the counting numbers and zero.
0,1,2,3,4,5…
We stopped at 5 when listing the first few counting numbers and whole numbers. We could have written more numbers if they were needed to make

the patterns clear.

Which of the following are ⓐ counting numbers? ⓑ whole numbers?

0,14,3,5.2,15,105

Solution

ⓐ The counting numbers start at 1, so 0 is not a counting number. The numbers 3,15,and105 are all counting numbers.
ⓑ Whole numbers are counting numbers and 0. The numbers 0,3,15,and105 are whole numbers.

The numbers 14 and 5.2 are neither counting numbers nor whole numbers. We will discuss these numbers later.

Use Variables and Algebraic Symbols

Greg and Alex have the same birthday, but they were born in different years. This year Greg is 20 years old and Alex is 23, so Alex is 3 years older than Greg. When Greg was 12, Alex was 15. When Greg is 35, Alex will be 38. No matter what Greg's age is, Alex's age will always be 3 years more, right?

In the language of algebra, we say that Greg's age and Alex's age are variable and the three is a constant. The ages change, or vary, so age is a variable. The 3 years between them always stays the same, so the age difference is the constant.

In algebra, letters of the alphabet are used to represent variables. Suppose we call Greg's age g. Then we could use $g+3$ to represent Alex's age. See [link].

Greg's age	Alex's age
12	15
20	23
35	38
g	$g+3$

Letters are used to represent variables. Letters often used for variables are x,y,a,b,andc.

Variables and Constants

A variable is a letter that represents a number or quantity whose value may change.

A constant is a number whose value always stays the same.

To write algebraically, we need some symbols as well as numbers and variables. There are several types of symbols we will be using. In Whole Numbers, we introduced the symbols for the four basic arithmetic operations: addition, subtraction, multiplication, and division. We will summarize them here, along with words we use for the operations and the result.

Operation	Notation	Say:	The result is…
Addition	$a+b$	aplusb	the sum of a and b

Subtraction	$a-b$	aminusb	the difference of a and b
Multiplication	$a\cdot b,(a)(b),(a)b,a(b)$	atimesb	The product of a and b
Division	$a\div b,a/b,ab,ba$	a divided by b	The quotient of a and b

In algebra, the cross symbol, $\times$, is not used to show multiplication because that symbol may cause confusion. Does 3xy mean $3\times y$ (three times y) or $3\cdot x\cdot y$ (three times xtimesy)? To make it clear, use • or parentheses for multiplication.

We perform these operations on two numbers. When translating from symbolic form to words, or from words to symbolic form, pay attention to the words *of* or *and* to help you find the numbers.

- The *sum* ***of*** 5 ***and*** 3 means add 5 plus 3, which we write as $5+3$.
- The *difference* ***of*** 9 ***and*** 2 means subtract 9 minus 2, which we write as $9-2$.
- The *product* ***of*** 4 ***and*** 8 means multiply 4 times 8, which we can write as $4\cdot 8$.
- The *quotient* ***of*** 20 ***and*** 5 means divide 20 by 5, which we can write as $20\div 5$.

When two quantities have the same value, we say they are equal and connect them with an *equal sign*.

Equality Symbol

$a=b$ is read a is equal to b

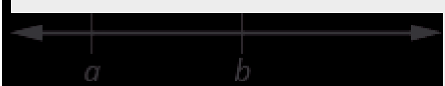
The symbol $=$ is called the equal sign.

An inequality is used in algebra to compare two quantities that may have different values. The number line can help you understand inequalities. Remember that on the number line the numbers get larger as they go from left to right. So if we know that b is greater than a, it means that b is to the right of a on the number line. We use the symbols "$<$" and "$>$" for inequalities.

Inequality

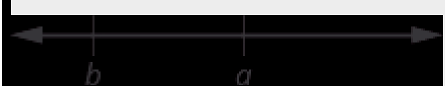
$a<b$ is read a is less than b

a is to the left of b on the number line



$a>b$ is read a is greater than b

a is to the right of b on the number line



The expressions $a < b$ and $a > b$ can be read from left-to-right or right-to-left, though in English we usually read from left-to-right. In general,
$a < b$ is equivalent to $b > a$. For example, $7 < 11$ is equivalent t

When we write an inequality symbol with a line under it, such as $a \leq b$, it means $a < b$ or $a = b$. We read this a is less than or equal to b. Also, if we put a slash through an equal sign, $\neq$, it means not equal.

We summarize the symbols of equality and inequality in [link].

Algebraic Notation	Say
$a = b$	a is equal to b
$a \neq b$	a is not equal to b
$a < b$	a is less than b
$a > b$	a is greater than b
$a \leq b$	a is less than or equal to b
$a \geq b$	a is greater than or equal to b

Symbols $<$ and $>$

The symbols $<$ and $>$ each have a smaller side

and a larger side.
smaller side $<$ larger side
larger side $>$ smaller side
The smaller side of the symbol faces the smaller number and the larger faces the larger number.

Translate from algebra to words:

ⓐ $20 \leq 35$
ⓑ $11 \neq 15-3$
ⓒ $9 > 10 \div 2$
ⓓ $x+2 < 10$

Solution

ⓐ
$20 \leq 35$
20 is less than or equal to 35

ⓑ

$11 \neq 15 - 3$

11 is not equal to 15 minus 3

ⓒ

$9 > 10 \div 2$

9 is greater than 10 divided by 2

ⓓ

$x + 2 < 10$

x plus 2 is less than 10

Grouping symbols in algebra are much like the commas, colons, and other punctuation marks in written language. They indicate which expressions are to be kept together and separate from other expressions. [link] lists three of the most commonly used grouping symbols in algebra.

Common Grouping Symbols	
parentheses	( )
brackets	[ ]
braces	{ }

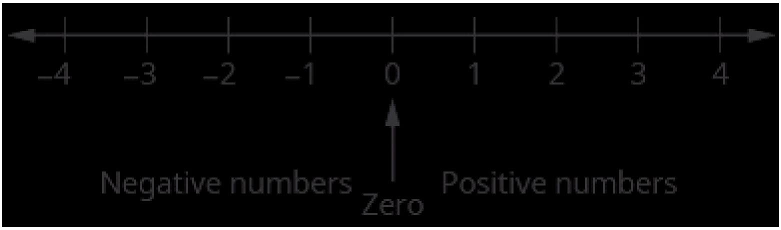
Here are some examples of expressions that include grouping symbols. We will simplify expressions like these later in this section.

$8(14-8)\quad 21-3[2+4(9-8)]\quad 24\div\{13-2[1(6-5)+4]\}$

The number line shows the location of positive and negative numbers. The numbers on a number line increase in value going from left to right and decrease in value going from right to left. All the marked numbers are called *integers*. The opposite of 3 is -3.

Use Negatives and Opposites

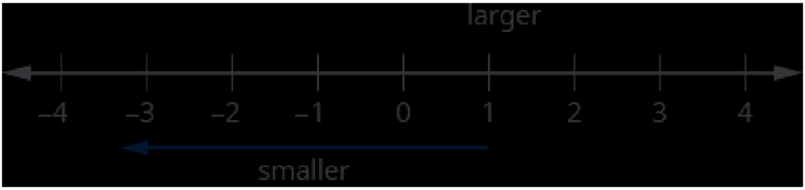
Our work so far has only included the counting numbers and the whole numbers. But if you have ever experienced a temperature below zero or accidentally overdrawn your checking account, you are already familiar with negative numbers. **Negative numbers** are numbers less than 0. The negative numbers are to the left of zero on the number line. See [link].



The arrows on the ends of the number line indicate that the numbers keep going forever. There is no biggest positive number, and there is no smallest negative number.

Is zero a positive or a negative number? Numbers larger than zero are positive, and numbers smaller than zero are negative. Zero is neither positive nor negative.

Consider how numbers are ordered on the number line. Going from left to right, the numbers increase in value. Going from right to left, the numbers decrease in value. See [link].



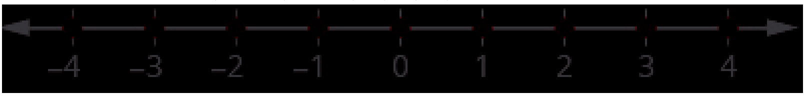
Doing the Manipulative Mathematics activity "Number Line-part 2" will help you develop a better understanding of integers.

Remember that we use the notation:

$a < b$ (read “a is less than b”) when a is to the left of b on the number line.

$a > b$ (read “a is greater than b”) when a is to the right of b on the number line.

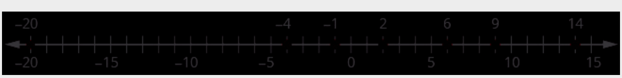
Now we need to extend the number line which showed the whole numbers to include negative numbers, too. The numbers marked by points in [link] are called the integers. The integers are the numbers $\ldots -3, -2, -1, 0, 1, 2, 3 \ldots$



Order each of the following pairs of numbers, using $<$ or $>$: ⓐ $14\_\_6$ ⓑ $-1\_\_9$ ⓒ $-1\_\_-4$ ⓓ $2\_\_-20$.

Solution

It may be helpful to refer to the number line shown.



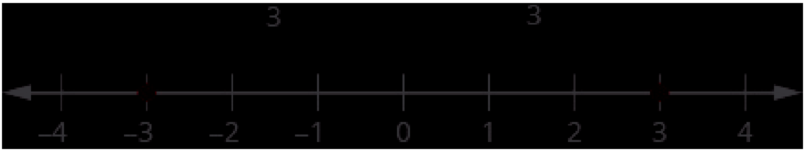
ⓐ 14 is to the right of 6 on the number line.	$14\_\_6$	$14>6$
ⓑ -1 is to the left of 9 on the number line.	$-1\_\_9$	$-1<9$
ⓒ -1 is to the right of -4 on the number line.	$-1\_\_-4$	$-1>-4$
ⓓ 2 is to the right of -20 on the number line.	$2\_\_-20$	$2>-20$

You may have noticed that, on the number line, the negative numbers are a mirror image of the positive numbers, with zero in the middle. Because the numbers 2 and -2 are the same distance from zero, they are called **opposites**. The opposite of 2 is -2, and the opposite of -2 is 2.

Opposite

The **opposite** of a number is the number that is the same distance from zero on the number line but on the opposite side of zero.

[link] illustrates the definition.



Sometimes in algebra the same symbol has different meanings. Just like some words in English, the specific meaning becomes clear by looking at how it is used. You have seen the symbol “ $-$ ” used in three different ways.

$10-4$Between two numbers, it indicates the operation ofsubtraction.We read$10-4$as“10minus4.”-8In front of a number, it indicates anegativenumber.We read -8as “negative eight.”$-x$In front of a variable, it indicates theopposite.We read $-x$as “the opposite of$x$.”$-(-2)$Here there are two“ $-$ ”signs. The one in the parentheses tells us the number isnegative2.The one outside the parentheses tells us to take theoppositeof -2.We read $-(-2)$as “the opposite of negative two.”

$10-4$	Between two numbers, it indicates the operation of *subtraction*. We read $10-4$ as "10 minus 4."
-8	In front of a number, it indicates a *negative* number. We read -8 as "negative eight."

$-x$	In front of a variable, it indicates the *opposite*. We read $-x$ as "the opposite of x."
$-(-2)$	Here there are two "$-$" signs. The one in the parentheses tells us the number is negative 2. The one outside the parentheses tells us to take the *opposite* of -2. We read $-(-2)$ as "the opposite of negative two."

Opposite Notation

$-a$ means the opposite of the number a.

The notation $-a$ is read as “the opposite of a.”

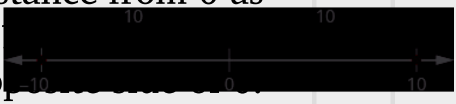
Find: ⓐ the opposite of 7 ⓑ the opposite of -10 ⓒ $-(-6)$.

Solution

ⓐ -7 is the same distance from 0 as 7, bu
sid

The opposite of 7 is -7.

ⓑ 10 is the same distance from 0 as
–
op



The opposite of -10 is 10.

ⓒ $-(-6)$

The opposite of $-(-6)$ is -6.

Our work with opposites gives us a way to define the integers.The whole numbers and their opposites are called the **integers**. The integers are the numbers $\ldots -3, -2, -1, 0, 1, 2, 3 \ldots$

Integers

The whole numbers and their opposites are called the **integers**.

The integers are the numbers

$\ldots -3, -2, -1, 0, 1, 2, 3 \ldots$

When evaluating the opposite of a variable, we must be very careful. Without knowing whether the variable represents a positive or negative number, we don't know whether $-x$ is positive or negative. We can see this in [link].

Evaluate ⓐ $-x$, when $x = 8$ ⓑ $-x$, when $x = -8$.

Solution

ⓐ

[redacted]	$-x$
[redacted]	
Write the opposite of	

8. [redacted]

[redacted]

$-x$

[redacted]

Write the opposite of 8

$-8.$

Add adn Subrtract Integers

So far, we have only used the counting numbers and the whole numbers.

Counting numbers $1,2,3\ldots$ Whole numbers $0,1,2,3\ldots.$

Our work with opposites gives us a way to define the **integers**. The whole numbers and their opposites are called the integers. The integers are the numbers $\ldots-3,-2,-1,0,1,2,3\ldots$

Integers

The whole numbers and their opposites are called the **integers**.

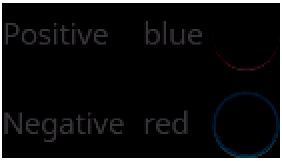
The integers are the numbers

$\ldots -3, -2, -1, 0, 1, 2, 3 \ldots,$

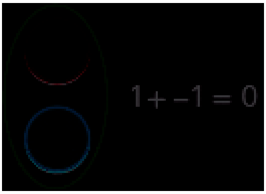
Most students are comfortable with the addition and subtraction facts for positive numbers. But doing addition or subtraction with both positive and negative numbers may be more challenging.

We will use two color counters to model addition and subtraction of negatives so that you can visualize the procedures instead of memorizing the rules.

We let one color (blue) represent positive. The other color (red) will represent the negatives.



If we have one positive counter and one negative counter, the value of the pair is zero. They form a neutral pair. The value of this neutral pair is zero.



We will use the counters to show how to add:
$5+3-5+(-3)-5+35+(-3)$

The first example, $5+3$, adds 5 positives and 3 positives—both positives.

The second example, $-5+(-3)$, adds 5 negatives and 3 negatives—both negatives.

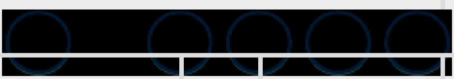
When the signs are the same, the counters are all the same color, and so we add them. In each case we get 8—either 8 positives or 8 negatives.

So what happens when the signs are different? Let’s add $-5+3$ and $5+(-3)$.

When we use counters to model addition of positive and negative integers, it is easy to see whether there are more positive or more negative counters. So we know whether the sum will be positive or negative.

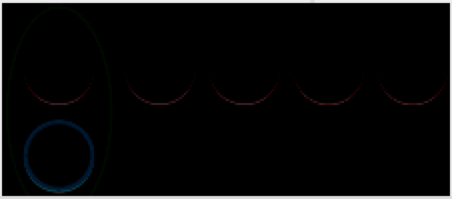
Add: ⓐ $-1+(-4)$ ⓑ $-1+5$ ⓒ $1+(-5)$.

ⓐ

	$-1+(-4)$
	
1 negative plus 4 negatives is 5 negatives	-5

ⓑ

[redacted]

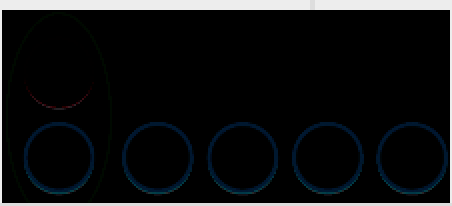


[redacted]

There are more positives, so the sum is po[redacted]

©

[redacted]



[redacted]

There are more

ne[redacted]
negative.

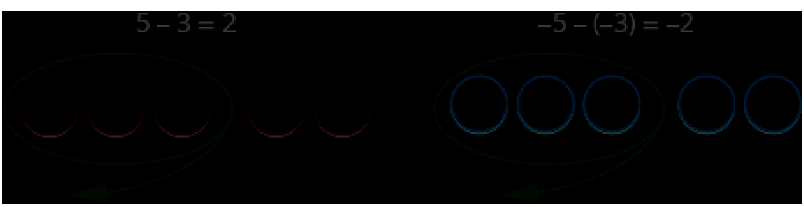
We will continue to use counters to model the subtraction. Perhaps when you were younger, you read “$5-3$” as “5 take away 3.” When you use counters, you can think of subtraction the same way!

We will use the counters to show to subtract:
$5-3$ $-5-(-3)$ $-5-3$ $5-(-3)$

The first example, $5-3$, we subtract 3 positives from 5 positives and end up with 2 positives.

In the second example, $-5-(-3)$, we subtract 3 negatives from 5 negatives and end up with 2 negatives.

Each example used counters of only one color, and the “take away” model of subtraction was easy to apply.



What happens when we have to subtract one positive and one negative number? We’ll need to use both blue and red counters as well as some neutral pairs. If we don’t have the number of counters needed to take away, we add neutral pairs. Adding a neutral pair does not change the value. It is like changing quarters to nickels—the value is the same, but it looks different.

Let’s look at $-5-3$ and $5-(-3)$.

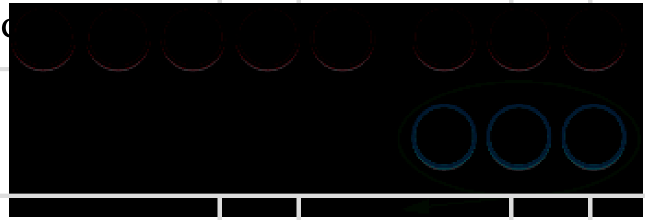
[redacted]

Model the first number.

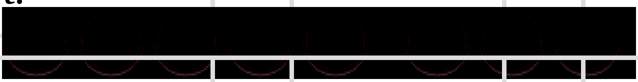
We now add the needed neutral pa

We remove the number of counters modeled by the

se



Count what is left.

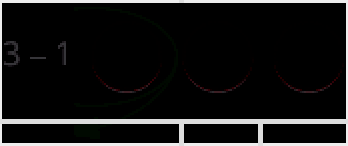


[redacted]

[redacted]

Subtract: ⓐ $3-1$ ⓑ $-3-(-1)$ ⓒ $-3-1$ ⓓ $3-(-1)$.

ⓐ



Take 1 positive from 3 positives and get 2 positives.

ⓑ

Take 1 positive from 3 negatives and get 2 negatives.

ⓒ

$-3-1$

Take 1 positive from the one ad[redacted]eutral pa[redacted]

-4

ⓓ

$3-(-1)$

Take 1 negative from the[redacted]dded ne[redacted]air.

4

Have you noticed that *subtraction of signed numbers can be done by adding the opposite*? In the last example, $-3-1$ is the same as $-3+(-1)$ and

$3-(-1)$ is the same as $3+1$. You will often see this idea, the Subtraction Property, written as follows:

Subtraction Property

$a-b=a+(-b)$

Subtracting a number is the same as adding its opposite.

Simplify: ⓐ $13-8$ and $13+(-8)$ ⓑ $-17-9$ and $-17+(-9)$ ⓒ $9-(-15)$ and $9+15$ ⓓ $-7-(-4)$ and $-7+4$.

ⓐ

$13-8 \text{and} 13+(-8)$ Subtract. $5 \quad 5$

ⓑ

$-17-9 \text{and} -17+(-9)$ Subtract. $-26 \quad -26$

ⓒ

$9-(-15) \text{and} 9+15$ Subtract. $24 \quad 24$

ⓓ

$-7-(-4) \text{and} -7+4$ Subtract. $-3 \quad -3$

What happens when there are more than three integers? We just use the order of operations as usual.

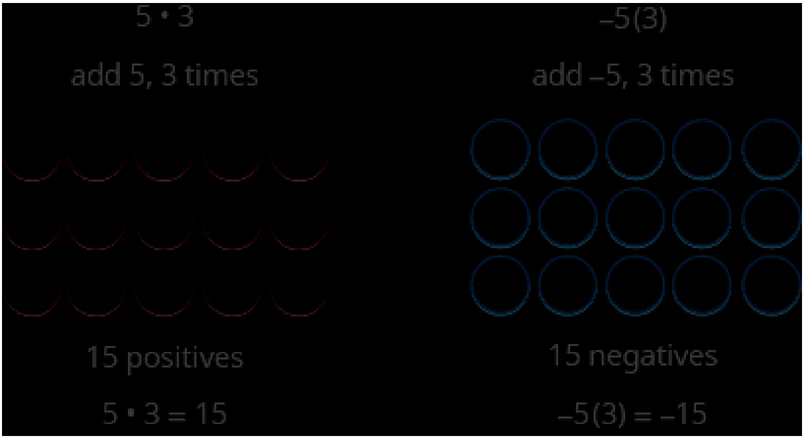
Simplify: $7-(-4-3)-9$.
$7-(-4-3)-9$ Simplify inside the parentheses first. $7-(-7)-9$ Subtract left to right. $14-9$ Subtract. 5

Multiply and Divide Integers

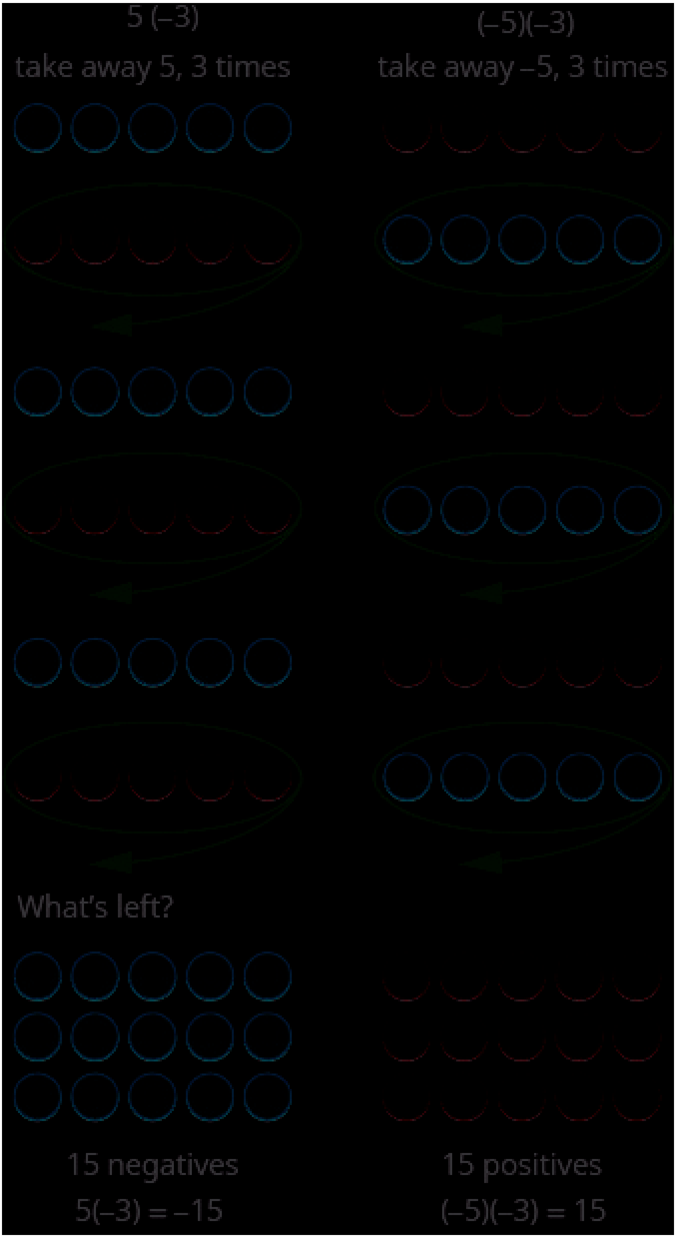
Since multiplication is mathematical shorthand for repeated addition, our model can easily be applied to show multiplication of integers. Let's look at this concrete model to see what patterns we notice. We will use the same examples that we used for addition and subtraction. Here, we will use the model just to help us discover the pattern.

We remember that $a \cdot b$ means add a, b times. Here, we are using the model just to help us discover the

pattern.



The next two examples are more interesting. What does it mean to multiply 5 by -3? It means subtract 5,3 times. Looking at subtraction as “taking away”, it means to take away 5, 3 times. But there is nothing to take away, so we start by adding neutral pairs on the workspace.



In summary:

$5 \cdot 3 = 15 - 5(3) = -155(-3) = -15(-5)(-3) = 15$

Notice that for multiplication of two signed numbers, when the:
signs are thesame, the product ispositive. signs aredifferent, the product isnegative.

What about division? Division is the inverse operation of multiplication. So, $15 \div 3 = 5$ because $5 \cdot 3 = 15$. In words, this expression says that 15 can be divided into 3 groups of 5 each because adding five three times gives 15. If you look at some examples of multiplying integers, you might figure out the rules for dividing integers.
$5 \cdot 3 = 15$so$15 \div 3 = 5$ $-5(3) = -15$so$-15 \div 3 = -5$
$(-5)(-3) = 15$so$15 \div (-3) = -5$ $5(-3) = -15$so
$-15 \div (-3) = 5$

Division follows the same rules as multiplication with regard to signs.

Multiplication of Signed Numbers
For multiplication of two signed numbers:

Same signs	Product	Example
Two positives	Positive	$7 \cdot 4 = 28$ $-8(-6) = 48$
Two negatives	Positive	

Different signs	Product	Example
Positive · negative Negative · positive	Negative Negative	$7(-9)=-63$ $-5\cdot 10=-50$

Multiply or divide: ⓐ $-100\div(-4)$ ⓑ $7\cdot 6$ ⓒ $4(-8)$ ⓓ $-27\div 3$.

ⓐ

$-100\div(-4)$ Divide, with signs that arethe same the quotient is positive.25

ⓑ

$7\cdot 6$ Multiply, with same signs.42

ⓒ

$4(-8)$ Multiply, with different signs. -32

ⓓ

$-27\div 3$ Divide, with different signs,the quotient is negative. -9

When we multiply a number by 1, the result is the

same number. What happens when we multiply a number by -1? Let's multiply a positive number and then a negative number by -1 to see what we get.
$-1\cdot4$ $-1(-3)$ Multiply. -4 3 -4 is the opposite of 4. 3 is the opposite of -3.

Each time we multiply a number by -1, we get its opposite!

Multiplication by -1
$-1a=-a$
Multiplying a number by -1 gives its opposite.

Multiply: ⓐ $-1\cdot7$ ⓑ $-1(-11)$.

Solution

ⓐ Multiply, noting that the signs are different	$-1\cdot7$ -7 -7 is the opposite of 7.

so the product is negative.

ⓑ Multiply, noting that the signs are the same so the product is positive.	$-1(-11)$ 11 11is the opposite of -11.

Identify Multiples of a Number

The numbers 2, 4, 6, 8, 10, 12 are called multiples of 2. A **multiple** of 2 can be written as the product of a counting number and 2.

Multiples of 2:

2,	4,	6,	8,	10,	12, …
$2 \cdot 1$	$2 \cdot 2$	$2 \cdot 3$	$2 \cdot 4$	$2 \cdot 5$	$2 \cdot 6$

Similarly, a multiple of 3 would be the product of a counting number and 3.

Multiples of 3:

3,	6,	9,	12,	15,	18, …
$3 \cdot 1$	$3 \cdot 2$	$3 \cdot 3$	$3 \cdot 4$	$3 \cdot 5$	$3 \cdot 6$

We could find the multiples of any number by continuing this process.

Counting Number	1	2	3	4	5	6	7	8	9	10	11	12
Multiples of 2	2	4	6	8	10	12	14	16	18	20	22	24
Multiples of 3	3	6	9	12	15	18	21	24	27	30	33	36
Multiples of 4	4	8	12	16	20	24	28	32	36	40	44	48
Multiples of 5	5	10	15	20	25	30	35	40	45	50	55	60
Multiples of 6	6	12	18	24	30	36	42	48	54	60	66	72
Multiples of 7	7	14	21	28	35	42	49	56	63	70	77	84
Multiples of 8	8	16	24	32	40	48	56	64	72	80	88	96
Multiples of 9	9	18	27	36	45	54	63	72	81	90	99	108

of
9

Multiple of a Number
A number is a **multiple** of n if it is the product of a counting number and n.

Another way to say that 15 is a multiple of 3 is to say that 15 is **divisible** by 3. That means that when we divide 3 into 15, we get a counting number. In fact, $15 \div 3$ is 5, so 15 is $5 \cdot 3$.

Use Common Divisibility Tests

Another way to say that 375 is a multiple of 5 is to say that 375 is divisible by 5. In fact, $375 \div 5$ is 75, so 375 is $5 \cdot 75$. Notice in [link] that 10,519 is not a multiple 3. When we divided 10,519 by 3 we did not get a counting number, so 10,519 is not divisible by 3.

Divisibility
If a number m is a multiple of n, then we say that m is divisible by n.

Since multiplication and division are inverse operations, the patterns of multiples that we found can be used as divisibility tests. [link] summarizes divisibility tests for some of the counting numbers between one and ten.

Divisibility Tests	
A number is divisible by	
2	if the last digit is 0,2,4,6,or8
3	if the sum of the digits is divisible by 3
5	if the last digit is 5 or 0
6	if divisible by both 2 and 3
10	if the last digit is 0

Determine whether 1,290 is divisible by 2,3,5,and10.

Solution

[link] applies the divisibility tests to 1,290. In

the far right column, we check the results of the divisibility tests by seeing if the quotient is a whole number.

Divisible by...?	Test	Divisible?	Check
2	Is last digit 0,2,4,6,or8? *Yes.*	yes	$1290 \div 2 = 645$
3	Is sum of digits divisible by3? $1 + 2 + 9 + 0 = 12$ *Yes.*	yes	$1290 \div 3 = 430$
5	Is last digit 5 or 0? *Yes.*	yes	$1290 \div 5 = 258$
10	Is last digit 0? *Yes.*	yes	$1290 \div 10 = 129$

Thus, 1,290 is divisible by 2,3,5,and10.

Determine whether 5,625 is divisible by 2,3,5,and10.

Solution

[link] applies the divisibility tests to 5,625 and tests the results by finding the quotients.

Divisible by...?	Test	Divisible?	Check
2	Is last digit 0,2,4,6,or8? *No.*	no	$5625 \div 2 = 2812.5$
3	Is sum of digits divisible by3? $5 + 6 + 2 + 5 = 18$ *Yes.*	yes	$5625 \div 3 = 1875$
5	Is last digit is 5 or 0? *Yes.*	yes	$5625 \div 5 = 1125$
10	Is last digit 0? *No.*	no	$5625 \div 10 = 562.5$

Thus, 5,625 is divisible by 3 and 5, but not 2,

or 10.

Find all the Factors of the Given Number

There are often several ways to talk about the same idea. So far, we've seen that if m is a multiple of n, we can say that m is divisible by n. We know that 72 is the product of 8 and 9, so we can say 72 is a multiple of 8 and 72 is a multiple of 9. We can also say 72 is divisible by 8 and by 9. Another way to talk about this is to say that 8 and 9 are factors of 72. When we write $72=8\cdot 9$ we can say that we have factored 72.

$$8\cdot 9 \quad = \quad 72$$

Factors
If $a\cdot b=m$, then a and b are factors of m, and m is the product of a and b.

In algebra, it can be useful to determine all of the

factors of a number. This is called factoring a number, and it can help us solve many kinds of problems.

Doing the Manipulative Mathematics activity “Model Multiplication and Factoring” will help you develop a better understanding of multiplication and factoring.

For example, suppose a choreographer is planning a dance for a ballet recital. There are 24 dancers, and for a certain scene, the choreographer wants to arrange the dancers in groups of equal sizes on stage.

In how many ways can the dancers be put into groups of equal size? Answering this question is the same as identifying the factors of 24. [link] summarizes the different ways that the choreographer can arrange the dancers.

Number of Groups	**Dancers per Group**	**Total Dancers**

1	24	$1\cdot24=24$
2	12	$2\cdot12=24$
3	8	$3\cdot8=24$
4	6	$4\cdot6=24$
6	4	$6\cdot4=24$
8	3	$8\cdot3=24$
12	2	$12\cdot2=24$
24	1	$24\cdot1=24$

What patterns do you see in [link]? Did you notice that the number of groups times the number of dancers per group is always 24? This makes sense, since there are always 24 dancers.

You may notice another pattern if you look carefully at the first two columns. These two columns contain the exact same set of numbers—but in reverse order. They are mirrors of one another, and in fact, both columns list all of the factors of 24, which are: $1,2,3,4,6,8,12,24$

We can find all the factors of any counting number by systematically dividing the number by each counting number, starting with 1. If the quotient is also a counting number, then the divisor and the quotient are factors of the number. We can stop when the quotient becomes smaller than the divisor.

Find all the factors of a counting number.

Divide the number by each of the counting numbers, in order, until the quotient is smaller than the divisor.

- If the quotient is a counting number, the divisor and quotient are a pair of factors.
- If the quotient is not a counting number, the divisor is not a factor.

List all the factor pairs. Write all the factors in order from smallest to largest.

Find all the factors of 72.

Solution

Divide 72 by each of the counting numbers starting with 1. If the quotient is a whole number, the divisor and quotient are a pair of factors.

Dividend	Divisor	Quotient	Factors
72	1	72	1, 72
72	2	36	2, 36
72	3	24	3, 24
72	4	18	4, 18
72	5	14.4	–
72	6	12	6, 12
72	7	~10.29	–
72	8	9	8, 9

The next line would have a divisor of 9 and a quotient of 8. The quotient would be smaller than the divisor, so we stop. If we continued, we would end up only listing the same factors again in reverse order. Listing all the factors from smallest to greatest, we have

1,2,3,4,6,8,9,12,18,24,36,and72

Additional Online Resources

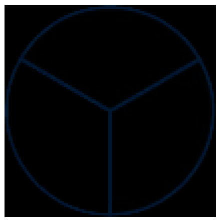
- Divisibility Rules
- Factors
- Ex 1: Determine Factors of a Number
- Ex 2: Determine Factors of a Number
- Ex 3: Determine Factors of a Number

In the circle, 23 of the circle is shaded—2 of the 3

equal parts.

Fraction Basics

A **fraction** is a way to represent parts of a whole. The fraction 23 represents two of three equal parts. See [link]. In the fraction 23, the 2 is called the **numerator** and the 3 is called the **denominator**. The line is called the fraction bar.



Fraction

A **fraction** is written ab, where $b \neq 0$ and
a is the **numerator** and *b* is the **denominator**.
A fraction represents parts of a whole. The denominator b is the number of equal parts the whole has been divided into, and the numerator a indicates how many parts are included.

Fractions that have the same value are **equivalent fractions**. The Equivalent Fractions

Property allows us to find equivalent fractions and

also simplify fractions.

> **Equivalent Fractions Property**
> If *a*, *b*, and *c* are numbers where $b \neq 0, c \neq 0$,
> then $ab = a \cdot cb \cdot c$ and $a \cdot cb \cdot c = ab$.

A fraction is considered simplified if there are no common factors, other than 1, in its numerator and denominator.

For example,

23 is simplified because there are no common factors of 2 and 3.

1015 is not simplified because 5 is a common factor of 10 and 15.

We simplify, or reduce, a fraction by removing the common factors of the numerator and denominator. A fraction is not simplified until all common factors have been removed. If an expression has fractions, it is not completely simplified until the fractions are simplified.

Sometimes it may not be easy to find common factors of the numerator and denominator. When

this happens, a good idea is to factor the numerator and the denominator into prime numbers. Then divide out the common factors using the Equivalent Fractions Property.

How To Simplify a Fraction

Simplify: -315770.

Step 1. Rewrite the numerator and denominator to show the common factors. If needed, use a factor tree.	Rewrite 315 and 770 as the product of the primes.	$-\frac{315}{770}$ $-\frac{3 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 5 \cdot 7 \cdot 11}$
Step 2. Simplify using the equivalent fractions property by dividing out common factors.	Mark the common factors 5 and 7. Divide out the common factors	$-\frac{3 \cdot 3 \cdot \cancel{5} \cdot \cancel{7}}{2 \cdot \cancel{5} \cdot \cancel{7} \cdot 11}$ $-\frac{3 \cdot 3}{2 \cdot 11}$
Step 3. Multiply the remaining factors, if necessary.		$-\frac{9}{22}$

Simplify: -69120.

-2340

We now summarize the steps you should follow to simplify fractions.

Simplify a fraction.

Rewrite the numerator and denominator to show the common factors.
If needed, factor the numerator and denominator into prime numbers first. Simplify using the Equivalent Fractions Property by dividing out common factors. Multiply any remaining factors.

Multiply Divide Fractions

Many people find multiplying and dividing fractions easier than adding and subtracting fractions.

To multiply fractions, we multiply the numerators and multiply the denominators.

Fraction Multiplication

If a, b, c, and d are numbers where $b \neq 0$, and $d \neq 0$, then
$ab \cdot cd = acbd$
To multiply fractions, multiply the numerators and multiply the denominators.

When multiplying fractions, the properties of positive and negative numbers still apply, of course. It is a good idea to determine the sign of the product as the first step. In [link], we will multiply negative and a positive, so the product will be negative.

When multiplying a fraction by an integer, it may be helpful to write the integer as a fraction. Any integer, a, can be written as $a1$. So, for example, $3 = 31$.

Multiply: $-125(-20x)$.

The first step is to find the sign of the product. Since the signs are the same, the product is positive.

[redacted]

Determine the sign of the product. The sig [redacted] are the same, so the product is positive.

Write $20x$ as a fraction.

[redacted]

Multiply.

[redacted]

Rewrite 20 to show the common factor 5 an [redacted] out.

Simplify.

[redacted]

Multiply: $113(-9a)$.

$-33a$

Multiply: 137(−14b).

−26b

Now that we know how to multiply fractions, we are almost ready to divide. Before we can do that, we need some vocabulary. The **reciprocal** of a fraction is found by inverting the fraction, placing the numerator in the denominator and the denominator in the numerator. The reciprocal of 23 is 32. Since 4 is written in fraction form as 41, the reciprocal of 4 is 14.

To divide fractions, we multiply the first fraction by the reciprocal of the second.

Fraction Division

If *a*, *b*, *c*, and *d* are numbers where $b \neq 0, c \neq 0$, and $d \neq 0$, then

$ab \div cd = ab \cdot dc$

To divide fractions, we multiply the first fraction

by the **reciprocal** of the second.

We need to say $b \neq 0$, $c \neq 0$, and $d \neq 0$, to be sure we don't divide by zero!

Find the quotient: $-718 \div (-1427)$.

$$-\frac{7}{18} \div \left(-\frac{14}{27}\right)$$

To divide, multiply the first fraction by the reciprocal of the second.

$$-\frac{7}{18}\left(-\frac{27}{14}\right)$$

Determine the sign of the product, and the

$$\frac{7 \cdot 27}{18 \cdot 14}$$

Rewrite showing

co $\frac{\cancel{7} \cdot \cancel{9} \cdot 3}{\cancel{9} \cdot 2 \cdot \cancel{7} \cdot 2}$ ors.

Remove common factors.

$$\frac{3}{2 \cdot 2}$$

Simplify.

$$\frac{3}{4}$$

Divide: $-727 \div (-3536)$.
415

Divide: $-514 \div (-1528)$.
23

The numerators or denominators of some fractions contain fractions themselves. A fraction in which the numerator or the denominator is a fraction is called a **complex fraction**.

> Complex Fraction
>
> A **complex fraction** is a fraction in which the numerator or the denominator contains a fraction.

Some examples of complex fractions are:

$$\frac{\frac{6}{7}}{3} \quad \frac{\frac{3}{4}}{\frac{5}{8}} \quad \frac{\frac{x}{2}}{\frac{5}{6}}$$

To simplify a complex fraction, remember that the fraction bar means division. For example, the complex fraction $\frac{\frac{3}{4}}{\frac{5}{8}}$ means $\frac{3}{4} \div \frac{5}{8}$.

Simplify: $\frac{\frac{x}{2}}{\frac{xy}{6}}$.
$\frac{\frac{x}{2}}{\frac{xy}{6}}$ Rewrite as division. $\frac{x}{2} \div \frac{xy}{6}$ Multiply the first fraction by the reciprocal of the second. $\frac{x}{2} \cdot \frac{6}{xy}$ Multiply. $\frac{x \cdot 6}{2 \cdot xy}$ Look for common factors. $\frac{x \cdot 3 \cdot 2}{2 \cdot x \cdot y}$ Divide common factors and simplify. $\frac{3}{y}$

Simplify: a8ab6.

34b

Add Subtract Fractions

When we multiplied fractions, we just multiplied the numerators and multiplied the denominators right straight across. To add or subtract fractions, they must have a common denominator.

Fraction Addition and Subtraction
If *a*, *b*, and *c* are numbers where $c \neq 0$, then
$ac + bc = a + bcandac - bc = a - bc$
To add or subtract fractions, add or subtract the numerators and place the result over the common denominator.

The **least common denominator** (LCD) of two fractions is the smallest number that can be used as a common denominator of the fractions. The LCD of the two fractions is the least common multiple

(LCM) of their denominators.

Least Common Denominator

The **least common denominator** (LCD) of two fractions is the least common multiple (LCM) of their denominators.

After we find the least common denominator of two fractions, we convert the fractions to equivalent fractions with the LCD. Putting these steps together allows us to add and subtract fractions because their denominators will be the same!

How to Add or Subtract Fractions

Add: $712+518$.

Step 1. Do they have a common denominator?	No.	
• No—rewrite each fraction with the LCD (least common denominator).	Find the LCD of 12, 18.	$12 = 2 \cdot 2 \cdot 3$ $18 = 2 \cdot \quad 3 \cdot 3$ $\text{LCD} = 2 \cdot 2 \cdot 3 \cdot 3$ $\text{LCD} = 36$ LCD is 36.
	We multiply the numerator and denominator of each fraction by the factor needed to get the denominator to be 36. Do not simplify the equivalent fractions! If you do, you'll get back to the original fractions and lose the common denominator!	$\frac{7}{12} + \frac{5}{18}$ $\frac{7 \cdot 3}{12 \cdot 3} + \frac{5 \cdot 2}{18 \cdot 2}$ $\frac{21}{36} + \frac{10}{36}$
Step 2. Add or subtract the fractions.	Add.	$\frac{31}{36}$
Step 3. Simplify, if possible.	Since 31 is prime, its only factors are 1 and 31. Since 31 does not go into 36, the answer is simplified.	

Add: $712 + 1115$.

7960

Add: $1315 + 1720$.

10360

Add or subtract fractions.

Do they have a common denominator?

- Yes—go to step 2.
- No—rewrite each fraction with the LCD (least common denominator).
    - Find the LCD.
    - Change each fraction into an equivalent fraction with the LCD as its denominator.

Add or subtract the fractions. Simplify, if possible.

We now have all four operations for fractions. [link] summarizes fraction operations.

Fraction Multiplication		Fraction Division
$ab \cdot cd = acbd$		$ab \div cd = ab \cdot dc$
Multiply the numerators		Multiply the first fraction

and multiply the denominators	by the reciprocal of the second.
Fraction Addition	**Fraction Subtraction**
$ac + bc = a + bc$	$ac - bc = a - bc$
Add the numerators and place the sum over the common denominator.	Subtract the numerators and place the difference over the common denominator.
To multiply or divide fractions, an LCD is NOT needed. To add or subtract fractions, an LCD is needed.	

When starting an exercise, always identify the operation and then recall the methods needed for that operation.

Simplify: ⓐ $5x6 - 310$ ⓑ $5x6 \cdot 310$.

First ask, “What is the operation?” Identifying the operation will determine whether or not we need a common denominator. Remember, we need a common denominator to add or subtract, but not to multiply or divide.

ⓐ

What is the operation? The operation is subtraction. Do the fractions have a common denominator? No.5x6 – 310 Find the LCD of6and10The LCD is 30.
6 = 2·310 = 2·5________LCD = 2·3·5LCD = 30
Rewrite each fraction as an equivalent fraction with the LCD. 5x·56·5 – 3·310·3 25x30 – 930
Subtract the numerators and place thedifference over the common denominators.25x – 930 Simplify,if possible.There are no common factors.The fraction is simplified.

ⓑ

What is the operation? Multiplication.25x6·310 To multiply fractions,multiply the numeratorsand multiply the denominators.25x·36·10 Rewrite, showing common factors.Remove common factors.5x·32·3·2·5 Simplify.x4

Notice, we needed an LCD to add 25x6 – 310, but not to multiply 25x6·310.

Simplify: ⓐ 3a4 – 89 ⓑ 3a4·89.

ⓐ $27a - 3236$ ⓑ $2a3$

Simplify: ⓐ$4k5 - 16$ ⓑ $4k5 \cdot 16$.

ⓐ $24k - 530$ ⓑ $2k15$

Decimal Basics

To add or subtract decimals, we line up the decimal points. By lining up the decimal points this way, we can add or subtract the corresponding place values. We then add or subtract the numbers as if they were whole numbers and then place the decimal point in the sum.

Add or subtract decimals.

Determine the sign of the sum or difference. Write the numbers so the decimal points line up

vertically. Use zeros as placeholders, as needed. Add or subtract the numbers as if they were whole numbers. Then place the
decimal point in the answer under the decimal points in the given numbers. Write the sum or difference with the appropriate sign.

Add or subtract: ⓐ $-23.5-41.38$ ⓑ $14.65-20$.

ⓐ

$-23.5-41.38$ The difference will be negative. To subtract, we add thenumerals. Write the numbers so the decimal points lineup vertically.$23.5+41.38$____ Put 0 as a placeholder after the 5 in23.5.Remember,$510=50100$so$0.5=0.50.23.50+41.3$ Add the numbers as if they were whole numbers.Then place the decimal point in the sum.$23.50+41.38$____64.88 Write the result with the correct sign. $-23.5-41.38=-64.88$

ⓑ

$14.65-20$ The difference will be negative. To subtract, wesubtract 14.65 from 20. Write the numbers so the decimal points line upvertically.$20-14.65$____ Remember, 20 is a

whole number, so place thedecimal point after the 0. Put in zeros to the right as placeholders.$20.00-14.65$____ Subtract and place the decimal point in the answer. $99110101020.00-14.65$__________ 5.35 Write the result with the correct sign.$14.65-20=-5.35$

Add or subtract: ⓐ $-4.8-11.69$ ⓑ$9.58-10.$

ⓐ -16.49 ⓑ -0.42

When we multiply signed decimals, first we determine the sign of the product and then multiply as if the numbers were both positive. We multiply the numbers temporarily ignoring the decimal point and then count the number of decimal points in the factors and that sum tells us the number of decimal places in the product. Finally, we write the product with the appropriate sign.

Multiply decimals.

Determine the sign of the product. Write in vertical format, lining up the numbers on the right. Multiply the numbers as if they were whole numbers, temporarily ignoring the decimal points. Place the decimal point. The number of decimal places in the product is the sum of the number of decimal places in the factors. Write the product with the appropriate sign.

Multiply: $(-3.9)(4.075)$.

	$(-3.9)(4.075)$
The signs are different. The product will be negative.	The product will be negative.
Write in vertical format, lining up the nu	$\begin{array}{r} 4.075 \\ \times\ 3.9 \\ \hline \end{array}$
Multiply.	

$$\begin{array}{r} 4.075 \\ \times\ 3.9 \\ \hline 36675 \\ 12225 \\ \hline 158925 \end{array}$$

Add the number of decimal places in the... Pl... 4 p...

$$\begin{array}{r} 4.075 \\ \times\ 3.9 \\ \hline 36675 \\ 12225 \\ \hline 15.8925 \end{array}$$

(–3.9) (4.075)

The signs are the different, so the product is negative.

$(-3.9)(4.075) = -15.8925$

Multiply: $-4.5(6.107)$.

-27.4815

Often, especially in the sciences, you will multiply decimals by powers of 10 (10, 100, 1000, etc). If you multiply a few products on paper, you may notice a pattern relating the number of zeros in the power of 10 to number of decimal places we move the decimal point to the right to get the product.

Multiply a decimal by a power of ten.

Move the decimal point to the right the same number of places as the
number of zeros in the power of 10. Add zeros at the end of the number as needed.

Multiply: 5.63 by ⓐ 10 ⓑ 100 ⓒ 1000.

By looking at the number of zeros in the multiple of ten, we see the number of places we need to move the decimal to the right.

ⓐ

[redacted]

There is 1 zero in 10, so move the [redacted]al point 1 place to the right.

56.3

ⓑ

[redacted]

There are 2 zeroes in 100, so move the de[redacted]nt 2 places to the right.

[redacted]

ⓒ

$5.63(1,000)$

There are 3 zeroes in 1,000, so move the de 5.63 t 3 places to the right.

A zero must be added to the end.

$5,630$

Just as with multiplication, division of signed decimals is very much like dividing whole numbers. We just have to figure out where the decimal point must be placed and the sign of the quotient. When dividing signed decimals, first determine the sign of the quotient and then divide as if the numbers were both positive. Finally, write the quotient with the appropriate sign.

We review the notation and vocabulary for division:

$$\underset{\text{dividend}}{a} \div \underset{\text{divisor}}{b} = \underset{\text{quotient}}{c} \qquad \underset{\text{divisor}}{b}\overset{\overset{\text{quotient}}{c}}{\overline{)\underset{\text{dividend}}{a}}}$$

We’ll write the steps to take when dividing decimals for easy reference.

Divide decimals.

Determine the sign of the quotient. Make the divisor a whole number by “moving” the decimal point all the way to the right. “Move” the decimal point in the dividend the same number of places—adding zeros as needed. Divide. Place the decimal point in the quotient above the decimal point in the dividend. Write the quotient with the appropriate sign.

Divide: $-25.65 \div (-0.06)$.

Remember, you can “move” the decimals in the divisor and dividend because of the Equivalent Fractions Property.

[redacted]

The signs are the same. The quotient is positive.

Make the divisor a whole number by “moving” the decimal point all the way to the right.

“Move” the decimal point in the dividend the number of places.

$0.06\overline{)25.65}$

Divide.

Place the decimal point in the

$$\begin{array}{r} 427.5 \\ 006.\overline{)2565.0} \\ \underline{-24} \\ 16 \\ \underline{-12} \\ 45 \\ \underline{-42} \\ 30 \\ \underline{-30} \end{array}$$

Write the quotient with the appropriate sign.

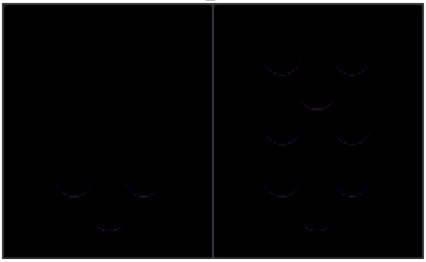
$-25.65 \div (-0.06) = 427.5$

The illustration shows a model of an equation with one variable. On the left side of the workspace is an unknown (envelope) and three counters, while on the right side of the workspace are eight counters. The illustration shows a model for solving an equation with one variable. On both sides of the

workspace remove three counters, leaving only the unknown (envelope) and five counters on the right side. The unknown is equal to five counters. The illustration shows a model for the equation $x+3=8$.

Solve Equations Subtraction and Addition Property of Equality

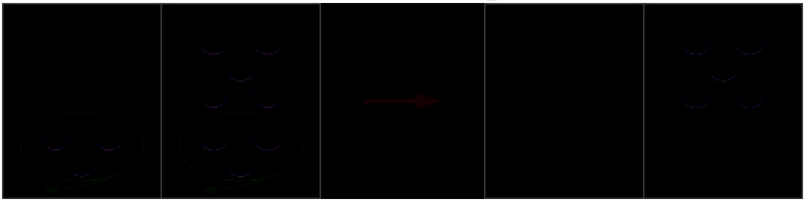
We are going to use a model to clarify the process of solving an equation. An envelope represents the variable – since its contents are unknown – and each counter represents one. We will set out one envelope and some counters on our workspace, as shown in [link]. Both sides of the workspace have the same number of counters, but some counters are "hidden" in the envelope. Can you tell how many counters are in the envelope?



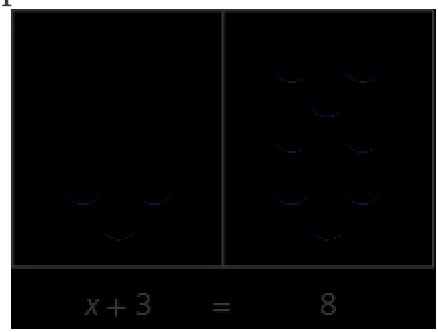
What are you thinking? What steps are you taking in your mind to figure out how many counters are in the envelope?

Perhaps you are thinking: "I need to remove the 3 counters at the bottom left to get the envelope by itself. The 3 counters on the left can be matched

with 3 on the right and so I can take them away from both sides. That leaves five on the right—so there must be 5 counters in the envelope.” See [link] for an illustration of this process.



What algebraic equation would match this situation? In [link] each side of the workspace represents an expression and the center line takes the place of the equal sign. We will call the contents of the envelope x.



Let’s write algebraically the steps we took to discover how many counters were in the envelope:

[redacted]

First, we took away three from each side.

[redacted]

Then we were left with five.

[redacted]

Check:

Five in the envelope plus three more does equal eight!
$5+3=8$

Our model has given us an idea of what we need to do to solve one kind of equation. The goal is to isolate the variable by itself on one side of the equation. To solve equations such as these mathematically, we use the **Subtraction Property of Equality**.

> Subtraction Property of Equality
> For any numbers *a*, *b*, and *c*,
> $\text{If } a=b, \text{ then } a-c=b-c$
> When you subtract the same quantity from both sides of an equation, you still have equality.

Doing the Manipulative Mathematics activity "Subtraction Property of Equality" will help you develop a better understanding of how to solve equations by using the Subtraction Property of Equality.

Let's see how to use this property to solve an equation. Remember, the goal is to isolate the variable on one side of the equation. And we check our solutions by substituting the value into the equation to make sure we have a true statement.

Solve: $y+37=-13$.

Solution

To get *y* by itself, we will undo the addition of 37 by using the Subtraction Property of Equality.

[redacted]

Subtract 37 from each side to [redacted] addition.

Simplify.

[redacted]

Check:

[redacted]

Substitute $y = -50$

[redacted]

Since $y = -50$ makes $y + 37 = -13$ a true statement, we have the solution to this equation.

Solve: $x + 19 = -27.$

$x = -46$

What happens when an equation has a number subtracted from the variable, as in the equation $x - 5 = 8$? We use another property of equations to solve equations where a number is subtracted from the variable. We want to isolate the variable, so to 'undo' the subtraction we will add the number to both sides. We use the **Addition Property of Equality**.

Addition Property of Equality
For any numbers *a*, *b*, and *c*,
$$\text{If } a = b, \text{then } a + c = b + c$$
When you add the same quantity to both sides of an equation, you still have equality.

In [link], 37 was added to the *y* and so we subtracted 37 to 'undo' the addition. In [link], we will need to 'undo' subtraction by using the Addition Property of Equality.

Solve: $a - 28 = -37$.

Solution

[redacted]

Add 28 to each side to 'undo' the [redacted]

Simplify.

[redacted]

Check:

[redacted]

Substitute $a = -9$

[redacted]

[redacted]

The solution to $a - 28 = -37$ is $a = -9$.

Solve: $n - 61 = -75$.

$n = -14$

Solve: $x - \frac{5}{8} = \frac{3}{4}$.

Solution

$$x - \frac{5}{8} = \frac{3}{4}$$

Use the Addition Property of Equality.

$$x - \frac{5}{8} + \frac{5}{8} = \frac{3}{4} + \frac{5}{8}$$

Find the LCD to add the fractions on the right.

$$x - \frac{5}{8} + \frac{5}{8} = \frac{6}{8} + \frac{5}{8}$$

Simplify.

$$x = \frac{11}{8}$$

Check:

$$x - \frac{5}{8} = \frac{3}{4}$$

Substitute $x = \frac{11}{8}$.

$$\frac{11}{8} - \frac{5}{8} \stackrel{?}{=} \frac{3}{4}$$

Subtract.

$\frac{6}{8} \stackrel{?}{=} \frac{3}{4}$

Simplify.

$\frac{3}{4} = \frac{3}{4} \checkmark$

The solution to $x - 58 = 34$ is $x = 118$.

The next example will be an equation with decimals.

Solve: $n - 0.63 = -4.2$.

Solution

[redacted]

Use the Addition Property of Equality. Add.

[redacted]

[redacted]

Check:

[redacted]

Let $n = -3.57$.

[redacted]

[redacted]

Solve: $c - 0.93 = -4.6$.

$c = -3.67$

The illustration shows a model of an equation with

one variable multiplied by a constant. On the left side of the workspace are two instances of the unknown (envelope), while on the right side of the workspace are six counters. The illustration shows a model of the equation $2x=6$.

Solve Equations with Multiplication and Division Property

You may have noticed that all of the equations we have solved so far have been of the form $x+a=b$ or $x-a=b$. We were able to isolate the variable by adding or subtracting the constant term on the side of the equation with the variable. Now we will see how to solve equations that have a variable multiplied by a constant and so will require division to isolate the variable.

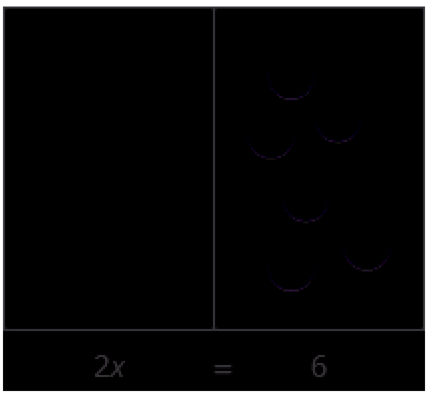
Let’s look at our puzzle again with the envelopes and counters in [link].

In the illustration there are two identical envelopes that contain the same number of counters.

Remember, the left side of the workspace must equal the right side, but the counters on the left side are “hidden” in the envelopes. So how many counters are in each envelope?

How do we determine the number? We have to separate the counters on the right side into two groups of the same size to correspond with the two envelopes on the left side. The 6 counters divided into 2 equal groups gives 3 counters in each group (since $6 \div 2 = 3$).

What equation models the situation shown in [link]? There are two envelopes, and each contains x counters. Together, the two envelopes must contain a total of 6 counters.



[redacted]

If we divide both sides of the equation by 2, as we di[redacted] the envelopes and counters, we get:

[redacted]

We found that each envelope contains 3 counters. Does this check? We know $2 \cdot 3 = 6$, so it works! Three counters in each of two envelopes does equal six!

This example leads to the **Division Property of Equality**.

The Division Property of Equality
For any numbers *a*, *b*, and *c*, and $c \neq 0$,
Ifa = b,thenac = bc
When you divide both sides of an equation by any non-zero number, you still have equality.

Doing the Manipulative Mathematics activity “Division Property of Equality” will help you develop a better understanding of how to solve equations by using the Division Property of

Equality.

The goal in solving an equation is to ‘undo’ the operation on the variable. In the next example, the variable is multiplied by 5, so we will divide both sides by 5 to ‘undo’ the multiplication.

Solve: $5x = -27$.

Solution

To isolate x, “undo” the mu[redacted] by 5.	[redacted]
Divide to ‘undo’ the m[redacted]	[redacted]
Simplify.	[redacted]

Check:

[redacted]

Substitute -275 for x.

[redacted]

[redacted]

Since this is a true statement, $x=-275$ is the solution to $5x=-27$.

Solve: $3y=-41$.

$y=-413$

Consider the equation $x4=3$. We want to know what number divided by 4 gives 3. So to “undo” the division, we will need to multiply by 4. The **Multiplication Property of Equality** will allow us

to do this. This property says that if we start with two equal quantities and multiply both by the same number, the results are equal.

The Multiplication Property of Equality

For any numbers *a*, *b*, and *c*,

If $a = b$, then $ac = bc$

If you multiply both sides of an equation by the same number, you still have equality.

Solve: $y - 7 = -14$.

Solution

Here y is divided by -7. We must multiply by -7 to isolate y.

[redacted]

Multiply both

sid[redacted]

Multiply.

[redacted]

Simplify.

[redacted]

Check: $y - 7 = -14$

Substitute $y = 98$.

[redacted]

Divide.

[redacted]

Solve: $a - 7 = -42$.

$a = 294$

Solve: $-n=9$.

Solution

[redacted]

Remember $-n$ is equivalent to [redacted]

Divide both sides by -1.

[redacted]

Divide.

[redacted]

Notice that there are two other ways to solve $-n=9$. We can also solve this equation by multiplying both sides by -1 and also by taking the

opposite of both sides.

Check:

[redacted]

Substitute $n = -9$.

[redacted]

Simplify.

[redacted]

Solve: $34x = 12$.

Solution

Since the product of a number and its reciprocal is 1, our strategy will be to isolate x by multiplying by the reciprocal of 34.

[redacted]

Multiply by the reciprocal of 34 [redacted]

Reciprocals multiply to 1. [redacted]

Multiply.

[redacted]

Notice that we could have divided both sides of the equation 34x = 12 by 34 to isolate x. While this would work, most people would find multiplying by the reciprocal easier.

Check:

[redacted]

Substitute x = 16.

[redacted]

[redacted]

In the next example, all the variable terms are on the right side of the equation. As always, our goal in solving the equation is to isolate the variable.

Solve: 815 = − 45x.

Solution

[redacted]

Multiply by the reciprocal of
−

[redacted]

Reciprocals

mu[redacted]

Multiply.

[redacted]

Check:

[redacted]

Let $x = -23$.

[redacted]

Solve: $56 = -83r$.

$r = -516$

Subtraction Property of EqualityAddition Property of Equality For any real numbersa,b,andc,For any real numbersa,b,andc, ifa = b,thena − c = b

$-c$. if $a=b$, then $a+c=b+c$. Division Property of EqualityMultiplication Property of Equality For any numbers a, b, and c, and $c \neq 0$, For any numbers a, b, and c, if $a=b$, then $ac=bc$. if $a=b$, then $ac=bc$.

When you add, subtract, multiply, or divide the same quantity from both sides of an equation, you still have equality.

Among every 100 community college students, 57 are female.

Basics of Percents

How many cents are in one dollar? There are 100 cents in a dollar. How many years are in a century? There are 100 years in a century. Does this give you a clue about what the word “percent” means? It is really two words, “per cent,” and means per one hundred. A **percent** is a ratio whose denominator is 100. We use the percent symbol %, to show percent.

Percent

A percent is a ratio whose denominator is 100.

According to data from the American Association of Community Colleges (2015), about 57% of community college students are female. This means 57 out of every 100 community college students are female, as [link] shows. Out of the 100 squares on the grid, 57 are shaded, which we write as the ratio 57100.

[missing_resource: CNX_BMath_Figure_06_01_001.jpg]

Similarly, 25% means a ratio of 25100,3% means a ratio of 3100 and 100% means a ratio of 100100. In words, "one hundred percent" means the total 100% is 100100, and since $100100=1$, we see that 100% means 1 whole.

According to the Public Policy Institute of California (2010),44% of parents of public school children would like their youngest child to earn a graduate degree. Write this percent as a ratio.

Solution

The amount we want to convert is 44%.	44%
Write the percent as a ratio. Remember that *percent* means per 100.	44100

Write the percent as a ratio.

According to a survey, 89% of college students have a smartphone.

89100

In 2007, according to a U.S. Department of Education report, 21 out of every 100 first-time freshmen college students at 4-year public institutions took at least one remedial course. Write this as a ratio and then as a percent.

Solution

The amount we want to convert is 21 out of 100.	21 out of 100
Write as a ratio.	21100
Convert the 21 per 100 to percent.	21%

Write as a ratio and then as a percent: The American Association of Community Colleges reported that 62 out of 100 full-time community college students balance their studies with full-time or part time employment.

62100,62%

Since percents are ratios, they can easily be expressed as fractions. Remember that percent means per 100, so the denominator of the fraction is 100.

Convert a percent to a fraction.

Write the percent as a ratio with the denominator 100. Simplify the fraction if possible.

Convert each percent to a fraction:

ⓐ36%
ⓑ125%

Solution

ⓐ	
	36%
Write as a ratio with denominator 100.	36100
Simplify.	925

ⓑ		
	125%	
Write as a ratio with denominator 100.	$\frac{125}{100}$	
Simplify.	$\frac{5}{4}$	

Convert each percent to a fraction:

ⓐ 48%
ⓑ 110%

ⓐ $\frac{12}{25}$
ⓑ $\frac{11}{10}$

The previous example shows that a percent can be greater than 1. We saw that 125% means $\frac{125}{100}$, or $\frac{5}{4}$. These are improper fractions, and their values are greater than one.

Convert each percent to a fraction:

ⓐ24.5%
ⓑ3313%

Solution

ⓐ

	24.5%
Write as a ratio with denominator 100.	24.5100
Clear the decimal by multiplying numerator and denominator by 10.	24.5(10)100(10)
Multiply.	2451000
Rewrite showing common factors.	5·495·200
Simplify.	49200

ⓑ

	3313%
Write as a ratio with denominator 100.	3313100
Write the numerator as an improper fraction.	1003100
Rewrite as fraction division, replacing 100 with 1001.	1003 ÷ 1001
Multiply by the reciprocal.	1003·1100
Simplify.	13

Convert each percent to a fraction:

ⓐ64.4%
ⓑ6623%

ⓐ161250
ⓑ23

In Decimals, we learned how to convert fractions to decimals. To convert a percent to a decimal, we first

convert it to a fraction and then change the fraction to a decimal.

Convert a percent to a decimal.

Write the percent as a ratio with the denominator 100. Convert the fraction to a decimal by dividing the numerator by the denominator.

Convert each percent to a decimal:

ⓐ 6%
ⓑ 78%

Solution

Because we want to change to a decimal, we will leave the fractions with denominator 100 instead of removing common factors.

ⓐ	
	6%
Write as a ratio with denominator 100.	6100
Change the fraction to a decimal by dividing the numerator by the denominator.	0.06

ⓑ	
	78%
Write as a ratio with denominator 100.	78100
Change the fraction to a decimal by dividing the numerator by the denominator.	0.78

Convert each percent to a decimal:

ⓐ 9%

ⓑ87%

ⓐ 0.09
ⓑ 0.87

Convert each percent to a decimal:

ⓐ135%
ⓑ12.5%

Solution

ⓐ

	135%
Write as a ratio with denominator 100.	135100
Change the fraction to a decimal by dividing the numerator by the denominator.	1.35

ⓑ	
	12.5%
Write as a ratio with denominator 100.	12.5100
Change the fraction to a decimal by dividing the numerator by the denominator.	0.125

Convert each percent to a decimal:

ⓐ115%
ⓑ23.5%

ⓐ 1.15
ⓑ 0.235

Let's summarize the results from the previous examples in [link], and look for a pattern we could use to quickly convert a percent number to a decimal number.

Percent	Decimal
6%	0.06
78%	0.78
135%	1.35
12.5%	0.125

Do you see the pattern?

To convert a percent number to a decimal number, we move the decimal point two places to the left and remove the % sign. (Sometimes the decimal point does not appear in the percent number, but just like we can think of the integer 6 as 6.0, we can think of 6% as 6.0%.) Notice that we may need to add zeros in front of the number when moving the decimal to the left.

[link] uses the percents in [link] and shows visually how to convert them to decimals by moving the decimal point two places to the left.

Percent	Decimal
006.%	0.06
078.%	0.78
135.%	1.35
012.5%	0.125

Among a group of business leaders, 77% believe that poor math and science education in the U.S. will lead to higher unemployment rates.

Convert the percent to: ⓐ a fraction ⓑ a decimal

Solution

ⓐ	
	77%
Write as a ratio with denominator 100.	77100

ⓑ	
	77100
Change the fraction to a decimal by dividing the numerator by the denominator.	0.77

Convert the percent to: ⓐ a fraction and ⓑ a decimal

Twitter's share of web traffic jumped 24% when one celebrity tweeted live on air.

ⓐ625
ⓑ0.24

Convert the percent to: ⓐ a fraction and ⓑ a decimal

The U.S. Census estimated that in 2013,44% of the population of Boston age 25 or older have a bachelor's or higher degrees.

ⓐ2250
ⓑ0.44

There are four suits of cards in a deck of cards—hearts, diamonds, clubs, and spades. The probability of randomly choosing a heart from a shuffled deck of cards is 25%. Convert the percent to:

ⓐ a fraction
ⓑ a decimal

(credit: Riles32807, Wikimedia Commons)

Solution

ⓐ	
	25%

Write as a ratio with denominator 100.	25100
Simplify.	14

ⓑ	14
Change the fraction to a decimal by dividing the numerator by the denominator.	0.25

Convert the percent to: ⓐ a fraction, and ⓑ a decimal

The probability that it will rain Monday is 30%.

ⓐ310
ⓑ0.3

Convert the percent to: ⓐ a fraction, and ⓑ a decimal

The probability of getting heads three times when tossing a coin three times is 12.5%.

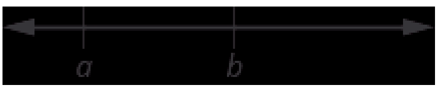
ⓐ12.5100
ⓑ0.125

Key Concepts

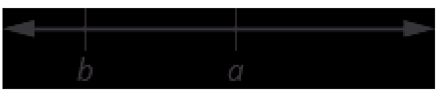
Operation	Notation	Say:	The result is…
Addition	$a+b$	aplusb	the sum of a and b
Multiplication	$a\cdot b$,(a)(b),(a)b,a(b)	atimesb	The product of a and b
Subtraction	$a-b$	aminusb	the

			difference of a and b
Division	$a \div b$,a/b,ab,ba	a divided by b	The quotient of a and b

- **Equality Symbol**
  - $a = b$ is read as a is equal to b
  - The symbol $=$ is called the equal sign.
- **Inequality**
  - $a < b$ is read a is less than b
  - a is to the left of b on the number line



  - $a > b$ is read a is greater than b
  - a is to the right of b on the number line



Algebraic Notation	**Say**
$a = b$	a is equal to b

$a \neq b$	a is not equal to b
$a < b$	a is less than b
$a > b$	a is greater than b
$a \leq b$	a is less than or equal to b
$a \geq b$	a is greater than or equal to b

- **Addition of Positive and Negative Integers** $5+3-5+(-3)8-8$both positive,both negative,sum positivesum negative $-5+35+(-3)-22$different signs,different signs,more negativesmore positivessum negativesum positive
- **Property of Absolute Value**: $|n| \geq 0$ for all numbers. Absolute values are always greater than or equal to zero!
- **Subtraction of Integers** $5-3-5-(-3)2-25$positives5negativestake away3positivestake away3negatives2 positives2 negatives$-5-35-(-3)-885$negatives, want to5positives, want tosubtract3positivessubtract3negativesneed neutral pairsneed neutral pairs
- **Subtraction Property:** Subtracting a number is the same as adding its opposite.

Divisibility Tests	
A number is divisible by	
2	if the last digit is **0, 2, 4, 6, or 8**
3	if the sum of the digits is divisible by **3**
5	if the last digit is **5 or 0**
6	if divisible by both **2** and **3**
10	if the last digit is **0**

- **Factors** If a·b = m, then a and b are factors of m, and m is the product of a and b.
- **Find all the factors of a counting number.**

  Divide the number by each of the counting numbers, in order, until the quotient is smaller than the divisor.

  1. If the quotient is a counting number, the divisor and quotient are a pair of factors.
  2. If the quotient is not a counting number, the divisor is not a factor.

  List all the factor pairs. Write all the factors in order from smallest to largest.

- **Determine if a number is prime.**

  Test each of the primes, in order, to see if it is a factor of the number. Start with 2 and stop

when the quotient is smaller than the divisor or when a prime factor is found. If the number has a prime factor, then it is a composite number. If it has no prime factors, then the number is prime.

- **Equivalent Fractions Property**
  If *a*, *b*, and *c* are numbers where $b \neq 0, c \neq 0$, then
  $ab = a\cdot cb\cdot c$ and $a\cdot cb\cdot c = ab$.
- **How to simplify a fraction.**

  Rewrite the numerator and denominator to show the common factors.
  If needed, factor the numerator and denominator into prime numbers first. Simplify using the Equivalent Fractions Property by dividing out common factors. Multiply any remaining factors.

- **Fraction Multiplication**
  If *a*, *b*, *c*, and *d* are numbers where $b \neq 0$, and $d \neq 0$, then
  $ab\cdot cd = acbd$.
  To multiply fractions, multiply the numerators and multiply the denominators.
- **Fraction Division**
  If *a*, *b*, *c*, and *d* are numbers where $b \neq 0, c \neq 0$, and $d \neq 0$, then
  $ab \div cd = ab\cdot dc$.
  To divide fractions, we multiply the first

fraction by the reciprocal of the second.

- **Fraction Addition and Subtraction**
  If a, b, and c are numbers where $c \neq 0$, then $ac + bc = a + bcandac - bc = a - bc$.
  To add or subtract fractions, add or subtract the numerators and place the result over the common denominator.
- **Placement of Negative Sign in a Fraction**
  For any positive numbers a and b,
  $-ab = a - b = -ab$.
- **How to simplify complex fractions.**

  Simplify the numerator. Simplify the denominator. Divide the numerator by the denominator. Simplify if possible.

Expressions, Real Numbers (P1)
By the end of this section, you will be able to:

- Simplify expressions with absolute value
- Understand order of operations
- Identify liketerms
- Evaluate expressions with integers
- Understand real vs irrational numbers
- Understand Identity and Inverse
- Use properties of real numbers

This Module supports section **P1 of Mat 1023**.
In case you missed something in class, or just want to review a specific topic covered in this Module, here is a list of topics covered:

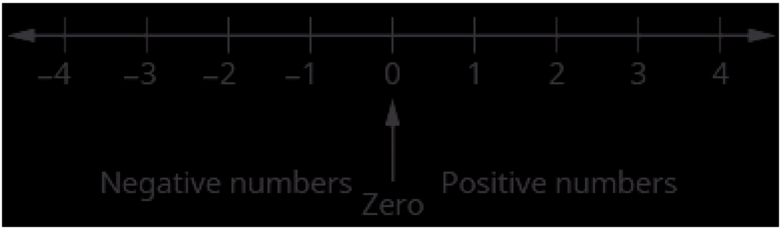
1. Simplify Expressions with Absolute Value [link]
2. Expressions and Order of Operations [link]
3. Evaluate Expressions with Integers [link]
4. Identify and Combine Like Terms [link]
5. Properties of Real Numbers [link]
6. Identity, Inverse, and Zero Properties [link]
7. Distributive Property [link]
8. Key Concepts [link]

The number line shows the location of positive and negative numbers. The opposite of 3 is -3. The numbers 5 and -5 are 5 units away from 0.

Simplify Expressions with Absolute Value

For more on the number line, refer to the Pre-knowledge Module [link].

A **negative number** is a number less than 0. The negative numbers are to the left of zero on the number line. See [link].

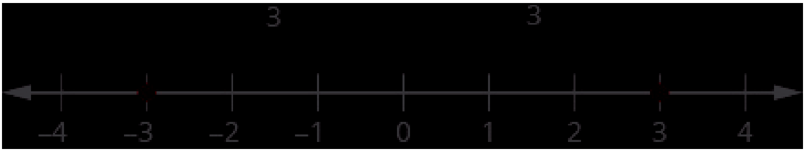


You may have noticed that, on the number line, the negative numbers are a mirror image of the positive numbers, with zero in the middle. Because the numbers 2 and -2 are the same distance from zero, each one is called the **opposite** of the other. The opposite of 2 is -2, and the opposite of -2 is 2.

Opposite

The **opposite** of a number is the number that is the same distance from zero on the number line but on the opposite side of zero.

[link] illustrates the definition.



Opposite Notation

$-a$ means the opposite of the number a The notation $-a$ is read as "the opposite of a."

We saw that numbers such as 3 and -3 are opposites because they are the same distance from 0 on the number line. They are both three units from 0. The distance between 0 and any number on the number line is called the **absolute value** of that number.

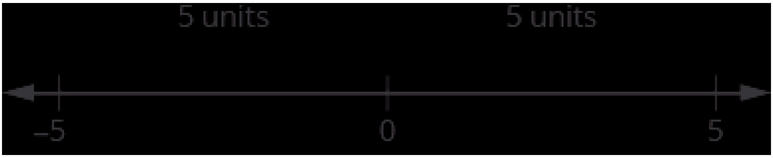
Absolute Value

The **absolute value** of a number is its distance

from 0 on the number line.
The absolute value of a number n is written as $|n|$ and $|n| \geq 0$ for all numbers.
Absolute values are always greater than or equal to zero.

For example,
$-5$is 5 units away from 0, so$|-5| = 5$. 5 is 5 units away from 0, so$|5| = 5$.

[link] illustrates this idea.



The absolute value of a number is never negative because distance cannot be negative. The only number with absolute value equal to zero is the number zero itself because the distance from 0 to 0 on the number line is zero units.

In the next example, we’ll order expressions with absolute values.

Fill in $<, >$, or $=$ for each of the following pairs of numbers:

ⓐ $|-5|\_-|-5|$ ⓑ $8\_-|-8|$ ⓒ $-9\_-|-9|$ ⓓ $-(-16)\_|-16|$.

ⓐ

$|-5|\_-|-5|$ Simplify. 5_-5 Order. $5>-5$ $|-5|>-|-5|$

ⓑ

$8\_-|-8|$ Simplify. 8_-8 Order. $8>-8$ $8>-|-8|$

ⓒ

$-9\_-|-9|$ Simplify. -9_-9 Order. $-9=-9$ $-9=-|-9|$

ⓓ

$-(-16)\_|-16|$ Simplify. 16_16 Order. $16=16$ $-(-16)=|-16|$

Fill in $<, >$, or $=$ for each of the following pairs of numbers:

ⓐ $-9\_-|-9|$ ⓑ $2\_-|-2|$ ⓒ $-8\_|-8|$ ⓓ $-(-9)\_|-9|$.

ⓐ $>$ ⓑ $>$ ⓒ $<$

ⓓ =

We now add absolute value bars to our list of grouping symbols. When we use the order of operations, first we simplify inside the absolute value bars as much as possible, then we take the absolute value of the resulting number.

Grouping Symbols
Parentheses()Braces{} Brackets[]Absolute value||

In the next example, we simplify the expressions inside absolute value bars first just like we do with parentheses.

Simplify: $24 - |19 - 3(6 - 2)|$.

$24 - |19 - 3(6 - 2)|$ Work inside parentheses first: subtract 2 from 6.$24 - |19 - 3(4)|$ Multiply 3(4).$24 - |19 - 12|$ Subtract inside the

absolute value bars.$24 - |7|$ Take the absolute value.$24 - 7$ Subtract.17

Simplify:$19 - |11 - 4(3 - 1)|$.

16

Simplify Expressions and Order of Operations

What is the difference in English between a phrase and a sentence? A phrase expresses a single thought that is incomplete by itself, but a sentence makes a complete statement. A sentence has a subject and a verb. In algebra, we have *expressions* and *equations*.

An **expression** is a number, a variable, or a combination of numbers and variables using operation symbols.

Expression	Words	English Phrase
$3+5$	3 plus 5	the sum of three and five
$n-1$	n minus one	the difference of n and one
$6\cdot 7$	6 times 7	the product of six and seven
$\frac{x}{y}$	x divided by y	the quotient of x and y

An **equation** is two expressions linked by an equal sign. When you read the words the symbols represent in an equation, you have a complete sentence in English. The equal sign gives the verb.

Equation	English Sentence
$3+5=8$	The sum of three and five is equal to eight.
$n-1=14$	n minus one equals fourteen.
$6\cdot 7=42$	The product of six and seven is equal to forty-two.
$x=53$	x is equal to fifty-three.
$y+9=2y-3$	y plus nine is equal to two y minus three.

To **simplify an expression** means to do all the math possible. For example, to simplify $4\cdot 2+1$ we would first multiply $4\cdot 2$ to get 8 and then add the 1 to get 9. A good habit to develop is to work down the page, writing each step of the process below the previous step. The example just described would look like this:

$$4\cdot 2+1$$
$$8+1$$
$$9$$

By not using an equal sign when you simplify an expression, you may avoid confusing expressions with equations.

Simplify an Expression

To **simplify an expression**, do all operations in the expression.

We've introduced most of the symbols and notation used in algebra, but now we need to clarify the **order of operations**. Otherwise, expressions may have different meanings, and they may result in different values.

For example, consider the expression $4+3\cdot 7$. Some students simplify this getting 49, by adding $4+3$ and then multiplying that result by 7. Others get 25, by multiplying $3\cdot 7$ first and then adding 4.

The same expression should give the same result. So mathematicians established some guidelines that are called the order of operations.

Use the order of operations.

Parentheses and Other Grouping Symbols

- Simplify all expressions inside the parentheses or other grouping symbols, working on the innermost parentheses first.

Exponents

- Simplify all expressions with exponents.

Multiplication and Division

- Perform all multiplication and division in order from left to right. These operations have equal priority.

Addition and Subtraction

- Perform all addition and subtraction in order from left to right. These operations have equal priority.

Students often ask, “How will I remember the order?” Here is a way to help you remember: Take the first letter of each key word and substitute the silly phrase “Please Excuse My Dear Aunt Sally”.
ParenthesesPleaseExponentsExcuseMultiplicationDivisionM

It’s good that “**M**y **D**ear” goes together, as this reminds us that **m**ultiplication and **d**ivision have equal priority. We do not always do multiplication before division or always do division before multiplication. We do them in order from left to right.

Similarly, “**A**unt **S**ally” goes together and so reminds

us that **a**ddition and **s**ubtraction also have equal priority and we do them in order from left to right.

Simplify: $18 \div 6 + 4(5 - 2)$.

[redacted]

Parentheses? Yes, subtract first.

[redacted]

Exponents? No.

Multiplication or division? Yes.

Divide first because we multiply and divide left to [redacted]

Any other multiplication or division? Yes.

Multiply.

[redacted]

Any other multiplication of division? No.

Any addition or subtraction? Yes. Add.

[redacted]

Simplify: $70 \div 10 + 4(6 - 2)$.

23

When there are multiple grouping symbols, we simplify the innermost parentheses first and work outward.

Simplify: $5 + 23 + 3[6 - 3(4 - 2)]$.

[redacted]

Are there any parentheses (or other gr[redacted]

Yes.

Focus on the parentheses that are ins[redacted]

brackets. Subtract.

Continue inside the brackets and multiply.

[redacted]

Continue inside the brackets and subtract.

[redacted]

The expression inside the brackets requires no further simplification.

Are there any exponents? Yes.

Si[redacted]

Is there any multiplication or division? Yes.

Multiply.

[redacted]

Is there any addition of

subtraction? Yes.

Add.

[redacted]

Add.

[redacted]

Simplify: $9+53-[4(9+3)]$.

86

Simplify: ⓐ $(-2)4$ ⓑ -24.

Notice the difference in parts (a) and (b). In part (a), the exponent means to raise what is in the parentheses, the -2 to the 4^{th} power. In part (b), the exponent means to raise just the 2 to the 4^{th} power and then take the opposite.

ⓐ

$(-2)4$ Write in expanded form. $(-2)(-2)(-2)(-2)$ Multiply. $4(-2)(-2)$ Multiply. $-8(-2)$ Multiply. 16

ⓑ

-24 Write in expanded form. $-(2\cdot2\cdot2\cdot2)$ We are asked to find the opposite of 24. Multiply. $-(4\cdot2\cdot2)$ Multiply. $-(8\cdot2)$ Multiply. -16

The last example showed us the difference between $(-2)4$ and -24. This distinction is important to prevent future errors. The next example reminds us to multiply and divide in order left to right.

Simplify: ⓐ $8(-9)\div(-2)3$ ⓑ $-30\div2+(-3)(-7)$.

ⓐ

$8(-9)\div(-2)3$ Exponents first. $8(-9)\div(-8)$ Multiply. $-72\div(-8)$ Divide. 9

ⓑ

$-30\div2+(-3)(-7)$ Multiply and divide left to right, so divide first. $-15+(-3)(-7)$ Multiply. $-15+21$ Add. 6

Idetnify and Combine Like Terms

Algebraic expressions are made up of terms. What do you think 3x + 6x would simplify to? If you thought 9x, you would be right!

We can see why this works by writing both terms as addition problems.

$$\underbrace{\overbrace{x+x+x}^{3x} + \overbrace{x+x+x+x+x+x}^{6x}}_{9x}$$

Add the coefficients and keep the same variable. It doesn’t matter what x is. If you have 3 of something and add 6 more of the same thing, the result is 9 of them. For example, 3 oranges plus 6 oranges is 9 oranges. We will discuss the mathematical properties behind this later.

Term

A **term** is a constant or the product of a constant and one or more variables.

Examples of terms are 7,y,5x2,9a, and b5.

Coefficient

The **coefficient** of a term is the constant that multiplies the variable in a term.

Think of the coefficient as the number in front of the variable. The coefficient of the term 3x is 3. When we write x, the coefficient is 1, since $x=1\cdot x$.

Some terms share common traits. When two terms are constants or have the same variable and exponent, we say they are **like terms**.

Look at the following 6 terms. Which ones seem to have traits in common?

5x7n243x9n2

We say,

7 and 4 are like terms.

5x and 3x are like terms.

n2 and 9n2 are like terms.

Combine Like Terms

Terms that are either constants or have the same variables raised to the same powers are called **like terms.**

Identify like terms. Rearrange the expression so like terms are together. Add or subtract the coefficients and keep the same variable for each group of like terms.

If there are like terms in an expression, you can simplify the expression by combining the like terms. We add the coefficients and keep the same variable.

Simplify.4x + 7x + xAdd the coefficients.12x

How To Combine Like Terms

Simplify: 2x2 + 3x + 7 + x2 + 4x + 5.

Step 1. Identify the like terms.	$2x^2 + 3x + 7 + x^2 + 4x + 5$ $\underline{\underline{2x^2}} + \underline{3x} + \utilde{7} + \underline{\underline{x^2}} + \underline{4x} + \utilde{5}$ $2x^2 + \quad + \quad + x^2 + \quad +$

Step 2. Rearrange the expression so the like terms are together.	$\underline{\underline{2x^2 + x^2}} + \underline{3x + 4x} + \underbrace{7 + 5}$ $2x^2 + x^2 + \quad +$
Step 3. Combine like terms.	$3x^2 + \quad +$

Simplify: $3x2+7x+9+7x2+9x+8$.

$10x2+16x+17$

Evaluate Variable Expressions with Integers

In the last few examples, we simplified expressions using the order of operations. Now we'll evaluate some expressions—again following the order of operations. To **evaluate an expression** means to find the value of the expression when the variable is

replaced by a given number.

Evaluate an Expression

To **evaluate an expression** means to find the value of the expression when the variable is replaced by a given number.

To evaluate an expression, substitute that number for the variable in the expression and then simplify the expression.

Evaluate when $x=4$: ⓐ x2 ⓑ 3x ⓒ 2x2 + 3x + 8.

ⓐ

[redacted]

[redacted]

Use definition of exponent.

[redacted]

Simplify.

[redacted]

ⓑ

[redacted]

[redacted]

Use definition of exponent.

[redacted]

Simplify.

[redacted]

ⓒ

[redacted]

[redacted]

Follow the order of operations.

[redacted]

[redacted]

[redacted]

Evaluate when x = 3, ⓐ x2 ⓑ 4x ⓒ 3x2 + 4x + 1.

ⓐ 9 ⓑ 64 ⓒ40

Remember that to evaluate an expression means to substitute a number for the variable in the expression. Now we can use negative numbers as well as positive numbers.

Evaluate $4x2 - 2xy + 3y2$ when $x = 2, y = -1$.

[redacted]

[redacted]

Simplify exponents.

[redacted]

Multiply.

[redacted]

Subtract.

[redacted]

Add.

[redacted]

This chart shows the number sets that make up the set of real numbers.

Properties of Real Numbers

We have already described numbers as *counting numbers*, *whole numbers*, and *integers*. What is the difference between these types of numbers? Difference could be confused with subtraction. How about asking how we distinguish between these types of numbers?

Counting numbers$1,2,3,4,\ldots.$ Whole numbers$0,1,2,3,4,\ldots.$ Integers$\ldots.-3,-2,-1,0,1,2,3,\ldots.$

What type of numbers would we get if we started with all the integers and then included all the fractions? The numbers we would have form the set of rational numbers. A **rational number** is a

number that can be written as a ratio of two integers.

In general, any decimal that ends after a number of digits (such as 7.3 or -1.2684) is a rational number. We can use the place value of the last digit as the denominator when writing the decimal as a fraction. The decimal for 13 is the number 0.3–. The bar over the 3 indicates that the number 3 repeats infinitely. Continuously has an important meaning in calculus. The number(s) under the bar is called the repeating block and it repeats continuously.

Since all integers can be written as a fraction whose denominator is 1, the integers (and so also the counting and whole numbers. are rational numbers.

Rational Number
A **rational number** is a number that can be written both as a ratio of integerspq, *where p and q are integers and* $q \neq 0$, *and as a decimal that stops or repeats.*
Its decimal form stops or repeats.

Are there any decimals that do not stop or repeat? Yes! The number π (the Greek letter *pi*, pronounced

“pie”), which is very important in describing circles, has a decimal form that does not stop or repeat. We use three dots (…) to indicate the decimal does not stop or repeat.
$\pi = 3.141592654...$

The square root of a number that is not a perfect square is a decimal that does not stop or repeat.

Irrational Number
An **irrational number** is a number that cannot be written as the ratio of two integers. I.e. A number whose decimal form does not stop or repeat cannot be written as a fraction of integers. We call this an **irrational number**.

Let’s summarize a method we can use to determine whether a number is rational or irrational.

Rational or Irrational
If the decimal form of a number

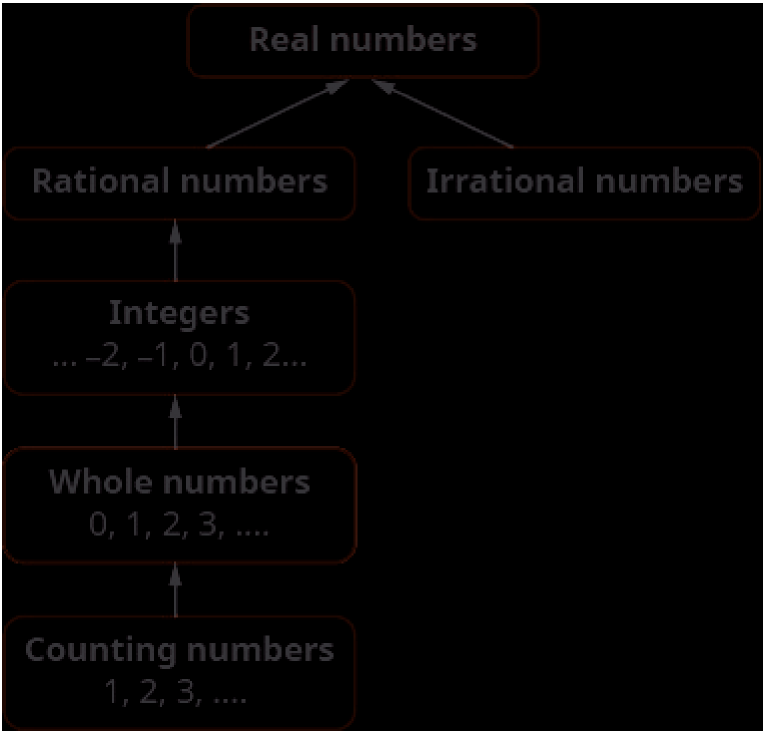
- *repeats or stops*, the number is a **rational number**.
- *does not repeat and does not stop*, the number is

an **irrational number.**

We have seen that all counting numbers are whole numbers, all whole numbers are integers, and all integers are rational numbers. The irrational numbers are numbers whose decimal form does not stop and does not repeat. When we put together the rational numbers and the irrational numbers, we get the set of **real numbers**.

Real Number
A **real number** is a number that is either rational or irrational.

Later in this course we will introduce numbers beyond the real numbers. [link] illustrates how the number sets we’ve used so far fit together.



Does the term “real numbers” seem strange to you? Are there any numbers that are not “real,” and, if so, what could they be? Can we simplify -25? Is there a number whose square is -25?
$()2 = -25?$

None of the numbers that we have dealt with so far has a square that is -25. Why? Any positive number squared is positive. Any negative number squared is positive. So we say there is no real number equal to -25. The square root of a negative number is not a real number.

Given the numbers $-7, 145, 8, 5, 5.9, -64$, list the ⓐ whole numbers ⓑ integers ⓒ rational numbers ⓓ irrational numbers ⓔ real numbers.

ⓐ Remember, the whole numbers are $0, 1, 2, 3, \ldots$, so 8 is the only whole number given.

ⓑ The integers are the whole numbers and their opposites (which includes 0). So the whole number 8 is an integer, and -7 is the opposite of a whole number so it is an integer, too. Also, notice that 64 is the square of 8 so $-64 = -8$. So the integers are $-7, 8$, and -64.

ⓒ Since all integers are rational, then $-7, 8$, and -64 are rational. Rational numbers also include fractions and decimals that repeat or stop, so 145 and 5.9 are rational. So the list of rational numbers is $-7, 145, 8, 5.9$, and -64.

ⓓ Remember that 5 is not a perfect square, so 5 is irrational.

ⓔ All the numbers listed are real numbers.

Using Commutatitve and Associative Properties

The order we add two numbers doesn't affect the result. If we add $8+9$ or $9+8$, the results are the same—they both equal 17. So, $8+9=9+8$. The order in which we add does not matter!

Similarly, when multiplying two numbers, the order does not affect the result. If we multiply $9·8$ or $8·9$ the results are the same—they both equal 72. So, $9·8=8·9$. The order in which we multiply does not matter!

These examples illustrate the Commutative Property.

Commutative Property
of AdditionIfaandbare real numbers, then$a+b=b+a$. of MultiplicationIfaandbare real numbers, then$a·b=b·a$.
When adding or multiplying, changing the *order* gives the same result.

The Commutative Property has to do with order. We subtract $9-8$ and $8-9$, and see that $9-8\neq 8-9$. Since changing the order of the subtraction does not give the same result, we know that *subtraction is not commutative*.

Division is not commutative either. Since $12 \div 3 \neq 3 \div 12$, changing the order of the division did not give the same result. The commutative properties apply only to addition and multiplication! Subtraction and division are *not* commutative.

When adding three numbers, changing the grouping of the numbers gives the same result. For example, $(7+8)+2=7+(8+2)$, since each side of the equation equals 17.

This is true for multiplication, too. For example, $(5 \cdot \frac{1}{3}) \cdot 3 = 5 \cdot (\frac{1}{3} \cdot 3)$, since each side of the equation equals 5.

These examples illustrate the Associative Property.

Associative Property
of AdditionIfa,b,andcare real numbers, then$(a+b)+c=a+(b+c)$. of MultiplicationIfa,b,andcare real numbers, then$(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
When adding or multiplying, changing the *grouping* gives the same result.

The Associative Property has to do with grouping. If we change how the numbers are grouped, the result will be the same. Notice it is the same three

numbers in the same order—the only difference is the grouping.

We saw that subtraction and division were not commutative. They are not associative either. $(10-3)-2\neq 10-(3-2)(24\div 4)\div 2\neq 24\div(4\div 2)$ $7-2\neq 10-16\div 2\neq 24\div 2$ $5\neq 93\neq 12$

When simplifying an expression, it is always a good idea to plan what the steps will be. In order to combine like terms in the next example, we will use the Commutative Property of addition to write the like terms together.

Simplify: $18p+6q+15p+5q$.
$18p+6q+15p+5q$ Use the Commutative Property of addition to$18p+15p+6q+5q$ reorder so that like terms are together. Add like terms.$33p+11q$

Simplify: $23r+14s+9r+15s$.

$32r+29s$

When we have to simplify algebraic expressions, we can often make the work easier by applying the Commutative Property or Associative Property first.

Simplify: $(513+34)+14$.
$(513+34)+14$ Notice that the last 2 terms have a commondenominator, so change the grouping.$513+(34+14)$ Add in parentheses first.$513+(44)$ Simplify the fraction.$513+1$ Add.1513 Convert to an improper fraction.1813

Identity, Inverse, and Zero Properties

What happens when we add 0 to any number? Adding 0 doesn't change the value. For this reason, we call 0 the **additive identity**. The **Identity Property of Addition** that states that for any real number $a, a+0=a$ and $0+a=a$.

What happens when we multiply any number by one? Multiplying by 1 doesn't change the value. So we call 1 the **multiplicative identity**. The **Identity Property of Multiplication** that states that for any real number $a, a\cdot 1=a$ and $1\cdot a=a$.

Identity Property
of AdditionFor any real numbera:$a+0=a$ $0+a=a$
0is theadditive identity of MultiplicationFor any real numbera:$a\cdot 1=a$ $1\cdot a=a$ 1is themultiplicative identity

What number added to 5 gives the additive identity, 0? We know

[redacted]

We call $-a$ the **additive inverse** of a. *The opposite of a number is its additive inverse.* A number and its opposite add to zero, which is the additive identity. This leads to the **Inverse Property of Addition** that states for any real number $a, a+(-a)=0$.

What number multiplied by 23 gives the multiplicative identity, 1? In other words, 23 times what results in 1? We know

$\frac{2}{3} \cdot \frac{3}{2} = 1$

We call 1a the **multiplicative inverse** of *a. The reciprocal of a number is its multiplicative inverse.* This leads to the **Inverse Property of Multiplication** that states that for any real number $a, a \neq 0, a \cdot 1a = 1$.

Inverse Property
of AdditionFor any real numbera,$a + (-a) = 0$ $-a$is theadditive inverseofa A number and itsoppositeadd to zero. of MultiplicationFor any real numbera,$a \neq 0, a \cdot 1a = 1$. 1ais themultiplicative inverseofa. A number and itsreciprocalmultiply to one.

The Identity Property of addition says that when we add 0 to any number, the result is that same number. What happens when we multiply a number by 0? Multiplying by 0 makes the product equal zero.

What about division involving zero? What is $0 \div 3$? Think about a real example: If there are no cookies in the cookie jar and 3 people are to share them, how many cookies does each person get? There are no cookies to share, so each person gets 0 cookies. So, $0 \div 3 = 0$.

We can check division with the related multiplication fact. So we know $0 \div 3 = 0$ because $0 \cdot 3 = 0$.

Now think about dividing *by* zero. What is the result of dividing 4 by 0? Think about the related multiplication fact:

[redacted]

Is there a number that multiplied by 0 gives 4? Since any real number multiplied by 0 gives 0, there is no real number that can be multiplied by 0 to obtain 4. We conclude that there is no answer to $4 \div 0$ and so we say that division by 0 is **undefined**.

Properties of Zero

Multiplication by Zero: For any real number *a*, $a \cdot 0 = 0$ $0 \cdot a = 0$ The product of any number and 0 is 0.

Division by Zero: For any real number *a*, $a \neq 0$ $\frac{0}{a} = 0$ Zero divided by any real number, except itself, is zero. $\frac{a}{0}$ is undefined Division by zero is undefined.

We will now practice using the properties of identities, inverses, and zero to simplify expressions.

Simplify: $-84n+(-73n)+84n$.

$-84n+(-73n)+84n$ Notice that the first and third terms areopposites; use the Commutative Property ofaddition to re-order the terms. $-84n+84n+(-73n)$ Add left to right.$0+(-73n)$ Add.$-73n$

Simplify: $-27a+(-48a)+27a$.

$-48a$

Now we will see how recognizing reciprocals is helpful. Before multiplying left to right, look for reciprocals—their product is 1.

Simplify: $715\cdot823\cdot157$.

715·823·157 Notice the first and third termsare reciprocals, so use the CommutativeProperty of multiplication to re-order thefactors.715·157·823 Multiply left to right.1·823 Multiply.823

The next example makes us aware of the distinction between dividing 0 by some number or some number being divided by 0.

Simplify: ⓐ $0n+5$, where $n \neq -5$ ⓑ $10-3p0$, where $10-3p \neq 0$.

ⓐ

$0n+5$ Zero divided by any real number except itself is 0.0

ⓑ

$10-3p0$ Division by 0 is undefined.undefined

Simplify: ⓐ $0m+7$, where $m \neq -7$ ⓑ

$18-6c0$, where $18-6c \neq 0$.

ⓐ 0 ⓑ undefined

Distributive Property

Suppose that three friends are going to the movies. They each need \$9.25—that's 9 dollars and 1 quarter—to pay for their tickets. How much money do they need all together?

You can think about the dollars separately from the quarters. They need 3 times \$9 so \$27 and 3 times 1 quarter, so 75 cents. In total, they need \$27.75. If you think about doing the math in this way, you are using the Distributive Property.

Distributive Property

Ifa,b,andcare real numbers, then$a(b+c)=ab+ac$ $(b+c)a=ba+ca$ $a(b-c)=ab-ac$ $(b-c)a=ba-ca$

In algebra, we use the Distributive Property to remove parentheses as we simplify expressions.

Simplify: $3(x+4)$.
$3(x+4)$ Distribute. $3\cdot x+3\cdot 4$ Multiply. $3x+12$

Simplify: $4(x+2)$.
$4x+8$

Some students find it helpful to draw in arrows to remind them how to use the Distributive Property. Then the first step in [link] would look like this:

Simplify: $8(38x+14)$.

$8\left(\frac{3}{8}x+\frac{1}{4}\right)$

Distribute.

$8\cdot\frac{3}{8}x+8\cdot\frac{1}{4}$

Multiply.

$3x+2$

Simplify: $6(56y+12)$.

$5y+3$

Using the Distributive Property as shown in the next example will be very useful when we solve money applications in later chapters.

Simplify: $100(0.3 + 0.25q)$.

[redacted]

Distribute.

[redacted]

Multiply.

[redacted]

When we distribute a negative number, we need to be extra careful to get the signs correct!

Simplify: $-11(4 - 3a)$.

$-11(4 - 3a)$ Distribute. $-11 \cdot 4 - (-11) \cdot 3a$

Multiply. $-44-(-33a)$ Simplify. $-44+33a$ Notice that you could also write the result as $33a-44$. Do you know why?

In the next example, we will show how to use the Distributive Property to find the opposite of an expression.

Simplify: $-(y+5)$.
$-(y+5)$ Multiplying by -1results in the opposite. $-1(y+5)$ Distribute. $-1\cdot y+(-1)\cdot 5$ Simplify. $-y+(-5)$ Simplify. $-y-5$

Simplify: $-(x-4)$.
$-x+4$

There will be times when we'll need to use the Distributive Property as part of the order of operations. Start by looking at the parentheses. If the expression inside the parentheses cannot be simplified, the next step would be multiply using the Distributive Property, which removes the parentheses. The next two examples will illustrate this.

Simplify: $8-2(x+3)$
We follow the order of operations. Multiplication comes before subtraction, so we will distribute the 2 first and then subtract. $8-2(x+3)$ Distribute. $8-2\cdot x-2\cdot 3$ Multiply. $8-2x-6$ Combine like terms. $-2x+2$

Simplify: $4(x-8)-(x+3)$.
$4(x-8)-(x+3)$ Distribute. $4x-32-x-3$

Combine like terms. $3x - 35$

All the properties of real numbers we have used in this chapter are summarized here.

Commutative Property

When adding or multiplying, changing the *order* gives the same result

of addition If a, b are real numbers, then $a + b = b + a$ of multiplication If a, b are real numbers, then $a \cdot b = b \cdot a$

Associative Property

When adding or multiplying, changing the *grouping* gives the same result.

of addition If a, b, and c are real numbers, then $(a + b) + c = a + (b + c)$ of multiplication If a, b, and c are real numbers, then $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

Distributive Property

If a, b, and c are real numbers, then $a(b + c) = ab + ac$ $(b + c)a = ba + ca$ $a(b - c) = ab - ac$ $(b - c)a = ba - ca$

Identity Property

of additionFor any real numbera:$a+0=a$ 0is theadditive identity$0+a=a$ of multiplicationFor any real numbera:$a\cdot 1=a$ 1is themultiplicative identity$1\cdot a=a$

Inverse Property

of additionFor any real numbera,$a+(-a)=0$ $-a$is theadditive inverseofa A number and itsoppositeadd to zero. of multiplicationFor any real numbera,$a\neq 0$$a\cdot\frac{1}{a}=1$ $\frac{1}{a}$is themultiplicative inverseofa A number and itsreciprocalmultiply to one.

Properties of Zero

For any real numbera,$a\cdot 0=0$ $0\cdot a=0$ For any real numbera,$a\neq 0$,$\frac{0}{a}=0$ For any real numbera,$\frac{a}{0}$is undefined

Key Concepts

- **Opposite Notation**
  $-a$means the opposite of the numbera The notation$-a$is read as “the opposite ofa.”
- **Absolute Value**

The absolute value of a number is its distance from 0 on the number line.
The absolute value of a number n is written as $|n|$ and $|n| \geq 0$ for all numbers.
Absolute values are always greater than or equal to zero.

- **Grouping Symbols**
  Parentheses()Braces{} Brackets[]Absolute value||
- **Subtraction Property**
  $a - b = a + (-b)$
  Subtracting a number is the same as adding its opposite.
- **Multiplication and Division of Signed Numbers**
  For multiplication and division of two signed numbers:

Same signs	Result
• Two positives	Positive
• Two negatives	Positive

If the signs are the same, the result is positive.

Different signs	Result
• Positive and negative	Negative
• Negative and positive	Negative

If the signs are different, the result is negative.

- **Multiplication by** -1
  $-1a = -a$

Multiplying a number by -1 gives its opposite.

- **How to Use Integers in Applications.**

  Read the problem. Make sure all the words and ideas are understood **Identify** what we are asked to find. **Write a phrase** that gives the information to find it. **Translate** the phrase to an expression. **Simplify** the expression. **Answer** the question with a complete sentence.

Commutative Property When adding or multiplying, changing the *order* gives the same result of addition If a, b are real numbers, then $a+b=b+a$ of multiplication If a, b are real numbers, then $a \cdot b=b \cdot a$
Associative Property When adding or multiplying, changing the *grouping* gives the same result. of addition If a, b, and c are real numbers, then $(a+b)+c=a+(b+c)$ of multiplication If a, b, and c are real numbers, then $(a \cdot b) \cdot c=a \cdot (b \cdot c)$
Distributive Property If a, b, and c are real numbers, then $a(b+c)=ab+ac$ $(b+c)a=ba+ca$ $a(b-c)=ab-ac$ $(b-c)a=ba-ca$

Identity Property

of additionFor any real numbera:a $+$ $0=$ a 0is theadditive identity0 $+$ a $=$ a of multiplicationFor any real numbera:a$\cdot$1 $=$ a 1is themultiplicative identity1$\cdot$a $=$ a

Inverse Property

of additionFor any real numbera,a $+(-$a$)=0$ $-$ais theadditive inverseofa A number and itsoppositeadd to zero. of multiplicationFor any real numbera,a $\neq$ 0a$\cdot$1a $=1$ 1ais themultiplicative inverseofa A number and itsreciprocalmultiply to one.

Properties of Zero

For any real numbera,a$\cdot$0 $=0$ 0$\cdot$a $=0$ For any real numbera,a $\neq$ 0,0a $=0$ For any real numbera,a0is undefined

Practice Makes Perfect

Simplify Expressions with Absolute Value

In the following exercises, fill in $<,>$, or $=$ for each of the following pairs of numbers.

ⓐ $|-7|$___$-|-7|$

ⓑ $6 \_\_ -|-6|$
ⓒ $|-11| \_\_ -11$
ⓓ $-(-13) \_\_ -|-13|$

ⓐ $>$ ⓑ $>$ ⓒ $>$ ⓓ $>$

ⓐ $-|2| \_\_ -|-2|$
ⓑ $-12 \_\_ -|-12|$
ⓒ $|-3| \_\_ -3$
ⓓ $|-19| \_\_ -(-19)$

ⓐ $=$ ⓑ $=$ ⓒ $>$ ⓓ $=$

In the following exercises, simplify.

$|15-7|-|14-6|$

0

$18-|12-4(4-1)+3|$

15

$25-[10-(3-12)]$

6

In the following exercises, evaluate each expression.

$y+(-14)$ when
ⓐ $y=-33$ ⓑ $y=30$

ⓐ -47 ⓑ 16

$(x+y)^2$ when
$x=-3, y=14$

121

$9a-2b-8$ when
$a=-6$ and $b=-3$

-56

Use Integers in Applications

In the following exercises, solve.

Temperature On January 15, the high temperature in Anaheim, California, was 84°.

That same day, the high temperature in Embarrass, Minnesota, was -12°. What was the difference between the temperature in Anaheim and the temperature in Embarrass?

96°

Football On the first down, the Chargers had the ball on their 25-yard line. On the next three downs, they lost 6 yards, gained 10 yards, and lost 8 yards. What was the yard line at the end of the fourth down?

21

Checking Account Mayra has \$124 in her checking account. She writes a check for \$152. What is the new balance in her checking account?

$-\$28$

Glossary

absolute value
: The absolute value of a number is its distance

from 0 on the number line.

integers
: The whole numbers and their opposites are called the integers.

negative numbers
: Numbers less than 0 are negative numbers.

opposite
: The opposite of a number is the number that is the same distance from zero on the number line but on the opposite side of zero.

Exponents and Scientific Notation (P2)
By the end of this section, you will be able to:

- Round Decimals
- Simplify expressions using the properties for exponents
- Use the definition of a negative exponent
- Use scientific notation

This Module supports section **P2 of Mat 1023**.
In case you missed something in class, or just want to review a specific topic covered in this Module, here is a list of topics covered:

1. Rounding Decimals [link]
2. Simplify Expressions Using Properties of Exponents [link]
3. Use the Definition of Negative Exponent [link]
4. Summary of Exponent Rules [link]
5. Using Scientific Notation[link]
6. Key Concepts [link]

Round Decimals

Decimals are another way of writing fractions whose denominators are powers of ten.

$0.1=\frac{1}{10}$ is "one tenth" $0.01=\frac{1}{100}$ is "one hundredth" $0.001=\frac{1}{1000}$ is "one thousandth" $0.0001=\frac{1}{10,000}$ is "one ten-thousandth"

Just as in whole numbers, each digit of a decimal corresponds to the place value based on the powers of ten. [link] shows the names of the place values to the left and right of the decimal point.

Place Value											
Hundred thousands	Ten thousands	Thousands	Hundreds	Tens	Ones		Tenths	Hundredths	Thousandths	Ten-thousandths	Hundred-thousandths
						•					

When we work with decimals, it is often necessary to round the number to the nearest required place value. We summarize the steps for rounding a decimal here.

Round decimals.

Locate the given place value and mark it with an arrow. Underline the digit to the right of the place value. Is the underlined digit greater than or equal to 5?

- Yes: add 1 to the digit in the given place value.
- No: do not change the digit in the given place value

Rewrite the number, deleting all digits to the right of the rounding digit.

Round 18.379 to the nearest ⓐ hundredth ⓑ tenth ⓒ whole number.

Round 18.379.

ⓐ to the nearest hundredth

Locate the hundredths place with an arrow.

[redacted]

18.379

Underline the digit to the right of the given pla[redacted]

18.379

Because 9 is greater than or equal to 5, add 1 t[redacted]

18.379

the[redacted]

Rewrite the number, deleting all digits to the

18.38

right of the rounding digit.

Notice that the deleted digits were NOT rep[redacted]

So, 18.379 rounded to the nearest hundredth is 18.38.

ⓑ to the nearest tenth

Locate the tenths place with an arrow.

[redacted]

18.379

Underline the digit to the right of the giv[redacted]

18.379

Because 7 is greater than or equal to 5, ad[redacted]

18.379

Rewrite the number, deleting all digits to the[redacted]

18.4

rounding digit.

Notice that the deleted digits were NOT rep[redacted]

So, 18.379 rounded to the nearest tenth is 18.4.

ⓒ to the nearest whole number

Locate the ones place with an arrow.

[redacted]

18.379

Underline the digit to the right of the gi[redacted]

Since 3 is not greater than or equal to 5, do[redacted]

Rewrite the number, deleting all digits to the[redacted] rounding digit.

[redacted]

Round 6.582 to the nearest ⓐ hundredth ⓑ tenth ⓒ whole number.

ⓐ 6.58 ⓑ 6.6 ⓒ 7

Round 15.2175 to the nearest ⓐ thousandth ⓑ hundredth ⓒ tenth.

ⓐ 15.218 ⓑ 15.22
ⓒ 15.2

Simplify Expressions Using the Properties for Exponents

Remember that an exponent indicates repeated multiplication of the same quantity. For example, in the expression am, the *exponent m* tells us how many times we use the *base a* as a factor.

$$a^m = a \cdot a \cdot a \cdot \ldots \cdot a \qquad (-9)^5 = (-9)(-9)(-9)(-9)(-9)$$

Let’s review the vocabulary for expressions with exponents.

Exponential Notation

a^m ← exponent, base ↑ — a^m means multiply a, m times

$$a^m = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{m \text{ factors}}$$

This is read *a* to the mth power.
In the expression am, the *exponent m* tells us how many times we use the *base a* as a factor.

When we combine like terms by adding and subtracting, we need to have the same base with the same exponent. But when you multiply and divide, the exponents may be different, and sometimes the bases may be different, too.

First, we will look at an example that leads to the **Product Property**.

[redacted]

What does this mean?

[redacted]

[redacted]

Notice that 5 is the sum of the exponents, 2 and 3. We see $x2 \cdot x3$ is $x2+3$ or $x5$.

The base stayed the same and we added the exponents. This leads to the Product Property for Exponents.

Product Property for Exponents

If *a* is a real number and *m* and *n* are integers, then

$am \cdot an = am+n$

To multiply with like bases, add the exponents.

Simplify each expression: ⓐ $y5 \cdot y6$ ⓑ $2x \cdot 23x$ ⓒ $2a7 \cdot 3a$.

ⓐ

$y^5 \cdot y^6$

Use the Product Property, am·an = am + n.

y^{5+6}

Simplify.

y^{11}

ⓑ

$2^x \cdot 2^{3x}$

Use the Product Property, am·an = am + n.

2^{x+3x}

Simplify.

2^{4x}

ⓒ

[redacted]

Rewrite, a = a1.

[redacted]

Use the Commutative Property and use the Product Property, am·an = am + n.

[redacted]

Simplify.

[redacted]

ⓓ

[redacted]

Add the exponents, since bases are the same.

[redacted]

Simplify.

[redacted]

Simplify each expression:

ⓐ b9·b8 ⓑ 42x·4x ⓒ 3p5·4p ⓓ x6·x4·x8.

ⓐ b17 ⓑ 43x ⓒ 12p6
ⓓ x18

Simplify each expression:

ⓐ x12·x4 ⓑ 10·10x ⓒ 2z·6z7 ⓓ b5·b9·b5.

ⓐ x16 ⓑ 10x+1 ⓐ 12z8
ⓓ b19

Now we will look at an exponent property for division. As before, we'll try to discover a property by looking at some examples.

Consider	x5x2	and	x2x3
What do they mean?	x·x·x·x·xx·x		x·xx·x·x
Use the Equivalent Fractions Property.	x·x·x·x·xx·x		x·x·1x·x·x
Simplify.	x3		1x

Notice, in each case the bases were the same and we subtracted exponents. We see x5x2 is x5 − 2 or x3. We see x2x3 is or 1x. When the larger exponent was in the numerator, we were left with factors in the numerator. When the larger exponent was in the denominator, we were left with factors in the denominator--notice the numerator of 1. When all the factors in the numerator have been removed, remember this is really dividing the factors to one, and so we need a 1 in the numerator. xx = 1. This leads to the **Quotient Property** for Exponents.

Quotient Property for Exponents

If *a* is a real number, a ≠ 0, and *m* and *n* are integers, then

aman = am − n,m > nandaman = 1an − m,n > m

Simplify each expression: ⓐ x9x7 ⓑ 31032 ⓒ b8b12 ⓓ 7375.

To simplify an expression with a quotient, we need to first compare the exponents in the numerator and denominator.

ⓐ

Since $9 > 7$, there are more factors of x in the numerator.	[redacted]
Use Quotient Property, aman = am − n.	[redacted]
Simplify.	[redacted]

ⓑ

Since 10 > 2, there are more factors of 3 in the numerator.

$$\frac{3^{10}}{3^2}$$

Use Quotient Property, aman = am − n.

$$3^{10-2}$$

Simplify.

$$3^8$$

Notice that when the larger exponent is in the numerator, we are left with factors in the numerator.

ⓒ

Since 12 > 8, there are more factors of b in the denominator.

$$\frac{b^8}{b^{12}}$$

Use Quotient Property, aman = 1an − m.

$$\frac{1}{b^{12-8}}$$

Simplify.

$$\frac{1}{b^4}$$

ⓓ

Since $5 > 3$, there are more factors of 3 in the denominator.

$$\frac{7^3}{7^5}$$

Use Quotient Property, $aman = 1an - m$.

$$\frac{1}{7^{5-3}}$$

Simplify.

$$\frac{1}{7^2}$$

Simplify.

$$\frac{1}{49}$$

Notice that when the larger exponent is in the denominator, we are left with factors in the denominator.

Simplify each expression: ⓐ x15x10 ⓑ 61465 ⓒ x18x22 ⓓ 12151230.

ⓐ x5 ⓑ 69 ⓒ 1x4
ⓓ 11215

A special case of the Quotient Property is when the exponents of the numerator and denominator are equal, such as an expression like amam. We know,xx=1, for any x(x≠0) since any number divided by itself is 1.

The Quotient Property for Exponents shows us how to simplify amam. when m>n and when n<m by subtracting exponents. What if m=n? We will simplifyamam in two ways to lead us to the definition of the **Zero Exponent Property**. In general, for a≠0:

$\frac{a^m}{a^m}$	$\frac{a^m}{a^m}$
a^{m-m}	$\frac{\cancel{a}\cdot\cancel{a}\cdot\cancel{a}\cdot\ldots\cdot\cancel{a}}{\cancel{a}\cdot\cancel{a}\cdot\cancel{a}\cdot\ldots\cdot\cancel{a}}$
a^0	1

We see amam simplifies to a0 and to 1. So a0=1. Any non-zero base raised to the power of zero equals 1.

Zero Exponent Property

If a is a non-zero number, then a0 = 1.

If a is a non-zero number, then a to the power of zero equals 1.

Any non-zero number raised to the zero power is 1.

In this text, we assume any variable that we raise to the zero power is not zero.

Simplify each expression: ⓐ 90 ⓑ n0.

The definition says any non-zero number raised to the zero power is 1.

ⓐ

90 Use the definition of the zero exponent.1

ⓑ

n0 Use the definition of the zero exponent.1

To simplify the expression n raised to the zero power we just use the definition of the zero exponent. The result is 1.

Simplify each expression: ⓐ 11^0 ⓑ q^0.

ⓐ 1 ⓑ 1

Use the Definition of a Negative Exponent

We saw that the Quotient Property for Exponents has two forms depending on whether the exponent is larger in the numerator or the denominator. What if we just subtract exponents regardless of which is larger?

Let’s consider $\frac{x^2}{x^5}$. We subtract the exponent in the denominator from the exponent in the numerator. We see $\frac{x^2}{x^5}$ is x^{2-5} or x^{-3}.

We can also simplify $\frac{x^2}{x^5}$ by dividing out common factors:

$$\frac{x^2}{x^5}$$

$$\frac{\cancel{x}\cdot\cancel{x}}{\cancel{x}\cdot\cancel{x}\cdot x\cdot x\cdot x}$$

$$\frac{1}{x^3}$$

This implies that x−3=1x3 and it leads us to the definition of a *negative exponent*. If *n* is an integer and a≠0, then a−n=1an.

Let’s now look at what happens to a fraction whose numerator is one and whose denominator is an integer raised to a negative exponent.

1a−n Use the definition of a negative exponent,a−n=1an.11an Simplify the complex fraction.1·an1 Multiply.an

This implies 1a−n=an and is another form of the definition of **Properties of Negative Exponents**.

Properties of Negative Exponents
If *n* is an integer and a≠0, then a−n=1an or 1a−n=an.

The negative exponent tells us we can rewrite the

expression by taking the reciprocal of the base and then changing the sign of the exponent.

Any expression that has negative exponents is not considered to be in simplest form. We will use the definition of a negative exponent and other properties of exponents to write the expression with only positive exponents.

For example, if after simplifying an expression we end up with the expression $x-3$, we will take one more step and write 1x3. The answer is considered to be in simplest form when it has only positive exponents.

Simplify each expression: ⓐ $x-5$ ⓑ $10-3$ ⓒ $1y-4$ ⓓ $13-2$.

ⓐ

$x-5$ Use the definition of a negative exponent,$a-n=1an$.1x5

ⓑ

$10-3$ Use the definition of a negative exponent,$a-n=1an$.1103 Simplify.11000

ⓒ

1y − 4 Use the property of a negative exponent,1a − n = an.y4

ⓓ
13 − 2 Use the property of a negative exponent,1a − n = an.32 Simplify.9

Simplify each expression: ⓐ z − 3 ⓑ 10 − 7 ⓒ 1p − 8 ⓓ 14 − 3.

ⓐ 1z3 ⓑ 1107 ⓒ p8 ⓓ 64

Suppose now we have a fraction raised to a negative exponent. Let’s use our definition of negative exponents to lead us to a new property.

(34) − 2 Use the definition of a negative exponent,a − n = 1an.1(34)2 Simplify the denominator.1916 Simplify the complex fraction.169 But we know that169is(43)2. This tells us that(34) − 2 = (43)2

To get from the original fraction raised to a negative exponent to the final result, we took the reciprocal of the base—the fraction—and changed the sign of

the exponent.

This leads us to the **Quotient to a Negative Power Property**.

Quotient to a Negative Power Property
If *a* and *b* are real numbers, $a\neq 0, b\neq 0$ and *n* is an integer, then
and$(ab)-n=(ba)n$

Simplify each expression: ⓐ $(57)-2$ ⓑ $(-xy)-3$.

ⓐ
$(57)-2$ Use the Quotient to a Negative Exponent Property,$(ab)-n=(ba)n$.Take the reciprocal of the fraction and change the sign of the exponent.$(75)2$ Simplify.4925

ⓑ
$(-xy)-3$ Use the Quotient to a Negative Exponent Property,$(ab)-n=(ba)n$.Take the reciprocal of the fraction and change the sign of the exponent.$(-yx)3$ Simplify.$-y3x3$

Simplify each expression: ⓐ $(23)-4$ ⓑ $(-mn)-2$.

ⓐ 8116 ⓑ n2m2

Now that we have negative exponents, we will use the Product Property with expressions that have negative exponents.

Simplify each expression: ⓐ $z-5·z-3$ ⓑ $(m4n-3)(m-5n-2)$ ⓒ $(2x-6y8)(-5x5y-3)$.

ⓐ

$z-5·z-3$ Add the exponents, since the bases are the same.$z-5-3$ Simplify.$z-8$ Use the definition of a negative exponent.1z8

ⓑ

$(m4n-3)(m-5n-2)$ Use the Commutative Property to get likebases together.$m4m-5·n-2n-3$ Add the exponents for each base.m

−1·n−5 Take reciprocals and change the signs of the exponents.1m1·1n5 Simplify.1mn5

ⓒ
(2x−6y8)(−5x5y−3) Rewrite with the like bases together.2(−5)·(x−6x5)·(y8y−3) Multiply the coefficients and add the exponentsof each variable.−10·x−1·y5 Use the definition of a negative exponent,a −n=1an.−10·1x·y5 Simplify.−10y5x

Simplify each expression:

ⓐ z−4·z−5 ⓑ (p6q−2)(p−9q−1) ⓒ (3u −5v7)(−4u4v−2).

ⓐ 1z9 ⓑ 1p3q3 ⓒ −12v5u

Now let’s look at an exponential expression that contains a power raised to a power. See if you can discover a general property.

(x2)3 What does this mean?x2·x2·x2

How many factors altogether?

[redacted]

So we have

[redacted]

Notice the 6 is the *product* of the exponents, 2 and 3. We see that $(x^2)^3$ is $x^{2\cdot3}$ or x^6.

We multiplied the exponents. This leads to the **Power Property for Exponents.**

Power Property for Exponents

If a is a real number and m and n are integers, then

$$(a^m)^n = a^{m\cdot n}$$

To raise a power to a power, multiply the exponents.

Simplify each expression: ⓐ $(y^5)^9$ ⓑ $(4^4)^7$ ⓒ $(y^3)^6(y^5)^4$.

ⓐ

[redacted]

Use the Power Property, $(a^m)^n = a^{m \cdot n}$.

[redacted]

Simplify.

[redacted]

ⓑ

[redacted]

Use the Power Property.

[redacted]

Simplify.

[redacted]

ⓒ

$(y^3)^6(y^5)^4$ Use the Power Property. $y^{18} \cdot y^{20}$

Add the exponents. y^{38}

Simplify each expression: ⓐ (b7)5 ⓑ (54)3 ⓒ (a4)5(a7)4.

ⓐ b35 ⓑ 512 ⓒ a48

We will now look at an expression containing a product that is raised to a power. Can you find this pattern?

(2x)3 What does this mean?2x·2x·2x We group the like factors together.2·2·2·x·x·x How many factors of 2 and ofx23·x3

Notice that each factor was raised to the power and (2x)3 is 23·x3.

The exponent applies to each of the factors! This leads to the **Product to a Power Property for Exponents**.

Product to a Power Property for Exponents
If *a* and *b* are real numbers and *m* is a whole number, then
(ab)m = ambm

To raise a product to a power, raise each factor to that power.

Simplify each expression: a $(-3mn)^3$ b $(-4a^2b)^0$ c $(6k^3)^{-2}$ d $(5x^{-3})^2$.

a

[redacted]

Use Power of a Product Property,
(a[redacted]bm.
Simplify.

[redacted]

b

$(-4a^2b)^0$ Use Power of a Product Property, $(ab)^m = a^m b^m$. $(-4)^0(a^2)^0(b)^0$ Simplify. $1 \cdot 1 \cdot 1$ Multiply. 1

ⓒ
(6k3) − 2 Use the Product to a Power Property, (ab)m = ambm.(6) − 2(k3) − 2 Use the Power Property,(am)n = am·n.6 − 2k − 6 Use the Definition of a negative exponent,a − n = 1an.162·1k6 Simplify.136k6

ⓓ
(5x − 3)2 Use the Product to a Power Property, (ab)m = ambm.52(x − 3)2 Simplify.25·x − 6 Rewritex − 6using,a − n = 1an.25·1x6 Simplify.25x6

Simplify each expression: ⓐ (2wx)5 ⓑ (− 11pq3)0 ⓒ (2b3) − 4 ⓓ (8a − 4)2.

ⓐ 32w5x5 ⓑ 1 ⓒ 116b12
ⓓ 64a8

Now we will look at an example that will lead us to the Quotient to a Power Property.

(xy)3 This meansxy·xy·xy Multiply the fractions.x·x·xy·y·y Write with exponents.x3y3

Notice that the exponent applies to both the numerator and the denominator.

We see that (xy)3 is x3y3.

This leads to the **Quotient to a Power Property for Exponents**.

Quotient to a Power Property for Exponents
If a and b are real numbers, $b \neq 0$, and m is an integer, then
(ab)m = ambm
To raise a fraction to a power, raise the numerator and denominator to that power.

Simplify each expression:

ⓐ (b3)4 ⓑ (kj) − 3 ⓒ (2xy2z)3 ⓓ (4p − 3q2)2.

ⓐ

$\left(\frac{b}{3}\right)^4$

Use Quotient to a Power Property, (ab)m=ambm.

$\frac{b^4}{3^4}$

Simplify.

$\frac{b^4}{81}$

ⓑ

$\left(\frac{k}{j}\right)^{-3}$

Raise the numerator and denominator to the po

$\frac{k^{-3}}{j^{-3}}$

Use the definition of negative exponent.

$\frac{1}{k^3} \cdot \frac{j^3}{1}$

Multiply.

$\frac{j^3}{k^3}$

ⓒ

(2xy2z)3 Use Quotient to a Power Property,(ab)m=ambm.(2xy2)3z3 Use the Product to a Power Property,(ab)m=ambm.8x3y6z3

ⓓ

(4p−3q2)2 Use Quotient to a Power Property,(ab)m=ambm.(4p−3)2(q2)2 Use the Product to a Power Property,(ab)m=ambm.42(p−3)2(q2)2 Simplify using the Power Property,(am)n=am·n.16p−6q4 Use the definition of negative exponent.16q4·1p6 Simplify.16p6q4

Simplify each expression:

ⓐ (p10)4 ⓑ (mn)−7 ⓒ(3ab3c2)4 ⓓ (3x−2y3)3.

ⓐ p410000 ⓑ n7m7
ⓒ 81a4b12c8 ⓓ 27x6y9

Simplify each expression:

ⓐ $(-2q)3$ ⓑ $(wx)-4$ ⓒ $(xy33z2)2$ ⓓ $(2m-2n-2)3$.

ⓐ $-8q3$ ⓑ $x4w4$ ⓒ $x2y69z4$
ⓓ $8n6m6$

We now have several properties for exponents. Let’s summarize them and then we’ll do some more examples that use more than one of the properties.

Summary of Exponent Properties

If *a* and *b* are real numbers, and *m* and *n* are integers, then

Property	Description
Product Property	$am \cdot an = am+n$
Power Property	$(am)n = am \cdot n$
Product to a Power	$(ab)m = ambm$
Quotient Property	$aman = am-n, a \neq 0$

Zero Exponent Property	$a0=1, a\neq 0$
Quotient to a Power Property	$(ab)m=ambm, b\neq 0$
Properties of Negative Exponents	$a-n=1an$ and $1a-n=an$
Quotient to a Negative Exponent	$(ab)-n=(ba)n$

Simplify each expression by applying several properties:

ⓐ $(3x2y)4(2xy2)3$ ⓑ $(x3)4(x-2)5(x6)5$ ⓒ $(2xy2x3y-2)2(12xy3x3y-1)-1$.

ⓐ

$(3x2y)4(2xy2)3$ Use the Product to a Power Property, $(ab)m=ambm$. $(34x8y4)(23x3y6)$ Simplify. $(81x8y4)(8x3y6)$ Use the Commutative Property. $81\cdot 8\cdot x8\cdot x3\cdot y4\cdot y6$ Multiply the constants and add the exponents. $648x11y10$

ⓑ

$(x3)4(x-2)5(x6)5$ Use the Power Property, $(am)n=am\cdot n$. $(x12)(x-10)(x30)$ Add the exponents in the numerator. $x2x30$ Use the Quotient Property, $aman=1an-m$. $1x28$

ⓒ
(2xy2x3y − 2)2(12xy3x3y − 1) − 1 Simplify inside the parentheses first.
(2y4x2)2(12y4x2) − 1 Use the Quotient to a Power Property,(ab)m = ambm.
(2y4)2(x2)2(12y4) − 1(x2) − 1 Use the Product to a Power Property,
(ab)m = ambm.4y8x4·12 − 1y − 4x − 2
Simplify.4y412x2 Simplify.y43x2

Simplify each expression:

ⓐ (c4d2)5(3cd5)4 ⓑ (a − 2)3(a2)4(a4)5 ⓒ (3xy2x2y − 3)2(9xy − 3x3y2) − 1.

ⓐ 81c24d30 ⓑ 1a18
ⓒ y15

Use Scientific Notation

Working with very large or very small numbers can

be awkward. Since our number system is base ten we can use powers of ten to rewrite very large or very small numbers to make them easier to work with. Consider the numbers 4,000 and 0.004.

Using place value, we can rewrite the numbers 4,000 and 0.004. We know that 4,000 means $4 \times 1{,}000$ and 0.004 means $4 \times 11{,}000$.

If we write the 1,000 as a power of ten in exponential form, we can rewrite these numbers in this way:

4,000	$4 \times 1{,}000$	4×103	
0.004	$4 \times 11{,}000$	4×1103	$4 \times 10 - 3$

When a number is written as a product of two numbers, where the first factor is a number greater than or equal to one but less than ten, and the second factor is a power of 10 written in exponential form, it is said to be in **scientific notation**.

Scientific Notation

A number is expressed in **scientific notation** when it is of the form
$a \times 10^n$ where $1 \leq a < 10$ and n is an integer.

It is customary in scientific notation to use as the $\times$ multiplication sign, even though we avoid using this sign elsewhere in algebra.

If we look at what happened to the decimal point, we can see a method to easily convert from decimal notation to scientific notation.

$4000. = 4 \times 10^3$	$0.004 = 4 \times 10^{-3}$
Moved the decimal point 3 places to the left.	Moved the decimal point 3 places to the right.

In both cases, the decimal was moved 3 places to get the first factor between 1 and 10.

The power of 10 is positive when the number is larger than 1: $4,000 = 4 \times 10^3$

The power of 10 is negative when the number is between 0 and 1: $0.004 = 4 \times 10^{-3}$

To convert a decimal to scientific notation.

Move the decimal point so that the first factor is greater than or equal to 1 but less than 10. Count the number of decimal places, *n*, that the decimal point was moved. Write the number as a product with a power of 10. If the original number is.

- greater than 1, the power of 10 will be 10n.
- between 0 and 1, the power of 10 will be 10 − n.

Check.

Write in scientific notation: ⓐ 37,000 ⓑ 0.0052.

ⓐ

The original number, 37,000, is greater than 1 so we will have a positive power of 10.	37,000

Move the decimal point to get 3.7, a number between 1 and 10.

37,000

Count the number of decimal places the point was moved.

4 places

Write as a product with a power of 10.

3.7×10^4

Check:

3.7×10^4 $3.7 \times 10{,}000$ $37{,}000$

$37{,}000 = 3.7 \times 10^4$

ⓑ

The original number, 0.0052, is between 0 and 1 so we will have a negative power of 10.

0.0052

Move the decimal point to get 5.2, a number between 1 and 10.

0.0052

Count the number of decimal places the point was moved.

3 places

Write as a product with a power of 10.

5.2×10^{-3}

Check:$5.2 \times 10 - 35.2 \times 11035.2 \times 110005.2 \times 0.0010.00$

$0.0052 = 5.2 \times 10^{-3}$

Write in scientific notation: ⓐ 96,000 ⓑ 0.0078.

ⓐ 9.6×104 ⓑ $7.8 \times 10 - 3$

Write in scientific notation: ⓐ 48,300 ⓑ 0.0129.

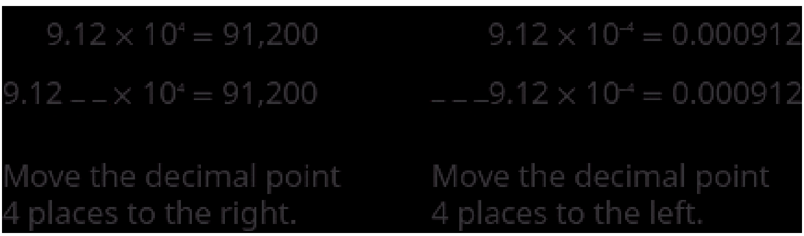
ⓐ 4.83×104

ⓑ $1.29 \times 10-2$

How can we convert from **scientific notation to decimal form**? Let’s look at two numbers written in scientific notation and see.

$9.12 \times 1049.12 \times 10-4$ $9.12 \times 10{,}0009.12 \times 0.0001$
$91{,}2000.000912$

If we look at the location of the decimal point, we can see an easy method to convert a number from scientific notation to decimal form.



In both cases the decimal point moved 4 places. When the exponent was positive, the decimal moved to the right. When the exponent was negative, the decimal point moved to the left.

Convert scientific notation to decimal form.

Determine the exponent, *n*, on the factor 10. Move the decimal *n* places, adding zeros if needed.

- If the exponent is positive, move the decimal point *n* places to the right.
- If the exponent is negative, move the decimal point $|n|$ places to the left.

Check.

Convert to decimal form: ⓐ 6.2×103 ⓑ $-8.9 \times 10 - 2$.

ⓐ

[redacted]

Determine the exponent, *n*, on the factor 10.

The exponent is 3.

Since the exponent is positive, move the de[redacted]aces

to the right.
Add zeros as needed for placeholders.

[redacted]

[redacted]

ⓑ

[redacted]

Determine the exponent, n, on the factor 10.

The exponent is -2.

Since the exponent is negative, move the de[redacted]es to the left.
Add zeros as needed for placeholders.

[redacted]

[redacted]

Convert to decimal form: ⓐ 1.3×103 ⓑ $-1.2 \times 10-4$.

ⓐ 1,300 ⓑ -0.00012

Convert to decimal form: ⓐ -9.5×104 ⓑ $7.5 \times 10-2$.

ⓐ $-950{,}000$ ⓑ 0.075

When scientists perform calculations with very large or very small numbers, they use scientific notation. Scientific notation provides a way for the calculations to be done without writing a lot of zeros. We will see how the Properties of Exponents are used to multiply and divide numbers in scientific notation.

Multiply or divide as indicated. Write answers in decimal form: ⓐ $(-4\times 105)(2\times 10-7)$ ⓑ $9\times 1033\times 10-2$.

ⓐ

$(-4\times 105)(2\times 10-7)$ Use the Commutative Property to rearrange the factors. $-4\cdot 2\cdot 105\cdot 10-7$ Multiply. $-8\times 10-2$ Change to decimal form by moving the decimal twoplaces left. -0.08

ⓑ

$9\times 1039\times 10-2$ Separate the factors, rewriting as the product of twofractions.$93\times 10310-2$ Divide.3×105 Change to decimal form by moving the decimal fiveplaces right.300,000

Multiply or divide as indicated. Write answers in decimal form:

ⓐ $(-3\times 105)(2\times 10-8)$ ⓑ $8\times 1024\times 10-2$.

ⓐ -0.006 ⓑ 20,000

Multiply or divide as indicated. Write answers in decimal form:

ⓐ; $(-3\times 10-2)(3\times 10-1)$ ⓑ $8\times 1042\times 10-1$.

ⓐ -0.009 ⓑ $400{,}000$

Access these online resources for additional instruction and practice with using multiplication properties of exponents.

- Exponential Notation
- Properties of Exponents
- Zero Exponent
- Simplify Exponent Expressions
- Quotient Rule for Exponents
- Scientific Notation
- Converting to Decimal Notation

Key Concepts

- **Exponential Notation**

a^m	a^m means multiply *a*, *m* times
base	$a^m = a \cdot a \cdot a \cdot \ldots\ldots \cdot a$

  This is read *a* to the mth power.
  In the expression am, the *exponent m* tells us how many times we use the *base a* as a factor.
- **Product Property for Exponents**
  If *a* is a real number and *m* and *n* are integers, then
  $am \cdot an = am + n$
  To multiply with like bases, add the exponents.
- **Quotient Property for Exponents**
  If a is a real number, $a \neq 0$, and *m* and *n* are integers, then
  $aman = am - n, m > n and aman = 1an - m, n > m$
- **Zero Exponent**

  - If *a* is a non-zero number, then $a0 = 1$.
  - If *a* is a non-zero number, then *a* to the power of zero equals 1.
  - Any non-zero number raised to the zero power is 1.

- **Negative Exponent**

  - If *n* is an integer and $a \neq 0$, then $a - n = 1an$ or $1a - n = an$.

- **Quotient to a Negative Exponent Property**
  If a,b are real numbers, $a \neq 0, b \neq 0$ and n is an integer, then
  $(ab)-n=(ba)n$
- **Power Property for Exponents**
  If a is a real number and m,n are integers, then
  $(am)n=am \cdot n$
  To raise a power to a power, multiply the exponents.
- **Product to a Power Property for Exponents**
  If *a* and *b* are real numbers and *m* is a whole number, then
  $(ab)m=ambm$
  To raise a product to a power, raise each factor to that power.
- **Quotient to a Power Property for Exponents**
  If a and are real numbers, $b \neq 0$, and m is an integer, then
  $(ab)m=ambm$
  To raise a fraction to a power, raise the numerator and denominator to that power.
- **Summary of Exponent Properties**
  If *a* and *b* are real numbers, and *m* and *n* are integers, then

Property	Description
Product Property	$am \cdot an=am+n$
Power Property	$(am)n=am \cdot n$
Product to a Power	$(ab)n=ambm$
Quotient Property	$aman=am-n, a \neq 0$
Zero Exponent Property	$a0=1, a \neq 0$

Quotient to a Power Property:	$(\frac{a}{b})^m = \frac{a^m}{b^m}, b \neq 0$
Properties of Negative Exponents	$a^{-n} = \frac{1}{a^n}$ and $\frac{1}{a^{-n}} = a^n$
Quotient to a Negative Exponent	$(\frac{a}{b})^{-n} = (\frac{b}{a})^n$

- **Scientific Notation**
  A number is expressed in scientific notation when it is of the form
  $a \times 10^n$ where $1 \leq a < 10$ and n is an integer.
- **How to convert a decimal to scientific notation.**

  Move the decimal point so that the first factor is greater than or equal to 1 but less than 10. Count the number of decimal places, n, that the decimal point was moved. Write the number as a product with a power of 10. If the original number is.

  - greater than 1, the power of 10 will be 10^n.
  - between 0 and 1, the power of 10 will be 10^{-n}.

  Check.

- **How to convert scientific notation to decimal form.**

Determine the exponent, n, on the factor 10. Move the decimal n places, adding zeros if needed.

- If the exponent is positive, move the decimal point n places to the right.
- If the exponent is negative, move the decimal point |n| places to the left.

Check.

Practice Makes Perfect

Simplify Expressions Using the Properties for Exponents

In the following exercises, simplify each expression using the properties for exponents.

ⓐ d3·d6 ⓑ 45x·49x ⓒ 2y·4y3 ⓓ w·w2·w3

ⓐ d9 ⓑ 414x ⓒ 8y4 ⓓ w6

mx·m3

---

mx + 3

ⓐ x18x3 ⓑ 51253 ⓒ q18q36 ⓓ 102103

ⓐ x15 ⓑ 59 ⓒ 1q18 ⓓ 110

ⓐ p21p7 ⓑ 41644 ⓒ bb9 ⓓ 446

ⓐ p14 ⓑ 412 ⓒ 1b8 ⓓ 145

ⓐ 200 ⓑ b0

ⓐ 1 ⓑ 1

ⓐ −270 ⓑ −(270)

ⓐ −1 ⓑ −1

Use the Definition of a Negative Exponent

In the following exercises, simplify each expression.

ⓐ a−2 ⓑ 10−3 ⓒ 1c−5 ⓓ 13−2

ⓐ 1a2 ⓑ 11000 ⓒ c5 ⓓ 9

ⓐ (49)−3 ⓑ (−uv)−5

ⓐ 72964 ⓑ −v5u5

ⓐ (−5)−2 ⓑ −5−2 ⓒ (−15)−2 ⓓ −(15)−2

ⓐ 125 ⓑ 125 ⓒ 25 ⓓ −25

In the following exercises, simplify each expression using the Product Property.

ⓐ b4b−8 ⓑ (w4x−5)(w−2x−4)
ⓒ (−6c−3d9)(2c4d−5)

ⓐ 1b4 ⓑ w2x9 ⓒ −12cd4

p5·p−2·p−4

1p

In the following exercises, simplify each expression using the Power Property.

ⓐ (m4)2 ⓑ (103)6 ⓒ (x3)−4

ⓐ m8 ⓑ 1018 ⓒ 1x12

ⓐ (y3)x ⓑ (5x)y ⓒ (q6)−8

ⓐ y3x ⓑ 5xy ⓒ 1q48

In the following exercises, simplify each expression using the Product to a Power Property.

ⓐ (−3xy)2 ⓑ (6a)0 ⓒ (5x2)−2 ⓓ (−4y−3)2

ⓐ 9x2y2 ⓑ 1 ⓒ 125x4
ⓓ 16y6

ⓐ (−5ab)3 ⓑ (−4pq)0 ⓒ (−6x3)−2 ⓓ (3y −4)2

ⓐ −125a3b3 ⓑ 1 ⓒ 136x6 ⓓ 9y8

In the following exercises, simplify each expression using the Quotient to a Power Property.

ⓐ (p2)5 ⓑ (xy)−6 ⓒ (2xy2z)3 ⓓ (4p−3q2)2

ⓐ p532 ⓑ y6x6 ⓒ 8x3y6z3
ⓓ 16p6q4

ⓐ (a3b)4 ⓑ (54m) − 2 ⓒ (3a-2b3c3)-2 ⓓ (p-1q4r-4)2

ⓐ a481b4 ⓑ 16m225 ⓒ a4c49b6 ⓓ q8r8p2

In the following exercises, simplify each expression by applying several properties.

ⓐ (5t2)3(3t)2 ⓑ (t2)5(t − 4)2(t3)7
ⓒ (2xy2x3y − 2)2(12xy3x3y − 1) − 1

ⓐ 1125t8 ⓑ 1t19 ⓒ y43x2

ⓐ (m2n)2(2mn5)4 ⓑ (− 2p − 2)4(3p4)2(− 6p3)2

ⓐ 16m8n22 ⓑ 4p6

Mixed Practice

In the following exercises, simplify each expression.

ⓐ $7n-1$ ⓑ $(7n)-1$ ⓒ $(-7n)-1$

ⓐ $7n$ ⓑ $17n$ ⓒ $-17n$

(x2)4·(x3)2

---

x14

(2m6)3

8m18

(10x2y)3

1,000x6y3

(23x2y)3

827x6y3

(8a3)2(2a)4

1,024a10

(3m2n)2(2mn5)4

144m8n22

(j − 2j5j4)3

1j3

(− 10n − 2)3(4n5)2(2n8)2

− 4000n12

Use Scientific Notation

In the following exercises, write each number in scientific notation.

ⓐ 340,000 ⓑ 0.041

ⓐ 34 × 104 ⓑ 41 × 10 − 3

In the following exercises, convert each number to

decimal form.

ⓐ -8.3×102 ⓑ $3.8 \times 10-2$

ⓐ -830 ⓑ 0.038

ⓐ 1.6×1010 ⓑ $8.43 \times 10-6$

ⓐ 16,000,000,000
ⓑ 0.00000843

In the following exercises, multiply or divide as indicated. Write your answer in decimal form.

ⓐ $(2 \times 102)(1 \times 10-4)$ ⓑ $5 \times 10-21 \times 10-10$

ⓐ 0.02 ⓑ 500,000,000

ⓐ $(3.5 \times 10-4)(1.6 \times 10-2)$ ⓑ
$8 \times 1064 \times 10-1$

ⓐ 0.0000056 ⓑ 20,000,000

Writing Exercises

Use the Product Property for Exponents to explain why $x \cdot x = x2$.

Explain why $-53 = (-5)3$ but $-54 \neq (-5)4$.

When you convert a number from decimal notation to scientific notation, how do you know if the exponent will be positive or negative?

Answers will vary.

Glossary

Product Property
: According to the Product Property, *a* to the *m* times *a* to the *n* equals *a* to the *m* plus *n*.

Power Property
: According to the Power Property, *a* to the *m* to the *n* equals *a* to the *m* times *n*.

Product to a Power
: According to the Product to a Power Property, *a* times *b* in parentheses to the *m* equals *a* to the *m* times *b* to the *m*.

Quotient Property

According to the Quotient Property, *a* to the *m* divided by *a* to the *n* equals *a* to the *m* minus *n* as long as *a* is not zero.

Zero Exponent Property

According to the Zero Exponent Property, *a* to the zero is 1 as long as *a* is not zero.

Quotient to a Power Property

According to the Quotient to a Power Property, *a* divided by *b* in parentheses to the power of *m* is equal to *a* to the *m* divided by *b* to the *m* as long as *b* is not zero.

Properties of Negative Exponents

According to the Properties of Negative Exponents, *a* to the negative *n* equals 1 divided by *a* to the *n* and 1 divided by *a* to the negative *n* equals *a* to the *n*.

Quotient to a Negative Exponent

Raising a quotient to a negative exponent occurs when *a* divided by *b* in parentheses to the power of negative *n* equals *b* divided by *a* in parentheses to the power of *n*.

Radicals and Exponents (P3)
By the end of this section, you will be able to:

- Simplify expressions with roots
- Add, subtract, multiply, divide roots
- Simplify variable expressions with roots
- Use Properties of Exponents to simplify roots

This Module supports section **P3 of Mat 1023**.
In case you missed something in class, or just want to review a specific topic covered in this Module, here is a list of topics covered:

1. Simplify Expressions with Roots [link]
2. Product Property to Simplify Roots [link]
3. Quotient Property to Simplify Roots[link]
4. Add, Subtract Square Roots[link]
5. Rationalize Denominators [link]
6. Simplify Expressions with Nth Roots[link]
7. Product Rule with Nth Roots[link]
8. Quotient Rule with Nth Roots[link]
9. Rational Exponents in a1n form [link]
10. Rational Exponents in amn form [link]
11. Key Concepts [link]

Simplify Expressions with Roots

Remember that when a real number *n* is multiplied by itself, we write n2 and read it ‘*n* squared’. This number is called the **square** of *n*, and *n* is called the **square root**. For example,
132is read “13 squared”169 is called thesquareof 13, since132=16913 is asquare rootof 169

Square and Square Root of a number
Square
Ifn2=m,thenmis thesquareofn.
Square Root
Ifn2=m,thennis asquare rootofm.

Notice $(-13)2 = 169$ also, so -13 is also a square root of 169. Therefore, both 13 and -13 are square roots of 169.

So, every positive number has two square roots—one positive and one negative. What if we only wanted the positive square root of a positive number? We use a *radical sign*, and write, m, which denotes the positive square root of *m*. The positive square root is also called the **principal square root**.

We also use the radical sign for the square root of

zero. Because $02=0$, $0=0$. Notice that zero has only one square root.

Square Root Notation
mis read "the square root ofm".Ifn2=m,thenn=m,forn≥0.

We know that every positive number has two square roots and the radical sign indicates the positive one. We write $169=13$. If we want to find the negative square root of a number, we place a negative in front of the radical sign. For example, $-169=-13$.

Simplify: ⓐ 144 ⓑ -289.

ⓐ

$144 \text{Since} 122=144.12$

ⓑ

$-289 \text{Since} 172=289$and the negative is

infront of the radical sign. -17

Can we simplify -49? Is there a number whose square is -49?
$()2=-49$

Any positive number squared is positive. Any negative number squared is positive. There is no real number equal to -49. The square root of a negative number is **not a real number.**

Simplify: ⓐ -196 ⓑ -64.

ⓐ
-196There is no real number whose square is -196. $-196$is not a real number.

ⓑ
-64The negative is in front of the radical. -8

Simplify: ⓐ -169 ⓑ -81.

ⓐ not a real number ⓑ -9

But what if we want to estimate 500? If we simplify the square root first, we'll be able to estimate it easily. There are other reasons, too, to simplify square roots as you'll see later in this c

Simplified Square Root

a is considered simplified if a (radicand) has no perfect square factors.

So 31 is simplified. But 32 is not simplified, because 16 is a perfect square factor of 32.

Use Product Property to Simplify Square Roots

The properties we will use to simplify expressions with square roots are similar to the properties of exponents. We know that $(ab)m=ambm$. The corresponding property of square roots says that $ab=a\cdot b$.

Product Property of Square Roots

If *a*, *b* are non-negative real numbers, then

ab = a·banda·b = ab

We use the Product Property of Square Roots to remove all perfect square factors from a radical. We will show how to do this in [link].

How To Use the Product Property to Simplify a Square Root

Simplify: 50.

Solution

Step 1. Find the largest perfect square factor of the radicand.	25 is the largest perfect square factor of 50.	$\sqrt{50}$
Rewrite the radicand as a product using the perfect square factor.	$50 = 25 \cdot 2$	
	Always write the perfect square factor first.	$\sqrt{25 \cdot 2}$

Step 2. Use the product rule to rewrite the radical as the product of two radicals.		$\sqrt{25} \cdot \sqrt{2}$

Step 3. Simplify the square root of the perfect square.		$5\sqrt{2}$

Simplify: 48.

43

Notice in the previous example that the simplified form of 50 is 52, which is the product of an integer and a square root. We always write the integer in front of the square root.

Simplify a square root using the product property.

Find the largest perfect square factor of the radicand. Rewrite the radicand as a product using the perfect-square factor. Use the product rule to rewrite the radical as the product of two radicals. Simplify the square root of the perfect square.

Simplify: 500.

Solution

500Rewrite the radicand as a product using thelargest perfect square factor.100·5Rewrite the radical as the product of tworadicals.100·5Simplify.105

The next example is much like the previous examples, but with variables.

Simplify: x3.

Solution

x3Rewrite the radicand as a product using thelargest perfect square factor.x2·xRewrite the radical as the product of tworadicals.x2·xSimplify.xx

Simplify: 25y5.

Solution

25y5Rewrite the radicand as a product using thelargest perfect square factor.25y4·yRewrite the radical as the product of tworadicals.25y4·ySimplify.5y2y

So we can multiply 3·5 in this way:
3·53·515

Even when the product is not a perfect square, we must look for perfect-square factors and simplify the radical whenever possible.

Simplify: ⓐ 2·6 ⓑ (43)(212).

Solution

ⓐ
2·6Multiply using the Product Property.12Simplify the radical.4·3Simplify.23

ⓑ
(43)(212)Multiply using the Product Property.836Simplify the

radical.8·6Simplify.48

Notice that in (b) we multiplied the coefficients and multiplied the radicals. Also, we did not simplify 12. We waited to get the product and then simplified.

Simplify: ⓐ (8x3)(3x) ⓑ (20y2)(5y3).

Solution

ⓐ
(8x3)(3x)Multiply using the Product Property.24x4Simplify the radical.4x4·6Simplify.2x26

ⓑ
(20y2)(5y3)Multiply using the Product Property.100y5Simplify the radical.10y2y

Quotient Property with Roots

Whenever you have to simplify a square root, the first step you should take is to determine whether

the radicand is a perfect square. A *perfect square fraction* is a fraction in which both the numerator and the denominator are perfect squares.

Simplify: 964.

Solution

964Since(38)2=96438

If the numerator and denominator have any common factors, remove them. You may find a perfect square fraction!

Simplify: 4580.

Solution

4580Simplify inside the radical first. Rewriteshowing the common factors of thenumerator and denominator.5·95·16Simplify the fraction by

removing commonfactors.916Simplify.
(34)2 = 91634

In the last example, our first step was to simplify the fraction under the radical by removing common factors. In the next example we will use the **Quotient Property** to simplify under the radical. We divide the like bases by subtracting their exponents, aman = am − n,a ≠ 0.

Simplify: m6m4.

Solution

m6m4Simplify the fraction inside the radical first.Divide the like bases by subtracting theexponents.m2Simplify.m

Remember the **Quotient to a Power Property**? It said we could raise a fraction to a power by raising the numerator and denominator to the power separately.

(ab)m = ambm,b ≠ 0

We can use a similar property to simplify a square root of a fraction. After removing all common factors from the numerator and denominator, if the fraction is not a perfect square we simplify the numerator and denominator separately.

Quotient Property of Square Roots

If *a*, *b* are non-negative real numbers and $b \neq 0$, then

$ab=abandab=ab$

Simplify: 2164.

Solution

2164We cannot simplify the fraction inside theradical. Rewrite using the quotientproperty.2164Simplify the square root of 64. Thenumerator cannot be simplified.218

We will use the Quotient Property for Exponents, $aman=am-n$, when we have variables with exponents in the radicands.

Simplify: 27m3196.

Solution

Step 1. Simplify the fraction in the radicand, if possible.	$\frac{27m^3}{196}$ cannot be simplified.	$\sqrt{\frac{27m^3}{196}}$
Step 2. Use the Quotient Property to rewrite the radical as the quotient of two radicals.	We rewrite $\sqrt{\frac{27m^3}{196}}$ as the quotient of $\sqrt{27m^3}$ and $\sqrt{196}$.	$\frac{\sqrt{27m^3}}{\sqrt{196}}$
Step 3. Simplify the radicals in the numerator and the denominator.	$9m^2$ and 196 are perfect squares.	$\frac{\sqrt{9m^2} \cdot \sqrt{3m}}{\sqrt{196}}$ $\frac{3m\sqrt{3m}}{14}$

Simplify: 45x5y4.

Solution

45x5y4We cannot simplify the fraction in theradicand. Rewrite using the QuotientProperty.45x5y4Simplify the radicals in the numerator andthe

denominator.9x4·5xy2Simplify.3x25xy2

Simplify: 81d925d4.

Solution

81d925d4Simplify the fraction in the radicand.81d525Rewrite using the Quotient Property.81d525Simplify the radicals in the numerator andthe denominator.81d4·d5Simplify.9d2d5

Simplify: 6y52y.

Solution

6y52yNeither radicand is a perfect square, sorewrite using the quotient property ofsquare roots.6y52yRemove common factors in the numeratorand denominator.2·3·y4·y2·ySimplify.3y4Simplify the radical.y23

We will use the Quotient Property of Square Roots ‘in reverse’ when the fraction we start with is the quotient of two square roots, and neither radicand is a perfect square. When we write the fraction in a single square root, we may find common factors in the numerator and denominator.

Simplify: 2775.

Solution

2775Neither radicand is a perfect square, sorewrite using the quotient property ofsquare roots.2775Remove common factors in the numeratorand denominator.3·93·25Simplify.92535

Add, Subtract Square Roots

We know that we must follow the order of operations to simplify expressions with square roots. The radical is a grouping symbol, so we work inside the radical first. We simplify 2+7 in this way:

2+7Add inside the radical.9Simplify.3

So if we have to add $2+7$, we must not combine them into one radical.
$2+7 \neq 2+7$

Trying to add square roots with different radicands is like trying to add unlike terms.

But, just like we can add$x+x$,we can add$3+3$. $x+x=2x$$3+3=23$

Adding square roots with the same radicand is just like adding like terms. We call square roots with the same radicand like square roots to remind us they work the same as like terms.

> Square roots with the same radicand are called **like square roots**.
> We add and subtract like square roots in the same way we add and subtract like terms. We know that $3x+8x$ is $11x$. Similarly we add $3x+8x$ and the result is $11x$.

Think about adding like terms with variables as you do the next few examples. When you have like radicands, you just add or subtract the coefficients. When the radicands are not like, you cannot combine the terms.

Simplify: $22 - 72$.

Solution

$22 - 72$Since the radicals are like, we subtract thecoefficients. -52

Simplify: $513 + 413 + 213$.

Solution

$513 + 413 + 213$Since the radicals are like, we add thecoefficients.1113

Remember that we always simplify square roots by removing the largest perfect-square factor. Sometimes when we have to add or subtract square roots that do not appear to have like radicals, we find like radicals after simplifying the square roots.

Simplify: 20 + 35.

Solution

20 + 35Simplify the radicals, when possible.4·5 + 3525 + 35Combine the like radicals.55

Just like we use the Associative Property of Multiplication to simplify 5(3x) and get 15x, we can simplify 5(3x) and get 15x. We will use the Associative Property to do this in the next example.

Simplify: 518 − 28.

Solution

518 − 28Simplify the radicals.5·9·2 − 2·4·25·3·2 − 2·2·2152 − 42Combine the like radicals.112

Simplify: $18n5 - 32n5$.

Solution

$18n5 - 32n5$Simplify the radicals.$9n4 \cdot 2n - 16n4 \cdot 2n3n22n - 4n22n$Combine the like radicals.$-n22n$

Rationalize Denominators

Before the calculator became a tool of everyday life, approximating the value of a fraction with a radical in the denominator was a very cumbersome process!

For this reason, a process called **rationalizing the denominator** was developed. A fraction with a radical in the denominator is converted to an equivalent fraction whose denominator is an integer. Square roots of numbers that are not perfect squares are irrational numbers. When we rationalize the denominator, we write an equivalent fraction with a rational number in the denominator.

The process of converting a fraction with a radical in the denominator to an equivalent fraction whose denominator is an integer is called **rationalizing the denominator**.

Suppose we need an approximate value for the fraction.12A five decimal place approximation to2is1.41421.11.41421Without a calculator, would you want to do this division?1.414211.0

But we can find a fraction equivalent to 12 by multiplying the numerator and denominator by 2.

$$\frac{1}{\sqrt{2}}$$

$$\frac{1 \cdot \sqrt{2}}{\sqrt{2} \cdot \sqrt{2}}$$

$$\frac{\sqrt{2}}{2}$$

Now if we need an approximate value, we divide 21.41421. This is much easier.

Even though we have calculators available nearly everywhere, a fraction with a radical in the denominator still must be rationalized. It is not considered simplified if the denominator contains a square root.

A square root is considered simplified if there are

- no perfect-square factors in the radicand
- no fractions in the radicand

- no square roots in the denominator of a fraction

To rationalize a denominator with a square root, we use the property that (a)2 = a. If we square an irrational square root, we get a rational number.

Simplify: ⓐ 43 ⓑ 320 ⓒ 36x.

To rationalize a denominator with one term, we can multiply a square root by itself. To keep the fraction equivalent, we multiply both the numerator and denominator by the same factor.

ⓐ

$\frac{4}{\sqrt{3}}$

Multiply both the numerator and denominator by 3.

$$\frac{4 \cdot \sqrt{3}}{\sqrt{3} \cdot \sqrt{3}}$$

Simplify.

$$\frac{4\sqrt{3}}{3}$$

ⓑ We always simplify the radical in the denominator first, before we rationalize it. This way the numbers stay smaller and easier to work with.

$$\sqrt{\frac{3}{20}}$$

The fraction is not a perfect square, so rewrite using the Quotient Property.

$$\frac{\sqrt{3}}{\sqrt{20}}$$

Simplify the denominator.

$$\frac{\sqrt{3}}{2\sqrt{5}}$$

Multiply the numerator and denominator by 5.

$$\frac{\sqrt{3}\cdot\sqrt{5}}{2\sqrt{5}\cdot\sqrt{5}}$$

Simplify.

$$\frac{\sqrt{15}}{2\cdot 5}$$

Simplify.

$$\frac{\sqrt{15}}{10}$$

ⓒ

$$\frac{3}{\sqrt{6x}}$$

Multiply the numerator and denominator by 6x

$$\frac{3\cdot\sqrt{6x}}{\sqrt{6x}\cdot\sqrt{6x}}$$

Simplify.

$$\frac{3\sqrt{6x}}{6x}$$

Simplify.

$$\frac{\sqrt{6x}}{2x}$$

When the denominator of a fraction is a sum or difference with square roots, we use the Product of Conjugates pattern to rationalize the denominator.

Product of Conjugates

(a − b)(a + b)(2 − 5)(2 + 5)a2 − b222 − (5)24 − 5 − 1

Simplify: 44 + 2.

Solution

$$\frac{4}{4+\sqrt{2}}$$

Multiply the numerator and denominator by the [illegible] e de[illegible]

$$\frac{4(4-\sqrt{2})}{(4+\sqrt{2})(4-\sqrt{2})}$$

Multiply the conjugates in the denominator.

$$\frac{4(4-\sqrt{2})}{4^2-(\sqrt{2})^2}$$

Simplify the denominator.

$$\frac{4(4-\sqrt{2})}{16-2}$$

Simplify the denominator.

$$\frac{4(4-\sqrt{2})}{14}$$

Remove common factors from the nu de

$$\frac{2(4-\sqrt{2})}{7}$$

We leave the numerator in factored form to make it easier to look for common factors after we have simplified the denominator.

Simplify: $3u-6$.

Solution

$$\frac{\sqrt{3}}{\sqrt{u}-\sqrt{6}}$$

Multiply the numerator and denominator by the ... de...

$$\frac{\sqrt{3}\left(\sqrt{u}+\sqrt{6}\right)}{\left(\sqrt{u}-\sqrt{6}\right)\left(\sqrt{u}+\sqrt{6}\right)}$$

Multiply the conjugates in the denominator.

$$\frac{\sqrt{3}\left(\sqrt{u}+\sqrt{6}\right)}{\left(\sqrt{u}\right)^2-\left(\sqrt{6}\right)^2}$$

Simplify the denominator.

$$\frac{\sqrt{3}\left(\sqrt{u}+\sqrt{6}\right)}{u-6}$$

Simplify Variable Expressions with Roots

So far we have only talked about squares and square roots. Let’s now extend our work to include higher powers and higher roots.

Let’s review some vocabulary first.

We write:We say:n2nsquaredn3ncubedn4nto the fourth powern5nto the fifth power

The terms ‘squared’ and ‘cubed’ come from the formulas for area of a square and volume of a cube.

It will be helpful to have a table of the powers of the integers from -5 to 5. See [link].

Number	Square	Cube	Fourth power	Fifth power
n	n^2	n^3	n^4	n^5
1	1	1	1	1
2	4	8	16	32
3	9	27	81	243
4	16	64	256	1024
5	25	125	625	3125
x	x^2	x^3	x^4	x^5
x^2	x^4	x^6	x^8	x^{10}

Number	Square	Cube	Fourth power	Fifth power
n	n^2	n^3	n^4	n^5
–1	1	–1	1	–1
–2	4	–8	16	–32
–3	9	–27	81	–243
–4	16	–64	256	–1024
–5	25	–125	625	–3125

Notice the signs in the table. All powers of positive numbers are positive, of course. But when we have a negative number, the *even* powers are positive and the *odd* powers are negative. We’ll copy the row with the powers of -2 to help you see this.

n		n^3		n^5
–2	4	–8	16	–32

power
Positive result

Odd power
Negative result

We will now extend the square root definition to higher roots.

nth Root of a Number

Ifbn $=$ a,thenbis annthroot ofa.The principalnthroot

ofais writtenan.nis called theindexof the radical.

Just like we use the word ‘cubed’ for *b*3, we use the term ‘cube root’ for a3.

We can refer to [link] to help find higher roots.
43=6434=81(−2)5=−32643=4814=3−325=−2

Could we have an even root of a negative number? We know that the square root of a negative number is not a real number. The same is true for any even root. *Even* roots of negative numbers are not real numbers. *Odd* roots of negative numbers are real numbers.

Properties of an

When *n* is an even number and

- a≥0, then an is a real number.
- a<0, then an is not a real number.

When *n* is an odd number, an is a real number for all values of *a*.

We will apply these properties in the next two examples.

Simplify: ⓐ 643 ⓑ 814 ⓒ 325.

ⓐ

643Since43 = 64.4

ⓑ

814Since(3)4 = 81.3

ⓒ

325Since(2)5 = 32.2

In this example be alert for the negative signs as well as even and odd powers.

Simplify: ⓐ −1253 ⓑ 164 ⓒ −2435.

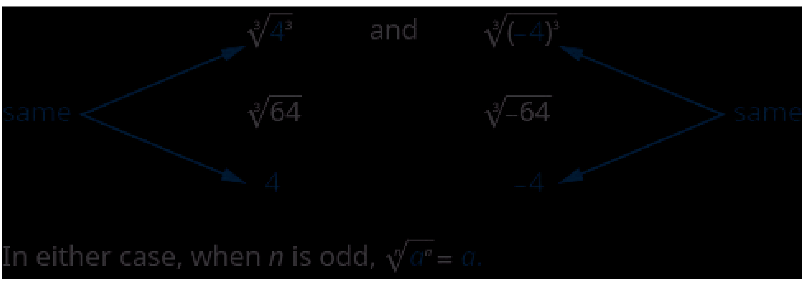
ⓐ

-1253Since$(-5)3=-125.-5$

ⓑ
-164Think,$(?)4=-16.$No real number raisedto the fourth power is negative.Not a real number.

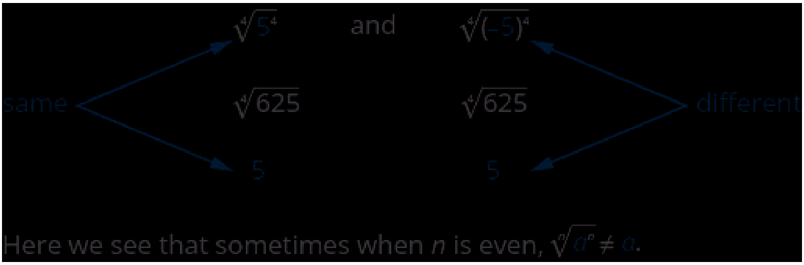
ⓒ
-2435Since$(-3)5=-243.-3$

The odd root of a number can be either positive or negative. For example,



But what about an even root? We want the principal root, so $6254=5$.

But notice,



How can we make sure the fourth root of -5 raised to the fourth power is 5? We can use the absolute value. $|-5| = 5$. So we say that when n is even ann $= |a|$. This guarantees the principal root is positive.

Simplifying Odd and Even Roots (nth root of Perfect nth power)

For any integer $n \geq 2$,

when the indexnis oddann $=$ awhen the indexnis evenann $= |a|$

We must use the absolute value signs when we take an even root of an expression with a variable in the radical.

Simplify: ⓐ x2 ⓑ n33 ⓒ p44 ⓓ y55.

ⓐ We use the absolute value to be sure to get

the positive root.

x2Since the indexnis even,ann=|a|.|x|

ⓑ This is an odd indexed root so there is no need for an absolute value sign.

m33Since the indexnis odd,ann=a.m

ⓒ

p44Since the indexnis evenann=|a|.|p|

ⓓ

y55Since the indexnis odd,ann=a.y

Simplify: ⓐ x6 ⓑ y16.

ⓐ
x6Since(x3)2=x6.(x3)2Since the indexnis evenan=|a|.|x3|

ⓑ
y16Since(y8)2=y16.(y8)2Since the indexnis evenann=|a|.y8In this case the absolute value sign isnot needed asy8is positive.

The next example uses the same idea for highter roots.

Simplify: ⓐ y183 ⓑ z84.

ⓐ
y183Since(y6)3 = y18.(y6)33Sincenis odd,ann = a.y6

ⓑ
z84Since(z2)4 = z8.(z2)44Sincez2is positive, we do not need anz2absolute value sign.

Access this online resource for additional instruction and practice with simplifying expressions with roots.

- Simplifying Variables Exponents with Roots using Absolute Values

Use Product and Quotient Rule to Simplify

We will simplify radical expressions in a way similar to how we simplified fractions. A fraction is simplified if there are no common factors in the numerator and denominator. To simplify a fraction, we look for any common factors in the numerator and denominator.

A radical expression, an, is considered simplified if it has no factors of mn. So, to simplify a radical expression, we look for any factors in the radicand that are powers of the index.

> Simplified Radical Expression
> For real numbers *a* and *m*, and $n \geq 2$,
> anis considered simplified ifahas no factors ofmn

For example, 5 is considered simplified because there are no perfect square factors in 5. But 12 is not simplified because 12 has a perfect square factor of 4.

Similarly, 43 is simplified because there are no

perfect cube factors in 4. But 243 is not simplified because 24 has a perfect cube factor of 8.

To simplify radical expressions, we will also use some properties of roots. The properties we will use to simplify radical expressions are similar to the properties of exponents. We know that (ab)n = anbn. The corresponding of **Product Property of Roots** says that abn = an·bn.

Product Property of *n*th Roots

If an and bn are real numbers, and n ≥ 2 is an integer, then

abn = an·bnandan·bn = abn

Simplify: 98.

Step 1. Find the largest factor in the radicand that is a perfect power of the index. Rewrite the radicand as a product of two factors, using that factor.	We see that 49 is the largest factor of 98 that has a power of 2. In other words 49 is the largest perfect square factor of 98. $98 = 49 \cdot 2$ Always write the perfect square factor first.	$\sqrt{98}$ $\sqrt{49 \cdot 2}$

Step 2. Use the product rule to rewrite the radical as the product of two radicals.	$\sqrt{49} \cdot \sqrt{2}$
Step 3. Simplify the root of the perfect power.	$7\sqrt{2}$

Simplify a radical expression using the Product Property.

Find the largest factor in the radicand that is a perfect power of the index. Rewrite the radicand as a product of two factors, using that factor. Use the product rule to rewrite the radical as the product of two radicals. Simplify the root of the perfect power.

Simplify: ⓐ x3 ⓑ x43 ⓒ x74.

ⓐ

x3Rewrite the radicand as a product usingthe largest perfect square factor.x2·xRewrite the radical as the product of tworadicals.x2·xSimplify.|x|x

ⓑ
x43 Rewrite the radicand as a productusing the largest perfect cube factor.x3·x3. Rewrite the radical as the product of tworadicals.x33·x3 Simplify.xx3

ⓒ
x74Rewrite the radicand as a productusing the greatest perfect fourth powerx4·x34factor.Rewrite the radical as the product of tworadicals.x44·x34Simplify.|x|x34

Using Quotient Property to Simplify

Whenever you have to simplify a radical expression, the first step you should take is to determine whether the radicand is a perfect power of the index. If not, check the numerator and denominator for any common factors, and remove them. You may find a fraction in which both the numerator and the denominator are perfect powers of the index.

Simplify: ⓐ 4580 ⓑ 16543 ⓒ 5804.

ⓐ

4580Simplify inside the radical first.Rewrite showing the common factors ofthe numerator and denominator.5·95·16Simplify the fraction by removingcommon factors.916Simplify. Note(34)2 = 916.34

ⓑ

16543Simplify inside the radical first.Rewrite showing the common factors ofthe numerator and denominator.2·82·273Simplify the fraction by removingcommon factors.8273Simplify. Note(23)3 = 827.23

ⓒ

5804Simplify inside the radical first.Rewrite showing the common factors ofthe numerator and denominator.5·15·164Simplify the fraction by removingcommon factors.1164Simplify. Note(12)4 = 116.12

In the next example we will use the Quotient Property to simplify under the radical. We divide the like bases by subtracting their exponents, aman = am − n,a ≠ 0

Simplify: ⓐ m6m4 ⓑ a8a53 ⓒ a10a24.

ⓐ
m6m4Simplify the fraction inside the radical first.Divide the like bases by subtracting theexponents.m2Simplify.|m|

ⓑ
a8a53Use the Quotient Property of exponents tosimplify the fraction under the radical first.a33Simplify.a

ⓒ
a10a24Use the Quotient Property of exponents tosimplify the fraction under the radical first.a84Rewrite the radicand using perfectfourth power factors.(a2)44Simplify.a2

Remember the Quotient to a Power Property? It said we could raise a fraction to a power by raising the numerator and denominator to the power separately.

(ab)m = ambm,b ≠ 0

We can use a similar property to simplify a root of a fraction. After removing all common factors from the numerator and denominator, if the fraction is

not a perfect power of the index, we simplify the numerator and denominator separately.

Quotient Property of Radical Expressions

If an and bn are real numbers,b≠0, and for any integer n≥2 then,

abn=anbnandanbn=abn

Simplify the fraction in the radicand, if possible. Use the Quotient Property to rewrite the radical as the quotient of two radicals. Simplify the radicals in the numerator and the denominator.

How to Simplify the Quotient of Radical Expressions

Simplify: 27m3196.

Step 1. Simplify the fraction in the radicand, if possible.	$\frac{27m^3}{196}$ cannot be simplified.	$\sqrt{\frac{27m^3}{196}}$

Step 2. Use the Quotient Property to rewrite the radical as the quotient of two radicals.	We rewrite $\sqrt{\frac{27m^3}{196}}$ as the quotient of $\sqrt{27m^3}$ and $\sqrt{196}$.	$\frac{\sqrt{27m^3}}{\sqrt{196}}$

Step 3. Simplify the radicals in the numerator and the denominator.	$9m^2$ and 196 are perfect squares.	$\frac{\sqrt{9m^2} \cdot \sqrt{3m}}{\sqrt{196}}$ $\frac{3m\sqrt{3m}}{14}$

Simplify: ⓐ -108323 ⓑ $96x743x24$.

Solution

ⓐ

-108323Neither radicand is a perfect cube, so usethe Quotient Property to write as one radical.-10823Simplify the fraction under the radical.-543Rewrite the radicand as a product usingperfect cube factors.$(-3)3\cdot23$Rewrite the radical as the product of two radicals.$(-3)33\cdot23$Simplify.-323

ⓑ

$96x743x24$Neither radicand is a perfect fourth power,so use the Quotient Property to write as one radical.$96x73x24$Simplify the fraction under the radical.$32x54$Rewrite the radicand as a product usingperfect fourth power factors.$24x4\cdot2x4$Rewrite the radical as the product of two radicals.

(2x)44·2x4Simplify.2|x|2x4

Simplify Expressions with a1n

Rational exponents are another way of writing expressions with radicals. When we use rational exponents, we can apply the properties of exponents to simplify expressions.

The Power Property for Exponents says that (am)n = am·n when *m* and *n* are whole numbers. Let’s assume we are now not limited to whole numbers.

Suppose we want to find a number *p* such that (8p)3 = 8. We will use the Power Property of Exponents to find the value of *p*.

(8p)3 = 8Multiply the exponents on the left.83p = 8Write the exponent 1 on the right.83p = 81Since the bases are the same, the exponents must be equal.3p = 1Solve forp.p = 13

So (813)3 = 8. But we know also (83)3 = 8. Then it must be that 813 = 83.

This same logic can be used for any positive integer exponent *n* to show that a1n = an.

Rational Exponent a1n
If an is a real number and $n \geq 2$, then
a1n=an

The denominator of the rational exponent is the **index** of the radical.

There will be times when working with expressions will be easier if you use rational exponents and times when it will be easier if you use radicals. In the first few examples, you’ll practice converting expressions between these two notations.

Write as a radical expression: ⓐ x12 ⓑ y13 ⓒ z14.
We want to write each expression in the form an. ⓐ x12The denominator of the rational exponent is 2, sothe index of the radical is 2. We do not show theindex when it is 2.x ⓑ

y13The denominator of the exponent is 3, so theindex is 3.y3

ⓒ

z14The denominator of the exponent is 4, so theindex is 4.z4

Simplify: ⓐ 2512 ⓑ 6413 ⓒ 25614.

ⓐ

2512Rewrite as a square root.25Simplify.5

ⓑ

6413Rewrite as a cube root.643Recognize 64 is a perfect cube.433Simplify.4

ⓒ

25614Rewrite as a fourth root.2564Recognize 256 is a perfect fourth power.444Simplify.4

Be careful of the placement of the negative signs in the next example. We will need to use the property a−n=1an in one case.

Simplify: ⓐ $(-16)^{\frac{1}{4}}$ ⓑ $-16^{\frac{1}{4}}$ ⓒ $(16)^{-\frac{1}{4}}$.

ⓐ

$(-16)^{\frac{1}{4}}$ Rewrite as a fourth root.
$\sqrt[4]{-16}$ $\sqrt[4]{(-2)^4}$ Simplify. No real solution.

ⓑ

$-16^{\frac{1}{4}}$ The exponent only applies to the 16. Rewrite as a fouth root. $-\sqrt[4]{16}$ Rewrite 16 as 2^4. $-\sqrt[4]{2^4}$ Simplify. -2

ⓒ

$(16)^{-\frac{1}{4}}$ Rewrite using the property $a^{-n}=\frac{1}{a^n}$. $\frac{1}{(16)^{\frac{1}{4}}}$ Rewrite as a fourth root. $\frac{1}{\sqrt[4]{16}}$ Rewrite 16 as 2^4. $\frac{1}{\sqrt[4]{2^4}}$ Simplify. $\frac{1}{2}$

Simplify Expressions with $a^{\frac{m}{n}}$

We can look at $a^{\frac{m}{n}}$ in two ways. Remember the Power Property tells us to multiply the exponents and so $(a^{\frac{1}{n}})^m$ and $(a^m)^{\frac{1}{n}}$ both equal $a^{\frac{m}{n}}$. If we write these expressions in radical form, we get

$$a^{\frac{m}{n}}=(a^{\frac{1}{n}})^m=(\sqrt[n]{a})^m \quad\text{and}\quad a^{\frac{m}{n}}=(a^m)^{\frac{1}{n}}=\sqrt[n]{a^m}$$

This leads us to the following definition.

Rational Exponent amn

For any positive integers *m* and *n*,

amn = (an)mandamn = amn

Simplify: ⓐ 12523 ⓑ 16 − 32 ⓒ 32 − 25.

We will rewrite the expression as a radical first using the defintion, amn = (an)m. This form lets us take the root first and so we keep the numbers in the radicand smaller than if we used the other form.

ⓐ

12523The power of the radical is the numerator of the exponent, 2.The index of the radical is the denominator of theexponent, 3. (1253)2Simplify.(5)225

ⓑ We will rewrite each expression first using a − n = 1an and then change to radical form. 16 − 32Rewrite usinga − n = 1an11632Change to radical form. The power of the radical is thenumerator of the exponent, 3. The index is the denominatorof the exponent, 2.1(16)3Simplify.143164

ⓒ

32 − 25Rewrite usinga − n = 1an.13225Change to radical form.1(325)2Rewrite the radicand as a power.1(255)2Simplify.12214

Use the Properties of Exponents to Simplify Expressions with Rational Exponents

The same properties of exponents that we have already used also apply to rational exponents. We will list the Properties of Exponenets here to have them for reference as we simplify expressions.

Properties of Exponents
If *a* and *b* are real numbers and *m* and *n* are rational numbers, then
Product Propertyam·an = am + nPower Property(am)n = am·nProduct to a Power(ab)m = ambmQuotient Propertyaman = am − n,a ≠ 0Zero Exponent Definitiona0 = 1,a ≠ 0Quotient to a Power Property(ab)m = ambm,b ≠ 0Negative Exponent Propertya − n = 1an,a ≠ 0

We will apply these properties in the next example.

Simplify: ⓐ x12·x56 ⓑ (z9)23 ⓒ x13x53.

ⓐ The Product Property tells us that when we multiply the same base, we add the exponents.

x12·x56The bases are the same, so we add theexponents.x12+56Add the fractions.x86Simplify the exponent.x43

ⓑ The Power Property tells us that when we raise a power to a power, we multiply the exponents.

(z9)23To raise a power to a power, we multiplythe exponents.z9·23Simplify.z6

ⓒ The Quotient Property tells us that when we divide with the same base, we subtract the exponents.

x13x53x13x53To divide with the same base, we subtractthe exponents.1x53−13Simplify.1x43

Access these online resources for additional instruction and practice with simplifying rational

exponents.

- Review-Rational Exponents
- Using Laws of Exponents on Radicals: Properties of Rational Exponents

Key Concepts

- **Square Root Notation**
    - m is read ‘the square root of *m*’
    - If $n^2 = m$, then $n = m$, for $n \geq 0$.

    [redacted]

    - The square root of *m*, m, is a positive number whose square is *m*.
    - **Simplified Square Root** a is considered simplified if a has no perfect-square factors.

- **Product Property of Square Roots** If *a*, *b* are non-negative real numbers, then
$ab = a \cdot b$
- **Simplify a Square Root Using the Product Property** To simplify a square root using the

Product Property:

Find the largest perfect square factor of the radicand. Rewrite the radicand as a product using the perfect square factor. Use the product rule to rewrite the radical as the product of two radicals. Simplify the square root of the perfect square.

- **Quotient Property of Square Roots** If *a*, *b* are non-negative real numbers and $b \neq 0$, then $ab = ab$

- **Simplify a Square Root Using the Quotient Property** To simplify a square root using the Quotient Property:

  Simplify the fraction in the radicand, if possible. Use the Quotient Rule to rewrite the radical as the quotient of two radicals. Simplify the radicals in the numerator and the denominator.

- ***n*th Root of a Number**

  - If $bn = a$, then *b* is an *nth* root of *a*.
  - The principal *nth* root of *a* is written an.
  - *n* is called the *index* of the radical.

- **Properties of an**

    - When *n* is an even number and
        - $a \geq 0$, then an is a real number
        - $a < 0$, then an is not a real number
    - When *n* is an odd number, an is a real number for all values of *a*.

- **Simplifying Odd and Even Roots**
    - For any integer $n \geq 2$,
        - when *n* is odd $ann = a$
        - when *n* is even $ann = |a|$
    - We must use the absolute value signs when we take an even root of an expression with a variable in the radical.

- **Simplified Radical Expression**
    - For real numbers *a*, *m* and $n \geq 2$ an is considered simplified if *a* has no factors of mn

- **Product Property of nth Roots**
    - For any real numbers, an and bn, and for any integer $n \geq 2$ $abn = an \cdot bn$ and $an \cdot bn = abn$

- **How to simplify a radical expression using the Product Property**

  Find the largest factor in the radicand that is a perfect power of the index.
  Rewrite the radicand as a product of two factors, using that factor. Use the product rule to rewrite the radical as the product of two radicals. Simplify the root of the perfect power.

- **Quotient Property of Radical Expressions**

  - If an and bn are real numbers, $b \neq 0$, and for any integer $n \geq 2$ then,
    abn=anbn and anbn=abn

- **How to simplify a radical expression using the Quotient Property.**

  Simplify the fraction in the radicand, if possible. Use the Quotient Property to rewrite the radical as the quotient of two radicals. Simplify the radicals in the numerator and the denominator.

- **Rational Exponent a1n**

  - If an is a real number and $n \geq 2$, then a1n=an.

- **Rational Exponent amn**
    - For any positive integers *m* and *n*, amn = (an)m and amn = amn
- **Properties of Exponents**
    - If *a, b* are real numbers and *m, n* are rational numbers, then
        - **Product Property** am·an = am + n
        - **Power Property** (am)n = am·n
        - **Product to a Power** (ab)m = ambm
        - **Quotient Property** aman = am − n,a ≠ 0
        - **Zero Exponent Definition** a0 = 1, a ≠ 0
        - **Quotient to a Power Property** (ab)m = ambm,b ≠ 0
        - **Negative Exponent Property** a − n = 1an,a ≠ 0

Practice Makes Perfect

Simplify Expressions with Roots

In the following exercises, simplify.

ⓐ 64 ⓑ − 81

ⓐ 8 ⓑ -9

ⓐ 49 ⓑ -0.01

ⓐ 23 ⓑ -0.1

ⓐ -121 ⓑ -289

ⓐ not real number ⓑ -17

ⓐ 2163 ⓑ 2564

ⓐ 6 ⓑ 4

ⓐ -83 ⓑ -814 ⓒ -325

ⓐ -2 ⓑ not real ⓒ -2

Simplify Variable Expressions with Roots

In the following exercises, simplify using absolute values as necessary.

ⓐ y44 ⓑ m77

ⓐ |y| ⓑ m

ⓐ 49x2 ⓑ −81x18

ⓐ 7|x| ⓑ −9|x9|

Simplify using Porperties

y19

---

y9y

36n13

6n6n

Use Quotient Property to Simplify

98p62p2

26169

2613

Add Subtract LIke Roots

6m − 2m

4m

23rs + 3rs − 5rs

33rs − 5rs

48 + 75

93

Multiply Roots

22·614

247

(62y)(350y3)

180y2

Divide Roots

20y52y

---

y210

Rationalize Denominator

1015

2153

Rationalize Deonominator

44 + 27

---

16 − 123 − 11

Higher Roots

ⓐ d99
ⓑ
[missing_resource: CNX_ElemAlg_Figure_09_08_015_img_new.jpg]

ⓐ d ⓑ
[missing_resource: CNX_ElemAlg_Figure_09_08_009_img_new.jpg]

w12w23

---

w3w3

4205−2205

2205

Rational Exponent

3215

2

ⓐ (−81)14
ⓑ −8114
ⓒ (81)−14

ⓐ not real ⓑ −3 ⓒ 13

ⓐ c14·c58
ⓑ (p12)34
ⓒ r45r95

ⓐ c78 ⓑ p9 ⓒ 1r

Glossary

square of a number
If $n^2 = m$, then m is the square of n.

square root of a number
If $n^2 = m$, then n is a square root of m.

Polynomials (P4)
In this section students will:

- Identify the degree and leading coefficient of polynomials.
- Add and subtract polynomials.
- Multiply polynomials.
- Use FOIL to multiply binomials.
- Perform operations with polynomials of several variables.

This Module supports section **P4 of Mat 1023**.
In case you missed something in class, or just want to review a specific topic covered in this Module, here is a list of topics covered:

Identifying the Degree and Leading Coefficient of Polynomials

We have learned that a *term* is a constant or the product of a constant and one or more variables. A **monomial** is an algebraic expression with one term. When it is of the form axm, where *a* is a constant and *m* is a whole number, it is called a monomial in one variable. Some examples of monomial in one variable are. Monomials can also have more than one variable such as and −4a2b3c2.

polynomial—A monomial, or two or more algebraic terms combined by addition or subtraction is a polynomial.
monomial—A polynomial with exactly one term is called a monomial.
binomial—A polynomial with exactly two terms is called a binomial.
trinomial—A polynomial with exactly three terms is called a trinomial.

Here are some examples of polynomials.

Polynomial	y+1	4a2−7ab+2b2	4x4+x3+8x2−9x+1	
Monomial	14	8y2	−9x3y5	−13a3b2c
Binomial	a+7b	4x2−y2	y2−16	3p3q−9p2q
Trinomial	x2−7x+12	9m2+2mn−8n2	6k4−k3+8k	z4+3z2−1

The **degree of a polynomial** and the degree of its terms are determined by the exponents of the variable. A monomial that has no variable, just a **constant**, is a special case. The **degree of a constant** is then 0. In the table above, the example of monomial "14" would have a degree of 0.

> The **degree of a term** is the sum of the exponents of its variables.
> The **degree of a constant** is 0.
> The **degree of a polynomial** is the highest degree of all its terms.

Let's start by looking at a monomial. The monomial 8ab2 has two variables *a* and *b*. To find the degree we need to find the sum of the exponents. The variable a doesn't have an exponent written, but remember that means the exponent is 1. The exponent of *b* is 2. The sum of the exponents, 1+2,

is 3 so the degree is 3.

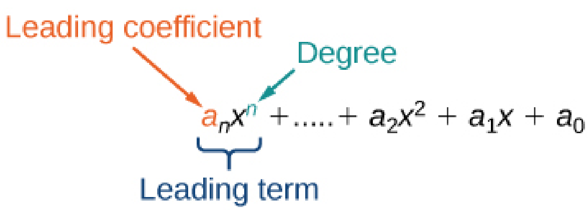
Determine whether each polynomial is a monomial, binomial, trinomial, or other polynomial. Then, find the degree of each polynomial.

ⓐ 7y2 − 5y + 3 ⓑ −2a4b2 ⓒ 3x5 − 4x3 − 6x2 + x − 8 ⓓ 2y − 8xy3 ⓔ 15

	Polynomial	Number of terms	Type	Degree of terms	Degree of polynomial
ⓐ	7y2 − 5y + 3	3	Trinomial	2, 1, 0	2
ⓑ	−2a4b2	1	Monomial	4, 2	6
ⓒ	3x5 − 4x3 − 6x2 + x − 8	5	Polynomial	5, 3, 2, 1, 0	5
ⓓ	2y − 8xy3	2	Binomial	1, 4	4
ⓔ	15	1	Monomial	0	0

A **polynomial**, which is a sum of or difference of terms, each consist of a variable raised to a nonnegative integer power. A number multiplied by a variable raised to an exponent, such as the 384 in 384x, is known as a **coefficient**. Coefficients can be positive, negative, or zero, and can be whole numbers, decimals, or fractions. Each product a i x i , such as 384xw, is a **term of a polynomial**. If a term does not contain a variable, it is called a *constant*.

The term with the highest degree is called the **leading term** because it is usually written first. The coefficient of the leading term is called the **leading coefficient**. When a polynomial is written so that the powers are descending, we say that it is in standard form.



Polynomials
A **polynomial** is an expression that can be written in the form
a n x n +…+ a 2 x 2 + a 1 x+ a 0
Each real number *ai* is called a **coefficient**. The number a 0 that is not multiplied by a variable is

called a *constant*. Each product $a_i x^i$ is a **term of a polynomial**. The highest power of the variable that occurs in the polynomial is called the **degree** of a polynomial. The **leading term** is the term with the highest power, and its coefficient is called the **leading coefficient**.

Given a polynomial expression, identify the degree and leading coefficient.

1. Find the highest power of *x* to determine the degree.
2. Identify the term containing the highest power of *x* to find the leading term.
3. Identify the coefficient of the leading term.

Identifying the Degree and Leading Coefficient of a Polynomial

For the following polynomials, identify the degree, the leading term, and the leading coefficient.

1. $3+2x^2-4x^3$
2. $5t^5-2t^3+7t$
3. $6p-p^3-2$

1. The highest power of x is 3, so the degree is 3. The leading term is the term containing that degree, $-4x^3$. The leading coefficient is the coefficient of that term, -4.
2. The highest power of t is 5, so the degree is 5. The leading term is the term containing that degree, $5t^5$. The leading coefficient is the coefficient of that term, 5.
3. The highest power of p is 3, so the degree is 3. The leading term is the term containing that degree, $-p^3$, The leading coefficient is the coefficient of that term, -1.

Adding and Subtracting Polynomials

We can add and subtract polynomials by combining like terms, which are terms that contain the same variables raised to the same exponents. For example, $5x^2$ and $-2x^2$ are like terms, and can be added to get $3x^2$, but $3x$ and $3x^2$ are not like terms, and therefore cannot be added.

Add or subtract: ⓐ $25y2+15y2$ ⓑ $16pq3-(-7pq3)$.

ⓐ
$25y2+15y2$ Combine like terms.$40y2$

ⓑ
$16pq3-(-7pq3)$ Combine like terms.$23pq3$

Given multiple polynomials, add or subtract them to simplify the expressions. We can think of adding and subtracting polynomials as just adding and subtracting a series of monomials. Look for the like terms—those with the same variables and the same exponent. The Commutative Property allows us to rearrange the terms to put like terms together.

1. Combine like terms.
2. Simplify and write in standard form.

Find the sum:$(7y2-2y+9)+(4y2-8y-7)$.

Identify like terms. $(7y^2$ ______ $-2y$ __ $+9)+(4y^2$ ______ $-8y$ __ $-7)$ Rewrite without the parentheses, rearranging to get the like terms together. $7y^2+4y^2$ ______________ $-2y-8y$ _____ $+9-7$ Combine like terms. $11y^2-10y+2$

Adding Polynomials

Find the sum.

$(12x^2+9x-21)+(4x^3+8x^2-5x+20)$

$4x^3+(12x^2+8x^2)+(9x-5x)+(-21+20)$ Combine like terms. $4x^3+20x^2+4x-1$ Simplify.

Analysis

We can check our answers to these types of problems using a graphing calculator. To check, graph the problem as given along with the simplified answer. The two graphs should be equivalent. Be sure to use the same window to compare the graphs. Using different windows can make the expressions seem equivalent when they are not.

Be careful with the signs as you distribute while subtracting the polynomials in the next example.

Find the difference: $(9w2-7w+5)-(2w2-4)$.

$(9w2-7w+5)-(2w2-4)$ Distribute and identify like terms.9w2______ −7w__ +5−2w2______+4 Rearrange the terms.9w2−2w2_______________−7w__+5+4 Combine like terms.$7w2-7w+9$

Subtracting Polynomials

Find the difference.

$(7 x 4 - x 2 + 6x + 1) - (5 x 3 - 2 x 2 + 3x + 2)$

$7 x 4 - 5 x 3 + (- x 2 + 2 x 2) + (6x - 3x) + (1 - 2)$ Combine like terms. $7 x 4 - 5 x 3 + x 2 + 3x - 1$ Simplify.

Analysis

Note that finding the difference between two polynomials is the same as adding the opposite of the second polynomial to the first.

Multiplying Polynomials

Since monomials are algebraic expressions, we can use the properties of exponents to multiply monomials. Review Module 2 if you need to review exponent rules.

Multiply: ⓐ $(3x2)(-4x3)$ ⓑ $(56x3y)(12xy2)$.

ⓐ

$(3x2)(-4x3)$ Use the Commutative Property to rearrange the terms.$3\cdot(-4)\cdot x2\cdot x3$ Multiply.
$-12x5$

ⓑ

$(56x3y)(12xy2)$ Use the Commutative Property

to rearrange the terms.56·12·x3·x·y·y2
Multiply.10x4y3

Multiplying a polynomial by a monomial is really just applying the **Distributive Property.**

Notice that before combining like terms, you have four terms. You multiply the two terms of the first binomial by the two terms of the second binomial—four multiplications.

[redacted]

We distributed the p to get:

[redacted]

What if we have $(x + 7)$ instead of p?

[redacted]

Distribute $(x + 7)$.

[redacted]

Distribute again.

[redacted]

Combine like terms.

[redacted]

Multiply: ⓐ $-2y(4y2+3y-5)$ ⓑ $3x3y(x2-8xy+y2)$.

ⓐ

[redacted]

Distribute.

[redacted]

Multiply.

[redacted]

ⓑ
$3x3y(x2-8xy+y2)$
Distribute.$3x3y\cdot x2+(3x3y)\cdot(-8xy)+(3x3y)\cdot y2$
Multiply.$3x5y-24x4y2+3x3y3$

Multiplying polynomials is a bit more challenging than adding and subtracting polynomials. We must use the distributive property to multiply each term in the first polynomial by each term in the second polynomial. We then combine like terms. We can also use a shortcut called the **FOIL** method when multiplying binomials. Certain special products follow patterns that we can memorize and use instead of multiplying the polynomials by hand each time. We will look at a variety of ways to multiply polynomials.

Multiplying Polynomials Using the Distributive Property

To multiply a number by a polynomial, we use the distributive property. The number must be distributed to each term of the polynomial. We can distribute the 2 in $2(x+7)$ to obtain the equivalent expression $2x+14$. When multiplying polynomials, the distributive property allows us to multiply each term of the first polynomial by each term of the second. We then add the products together and

combine like terms to simplify.

Given the multiplication of two polynomials, use the distributive property to simplify the expression.

1. Multiply each term of the first polynomial by each term of the second.
2. Combine like terms.
3. Simplify.

Multiplying Polynomials Using the Distributive Property

Find the product.

$(2x+1)(3x^2-x+4)$

$2x(3x^2-x+4)+1(3x^2-x+4)$
Use the distributive property. $(6x^3-2x^2+8x)+(3x^2-x+4)$ Multiply. $6x^3+(-2x^2+3x^2)+(8x-x)+4$
Combine like terms. $6x^3+x^2+7x+4$
Simplify.

Analysis

We can use a table to keep track of our work, as shown in [link]. Write one polynomial across the top and the other down the side. For each box in the table, multiply the term for that row by the term for that column. Then add all of the terms together, combine like terms, and simplify.

	$3x^2$	$-x$	$+4$
$2x$	$6x^3$	$-2x^2$	$8x$
$+1$	$3x^2$	$-x$	4

Multiply $(b+3)(2b^2-5b+8)$ using ⓐ the Distributive Property and ⓑ the Vertical Method.

ⓐ

[redacted]

Distribute.

[redacted]

Multiply.

[redacted]

Combine like terms.

[redacted]

ⓑ It is easier to put the polynomial with fewer terms on the bottom because we get fewer partial products this way.

Multiply $(2b2 - 5b + 8)$ by 3.

Mu [redacted] 8) by [redacted]

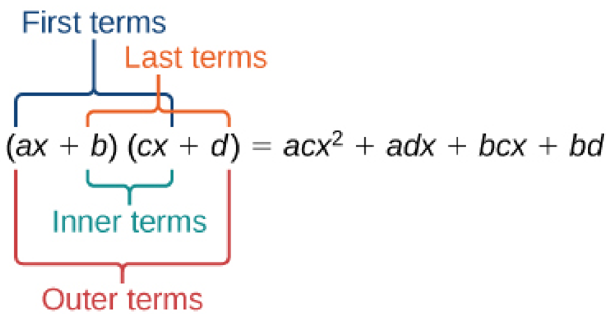
Add like terms.

[redacted]

$2b^3 + b^2 - 7b + 24$

Using FOIL to Multiply Binomials

A shortcut called FOIL is sometimes used to find the product of **two binomials**. It is called FOIL because we multiply the **f**irst terms, the **o**uter terms, the **i**nner terms, and then the **l**ast terms of each binomial.



The FOIL method arises out of the distributive property. We are simply multiplying each term of the first binomial by each term of the second binomial, and then combining like terms.

Given two binomials, use FOIL to simplify the expression.

Step 1. Multiply the *First* terms.

Step 2. Multiply the *Outer* terms.

Step 3. Multiply the *Inner* terms.

Step 4. Multiply the *Last* terms.

Step 5. Combine like terms, when possible.

$$\overset{\text{first}}{(a} + \overset{\text{last}}{b)}\overset{\text{first}}{(c} + \overset{\text{last}}{d)}$$

inner

outer

Say it as you multiply!
FOIL
First
Outer
Inner
Last

Using FOIL to Multiply Binomials

Use FOIL to find the product.

(2x-18)(3x+3)

Find the product of the first terms.

$2x - 18 \quad 3x + 3 \qquad 2x \cdot 3x = 6x^2$

Find the product of the outer terms.

$2x - 18 \quad 3x + 3 \qquad 2x \cdot 3 = 6x$

Find the product of the inner terms.

$2x - 18 \quad 3x + 3 \qquad -18 \cdot 3x = -54x$

Find the product of the last terms.

$2x - 18 \quad 3x + 3 \qquad -18 \cdot 3 = -54$

$6x^2 + 6x - 54x - 54$ Add the products. $6x^2 + (6x - 54x) - 54$ Combine like terms. $6x^2 - 48x - 54$ Simplify.

The FOIL method is usually the quickest method for multiplying two binomials, but it *only* works for binomials. You can use the Distributive Property to find the product of any two polynomials. Another method that works for all polynomials is the Vertical Method. It is very much like the method you use to multiply whole numbers. Look carefully at this example of multiplying two-digit numbers.

23	
×46	
138 partial product	Start by multiplying 23 by 6 to get 138.
92 partial product	Next, multiply 23 by 4, lining up the partial product in the correct columns.
1058 product	Last you add the partial products.

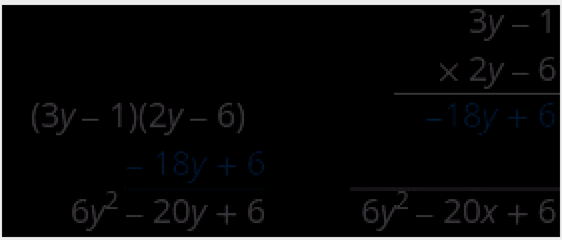
Multiply using the Vertical Method: $(3y - 1)(2y - 6)$.

Solution

It does not matter which binomial goes on the top.

Multiply $3y-1$ by -6. Multiply $3y-1$ by $2y$. Add like terms. $3y-1 \times 2y-6$______ $-18y$ $+6$ $6y2-2y$ ________$6y2-20y+6$ partial productpartial productproduct

Notice the partial products are the same as the terms in the FOIL method.



Sum or Difference (aka Difference of Squares or Conjugates)

Another special product is called the **difference of squares**, or Sum and Difference of Two Terms (as its called in the Algebra book. A pair of binomials that each have the same first term and the same last term, but one is a sum and one is a difference is called a **conjugate pair** and is of the form $(a-b),(a+b)$.

Notice how each pair has one sum and one difference.

$$\underbrace{(x-9)}_{\text{Difference}}\underbrace{(x+9)}_{\text{Sum}} \qquad \underbrace{(y-8)}_{\text{Difference}}\underbrace{(y+8)}_{\text{Sum}} \qquad \underbrace{(2x-5)}_{\text{Difference}}\underbrace{(2x+5)}_{\text{Sum}}$$

$$\begin{gathered}(x+9)(x-9)\\ x^2-9x+9x-81\\ x^2-81\end{gathered} \qquad \begin{gathered}(2x-5)(2x+5)\\ 4x^2+10x-10x-25\\ 4x^2-25\end{gathered}$$

Because the sign changes in the second binomial, the outer and inner terms cancel each other out, and we are left only with the square of the first term minus the square of the last term.

Difference of Squares or Sum and Difference

When a binomial is multiplied by a binomial with the same terms separated by the opposite sign, the result is the square of the first term minus the square of the last term.

$$(a-b)(a+b)=a^2-b^2$$

$$\underbrace{(a-b)(a+b)}_{\text{conjugates}} = \underbrace{a^2 \overset{\text{difference}}{-} b^2}_{\text{squares}}$$

$$(a+b)(a-b)=a^2-b^2$$

Multiply: $(x-8)(x+8)$.

Solution

First, recognize this as a product of conjugates. The binomials have the same first terms, and the same last terms, and one binomial is a sum and the other is a difference.

It fits the pattern.

[redacted]

Square the first term, x.

[redacted]

Square the last term, 8.

[redacted]

The product is a difference of squares.

[redacted]

Multiply $(2x+7)(2x-7)$.
$4x^2-49$

Binomial (Perfect) Square Trinomials

Certain binomial products have special forms. When a binomial is squared, the result is called a **perfect square trinomial**. We can find the square by multiplying the binomial by itself. However, there is a special form that each of these perfect square trinomials takes, and memorizing the form makes squaring binomials much easier and faster. Let's look at a few perfect square trinomials to familiarize ourselves with the form.

$$(x+5)^2 = x^2+10x+25 \quad (x-3)^2 = x^2-6x+9 \quad (4x-1)^2 = 16x^2-8x+1$$

Notice that the first term of each trinomial is the square of the first term of the binomial and,

similarly, the last term of each trinomial is the square of the last term of the binomial. The middle term is double the product of the two terms. Lastly, we see that the first sign of the trinomial is the same as the sign of the binomial.

Perfect Square Trinomials

When a binomial is squared, the result is the first term squared added to double the product of both terms and the last term squared.

$(x+a)^2 = (x+a)(x+a) = x^2 + 2ax + a^2$

Or

$(a+b)^2 = a^2 + 2ab + b^2$

$$\underbrace{(a+b)^2}_{(\text{binomial})^2} = \underbrace{a^2}_{(\text{first term})^2} + \underbrace{2ab}_{2(\text{product of terms})} + \underbrace{b^2}_{(\text{last term})^2}$$

$(a-b)^2 = a^2 - 2ab + b^2$

$$\underbrace{(a-b)^2}_{(\text{binomial})^2} = \underbrace{a^2}_{(\text{first term})^2} - \underbrace{2ab}_{2(\text{product of terms})} + \underbrace{b^2}_{(\text{last term})^2}$$

Given a binomial, square it using the formula for perfect square trinomials.

1. Square the first term of the binomial.
2. Square the last term of the binomial.
3. For the middle term of the trinomial, double the product of the two terms.
4. Add and simplify.

Multiply: ⓐ (x+5)2 ⓑ (2x−3y)2.

ⓐ

[redacted]

Square the first term.

[redacted]

Square the last term.

[redacted]

Double their product.

[redacted]

Simplify.

[redacted]

ⓑ

[redacted]

Use the pattern.

[redacted]

Simplify.

[redacted]

Expand $(4x-1)^2$.

$16x^2-8x+1$

Performing Operations with Polynomials of Several Variables

We have looked at polynomials containing only one variable. However, a polynomial can contain several

variables. All of the same rules apply when working with polynomials containing several variables. Consider an example:

$(a+2b)(4a-b-c)$ $a(4a-b-c)+2b(4a-b-c)$ Use the distributive property. $4a^2-ab-ac+8ab-2b^2-2bc$ Multiply. $4a^2+(-ab+8ab)-ac-2b^2-2bc$ Combine like terms. $4a^2+7ab-ac-2bc-2b^2$ Simplify.

Find the sum: $(u^2-6uv+5v^2)+(3u^2+2uv)$.

$(u^2-6uv+5v^2)+(3u^2+2uv)$
Distribute. $u^2-6uv+5v^2+3u^2+2uv$
Rearrange the terms to put like terms together. $u^2+3u^2-6uv+2uv+5v^2$ Combine like terms. $4u^2-4uv+5v^2$

Multiplying Polynomials Containing Several Variables

Multiply $(x+4)(3x-2y+5)$.

Follow the same steps that we used to multiply

polynomials containing only one variable.

$x(3x-2y+5)+4(3x-2y+5)$

Use the distributive property. $3x^2-2xy+5x+12x-8y+20$ Multiply. $3x^2-2xy+(5x+12x)-8y+20$ Combine like terms. $3x^2-2xy+17x-8y+20$ Simplify.

Multiply $(3x-1)(2x+7y-9)$.

$6x^2+21xy-29x-7y+9$

Access these online resources for additional instruction and practice with polynomials.

- Adding and Subtracting Polynomials
- Multiplying Polynomials
- Special Products of Polynomials

Key Concepts

- A polynomial is a sum of terms each consisting of a variable raised to a non-negative integer power. The degree is the highest power of the variable that occurs in the polynomial. The leading term is the term containing the highest degree, and the leading coefficient is the coefficient of that term. See [link].
- We can add and subtract polynomials by combining like terms. See [link] and [link].
- To multiply polynomials, use the distributive property to multiply each term in the first polynomial by each term in the second. Then add the products. See [link].
- FOIL (First, Outer, Inner, Last) is a shortcut that can be used to multiply binomials. See [link].
- Perfect square trinomials and difference of squares are special products. See [link] and [link].
- Follow the same rules to work with polynomials containing several variables. See [link].

Binomial Squares	Sum or Difference (Product of Conjugates)
$(a+b)^2 = a^2 + 2ab + b^2$	$(a-b)(a+b) = a^2 - b^2$
$(a-b)^2 = a^2 - 2ab + b^2$	

• Squaring a binomial	• Multiplying conjugates
• Product is a **trinomial**	• Product is a **binomial.**
• Inner and outer terms with FOIL are **the same.**	• Inner and outer terms with FOIL are **opposites.**
• Middle term is **double the product** of the terms	• There is **no** middle term.

Section Exercises

Algebraic

For the following exercises, identify the degree of the polynomial.

$7x-2x^2+13$

2

$-625a^8+16b^4$

8

x^2+4x+4

2

For the following exercises, find the sum or difference.

$(12x^2+3x)-(8x^2-19)$

$4x^2+3x+19$

$(6w^2+24w+24)-(3w^2-6w+3)$

$3w^2+30w+21$

$(11b^4-6b^3+18b^2-4b+8)-(3b^3+6b^2+3b)$

$11b^4-9b^3+12b^2-7b+8$

For the following exercises, find the product.

$(4x+2)(6x-4)$

$24x^2-4x-8$

$(6b^2-6)(4b^2-4)$

$24b^4-48b^2+24$

$(9v-11)(11v-9)$

$99v^2-202v+99$

$(8n-4)(n^2+9)$

$8n^3-4n^2+72n-36$

For the following exercises, expand the binomial.

$(3y-7)^2$

$9y^2-42y+49$

$(4p+9)^2$

$16p^2+72p+81$

$(3y-6)^2$

$9y^2-36y+36$

For the following exercises, multiply the binomials.

$(4c+1)(4c-1)$

$16c^2-1$

$(4+4m)(4-4m)$

$-16m^2+16$

$(11q-10)(11q+10)$

$121q^2-100$

For the following exercises, multiply the polynomials.

$(4t^2 + t - 7)(4t^2 - 1)$

$16t^4 + 4t^3 - 32t^2 - t + 7$

$(y - 2)(y^2 - 4y - 9)$

$y^3 - 6y^2 - y + 18$

$(4t - 5u)^2$

$16t^2 - 40tu + 25u^2$

$(4r - d)(6r + 7d)$

$24r^2 + 22rd - 7d^2$

Real-World Applications

A developer wants to purchase a plot of land to build a house. The area of the plot can be described by the following expression: $(4x + 1)(8x - 3)$ where x is measured in meters. Multiply the binomials to find the area of the

plot in standard form.

$32x^2 - 4x - 3\ \text{m}^2$

A prospective buyer wants to know how much grain a specific silo can hold. The area of the floor of the silo is $(2x+9)^2$. The height of the silo is $10x+10$, where *x* is measured in feet. Expand the square and multiply by the height to find the expression that shows how much grain the silo can hold.

Glossary

binomial
: a polynomial containing two terms

coefficient
: any real number a_i in a polynomial in the form $a_n x^n + \ldots + a_2 x^2 + a_1 x + a_0$

degree
: the highest power of the variable that occurs in a polynomial

difference of squares
: the binomial that results when a binomial is multiplied by a binomial with the same terms, but the opposite sign

leading coefficient
: the coefficient of the leading term

leading term
: the term containing the highest degree

monomial
: a polynomial containing one term

perfect square trinomial
: the trinomial that results when a binomial is squared

polynomial
: a sum of terms each consisting of a variable raised to a nonnegative integer power

term of a polynomial
: any $a_i x^i$ of a polynomial in the form $a_n x^n + \dots + a_2 x^2 + a_1 x + a_0$

trinomial
: a polynomial containing three terms

Factoring Polynomials (P5)
In this section students will:

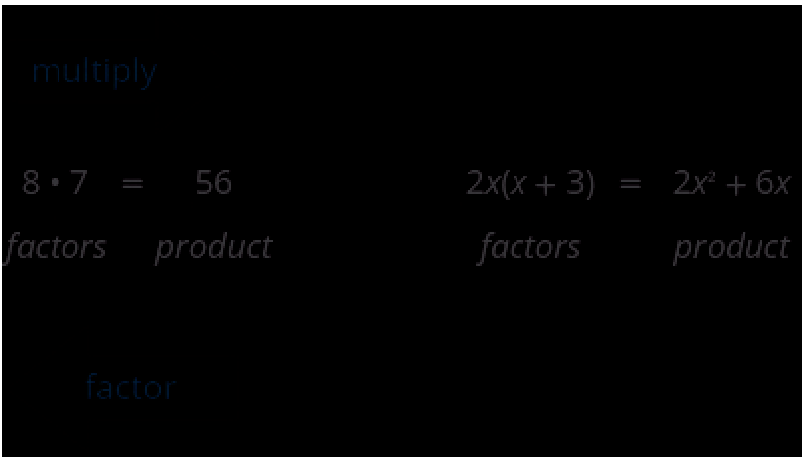
- Factor the greatest common factor of a polynomial.
- Factor a trinomial.
- Factor by grouping.
- Factor a perfect square trinomial.
- Factor a difference of squares.
- Factor the sum and difference of cubes.
- Factor expressions using fractional or negative exponents.

This Module supports section **P5 of Mat 1023**.
In case you missed something in class, or just want to review a specific topic covered in this Module, here is a list of topics covered:

1. Factoring the Greatest Common Factor of a Polynomial [link]
2. Factor by Grouping [link]
3. Factor Difference of Squares [link]
4. Factor Perfect Square Trinomial [link]
5. Factor Sum and Difference of Cubes [link]
6. General Strategy for Factor [link]
7. Factor Expressions with Negative or Fraction Exponent [link]
8. Key Concepts [link]

Greatest Common Factor of a Polynomial

We have learned how to factor numbers to find the least common multiple (LCM) of two or more numbers. Now we will factor expressions and find the **greatest common factor** of two or more expressions. The method we use is similar to what we used to find the LCM.



Greatest Common Factor
The **greatest common factor** (GCF) of two or more expressions is the largest expression that is a factor of all the expressions.

Find the greatest common factor of 21x3,9x2,15x.

Factor each coefficient into primes and write the [redacted] va [redacted] ex [redacted] form.
Circle the common factors in each column.
Bring down the common factors.
Multiply the factors.

[redacted]

The GCF of 21x3, 9x2 and 15x is 3x.

When we study fractions, we learn that the **greatest common factor** (GCF) of two numbers is the largest number that divides evenly into both numbers. For

instance, 4 is the GCF of 16 and 20 because it is the largest number that divides evenly into both 16 and 20 The GCF of polynomials works the same way: $4x$ is the GCF of $16x$ and $20x^2$ because it is the largest polynomial that divides evenly into both $16x$ and $20x^2$.

When factoring a polynomial expression, our first step should be to check for a GCF. Look for the GCF of the coefficients, and then look for the GCF of the variables.

Greatest Common Factor

The **greatest common factor** (GCF) of polynomials is the largest polynomial that divides evenly into the polynomials.

Given a polynomial expression, factor out the greatest common factor.

1. Identify the GCF of the coefficients.
2. Identify the GCF of the variables.
3. Combine to find the GCF of the expression.
4. Determine what the GCF needs to be multiplied by to obtain each term in the expression.
5. Write the factored expression as the product of

the GCF and the sum of the terms we need to multiply by.

We state the Distributive Property here just as you saw it in earlier chapters and “in reverse.”

Distributive Property

If *a*, *b*, and *c* are real numbers, then

$a(b+c)=ab+ac\text{and}ab+ac=a(b+c)$

The form on the left is used to multiply. The form on the right is used to factor.

So how do you use the Distributive Property to factor a polynomial? You just find the GCF of all the terms and write the polynomial as a product!

How to Use the Distributive Property to factor a polynomial

Factor: $8m3-12m2n+20mn2$.

Step 1. Find the GCF of all the terms of the polynomial.	Find the GCF of $8m^3, 12m^2n, 20mn^2$	$8m^3 = 2 \cdot 2 \cdot 2 \cdot m \cdot m \cdot m$ $12m^2n = 2 \cdot 2 \cdot 3 \cdot m \cdot m \cdot n$ $20mn^2 = 2 \cdot 2 \cdot 5 \cdot m \cdot n \cdot n$ $GCF = 2 \cdot 2 \cdot m$ $GCF = 4m$
Step 2. Rewrite each term as a product using the GCF.	Rewrite $8m^3, 12m^2n, 20mn^2$ as products of their GCF, $4m$. $8m^3 = 4m \cdot 2m^2$ $12m^2n = -4m \cdot 3m\,n$ $20mn^2 = 4m \cdot 5n^2$	$8m^3 - 12m^2n + 20mn^2$ $4m \cdot 2m^2 - 4m \cdot 3m\,n + 4m \cdot 5n^2$
Step 3. Use the "reverse" Distributive Property to factor the expression.		$4m(2m^2 - 3m\,n + 5n^2)$
Step 4. Check by multiplying the factors.		$4m(2m^2 - 3m\,n + 5n^2)$ $4m \cdot 2m^2 - 4m \cdot 3m\,n + 4m \cdot 5n^2$ $8m^3 - 12m^2n + 20mn^2$ ✓

Factoring the Greatest Common Factor

Factor $6x^3y^3 + 45x^2y^2 + 21xy$.

First, find the GCF of the expression. The GCF of 6, 45, and 21 is 3. The GCF of x^3, x^2, and x is x. (Note that the GCF of a set of expressions in the form x^n will always be the exponent of lowest degree.) And the GCF

of y^3, y^2, and y is y. Combine these to find the GCF of the polynomial, $3xy$.

Next, determine what the GCF needs to be multiplied by to obtain each term of the polynomial. We find that $3xy(2x^2y^2)=6x^3y^3$, $3xy(15xy)=45x^2y^2$, and $3xy(7)=21xy$.

Finally, write the factored expression as the product of the GCF and the sum of the terms we needed to multiply by.

$(3xy)(2x^2y^2+15xy+7)$

Analysis

After factoring, we can check our work by multiplying. Use the distributive property to confirm that $(3xy)(2x^2y^2+15xy+7)=6x^3y^3+45x^2y^2+21xy$.

Factor $x(b^2-a)+6(b^2-a)$ by pulling out the GCF.

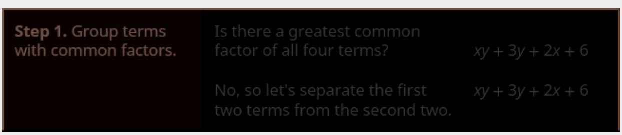
$(b^2-a)(x+6)$

Factor by Grouping

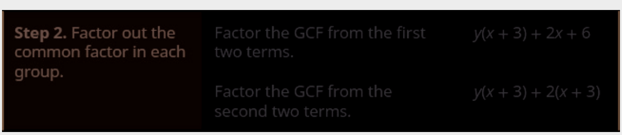
Sometimes there is no common factor of all the terms of a polynomial. When there are four terms we separate the polynomial into two parts with two terms in each part. Then look for the GCF in each part. If the polynomial can be factored, you will find a common factor emerges from both parts. Not all polynomials can be factored. Just like some numbers are prime, some polynomials are prime.

How to Factor a Polynomial by Grouping

Factor by grouping: $xy + 3y + 2x + 6$.



Step 1. Group terms with common factors.	Is there a greatest common factor of all four terms?	$xy + 3y + 2x + 6$
	No, so let's separate the first two terms from the second two.	$xy + 3y + 2x + 6$



Step 2. Factor out the common factor in each group.	Factor the GCF from the first two terms.	$y(x + 3) + 2x + 6$
	Factor the GCF from the second two terms.	$y(x + 3) + 2(x + 3)$

Step 3. Factor the common factor from the expression.	Notice that each term has a common factor of $(x + 3)$.	$y(x+3) + 2(x+3)$
	Factor out the common factor.	$(x+3)(y+2)$

Step 4. Check.	Multiply $(x+3)(y+2)$. Is the product the original expression?	$(x+3)(y+2)$ $xy + 2x + 3y + 6$ $xy + 3y + 2x + 6$ ✓

Factor by grouping.

Group terms with common factors. Factor out the common factor in each group. Factor the common factor from the expression. Check by multiplying the factors.

Factor by grouping: ⓐ $x2 + 3x - 2x - 6$ ⓑ $6x2 - 3x - 4x + 2$.

ⓐ

There is no GCF in all four terms.$x2 + 3x - 2x - 6$Separate into two parts.$x2 + 3x - 2x - 6$Factor the GCF from both parts. Be carefulwith the signs when factoring the GCF

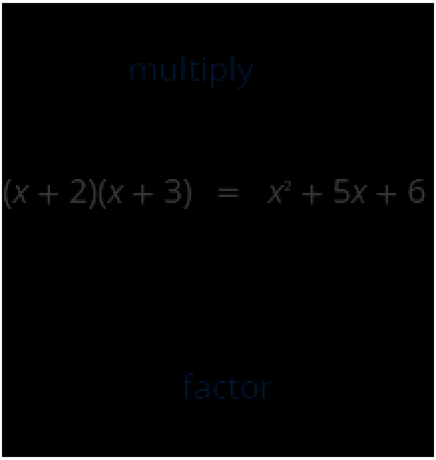
fromthe last two terms.$x(x+3)-2(x+3)$Factor out the common factor.$(x+3)(x-2)$Check on your own by multiplying.

ⓑ
There is no GCF in all four terms.$6x2-3x-4x+2$Separate into two parts.$6x2-3x-4x+2$Factor the GCF from both parts.$3x(2x-1)-2(2x-1)$Factor out the common factor.$(2x-1)(3x-2)$Check on your own by multiplying.

Factor Trinomials

Does the order of the factors matter?
No. Multiplication is commutative, so the order of the factors does not matter.

You have already learned how to multiply binomials using FOIL. Now you'll need to “undo” this multiplication. To factor the trinomial means to start with the product, and end with the factors.



To figure out how we would factor a trinomial of the form x^2+bx+c, such as x^2+5x+6 and factor it to $(x+2)(x+3)$, let’s start with two general binomials of the form $(x+m)$ and $(x+n)$.

	$(x+m)(x+n)$
Foil to find the product.	$x^2+mx+nx+mn$
Factor the GCF from the middle terms.	$x^2+(m+n)x+mn$
Our trinomial is of the form x^2+bx+c.	

$$x^2 + bx + c$$
$$x^2 + (m+n)x + mn$$

This tells us that to factor a trinomial of the form x^2+bx+c, we need two factors $(x+m)$ and $(x+n)$ where the two numbers *m* and *n* multiply to *c* and add to *b*.

How to Factor a Trinomial of the form x^2+bx+c

Factor: $x^2+11x+24$.

Step 1. Write the factors as two binomials with first terms *x*.	Write two sets of parentheses and put *x* as the first term.	$x^2 + 11x + 24$ $(x \quad)(x \quad)$

Step 2. Find two numbers *m* and *n* that multiply to *c*, $m \cdot n = c$ add to *b*, $m + n = b$	Find two numbers that multiply to 24 and add to 11.	

Factors of 24	Sum of factors
1, 24	$1 + 24 = 25$
2, 12	$2 + 12 = 14$
3, 8	$3 + 8 = 11$*
4, 6	$4 + 6 = 10$

Step 3. Use *m* and *n* as the last terms of the factors.	Use 3 and 8 as the last terms of the binomials.	$(x + 3)(x + 8)$

Step 4. Check by multiplying the factors.	$(x+3)(x+8)$ $x^2+8x+3x+24$ $x^2+11x+24$ ✓

Let’s summarize the method we just developed to factor trinomials of the form x2 + bx + c.

Strategy for Factoring Trinomials of the Form x2 + bx + c

When we factor a trinomial, we look at the signs of its terms first to determine the signs of the binomial factors.

x2 + bx + c(x + m)(x + n)Whencis positive,mandnhave the same sign.bpositivebnegativem,npositivem,nnegativex2 + 5x + 6x2 − 6x + 8(x + 2)(x + 3)(x − 4)(x − 2)same signssame signsWhencis negative,mandnhave opposite signs.x2 + x − 12x2 − 2x − 15(x + 4)(x − 3)(x − 5)(x + 3)opposite signsopposite signs

Notice that, in the case when *m* and *n* have opposite signs, the sign of the one with the larger absolute value matches the sign of *b*.

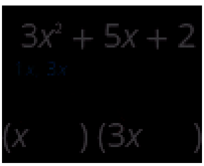
What happens when the leading coefficient is not 1 and there is no GCF? There are several methods that

can be used to factor these trinomials. First we will use the Trial and Error method.

Let’s factor the trinomial $3x^2+5x+2$.

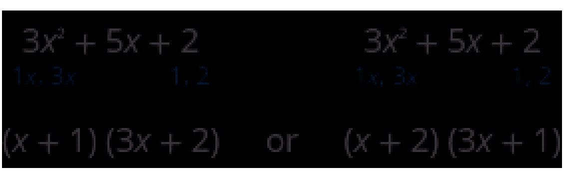
From our earlier work, we expect this will factor into two binomials.
$3x^2+5x+2 \quad (\;)(\;)$

We know the first terms of the binomial factors will multiply to give us $3x^2$. The only factors of $3x^2$ are $1x, 3x$. We can place them in the binomials.



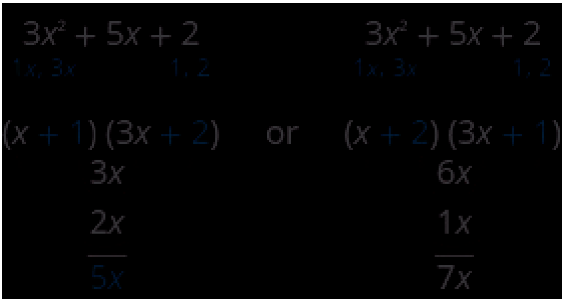
Check: Does $1x \cdot 3x = 3x^2$?

We know the last terms of the binomials will multiply to 2. Since this trinomial has all positive terms, we only need to consider positive factors. The only factors of 2 are 1, 2. But we now have two cases to consider as it will make a difference if we write 1, 2 or 2, 1.



Which factors are correct? To decide that, we

multiply the inner and outer terms.



Since the middle term of the trinomial is 5x, the factors in the first case will work. Let’s use FOIL to check.

$(x+1)(3x+2)$ $3x^2+2x+3x+2$ $3x^2+5x+2$ ✓

Our result of the factoring is:

$3x^2+5x+2$ $(x+1)(3x+2)$

How to Factor a Trinomial Using Trial and Error

Factor completely using trial and error: $3y^2+22y+7$.

Step 1. Write the trinomial in descending order.	The trinomial is already in descending order.	$3y^2 + 22y + 7$

Step 2. Factor any GCF.	There is no GCF.	
Step 3. Find all the factor pairs of the first term.	The only of $3y^2$ are $1y, 3y$. Since there is only one pair, we can put them in the parentheses.	$3y^2 + 22y + 7$ $1y, 3y$ $3y^2 + 22y + 7$ $1y, 3y$ $(y \quad)(3y \quad)$
Step 4. Find all the factor pairs of the third term.	The only factors of 7 are 1, 7.	$3y^2 + 22y + 7$ $1y, 3y \qquad 1, 7$ $(y \quad)(3y \quad)$
Step 5. Test all the possible combinations of the factors until the correct product is found.	$3y^2 + 22y + 7$ $1y, 3y \qquad 1, 7$ $(y + 1)(3y + 7)$ $3y$ $7y$ $10y$ No! We need $22y$ $3y^2 + 22y + 7$ $1y, 3y \qquad 1, 7$ $(y + 7)(3y + 1)$ $21y$ $+y$ $22y$	see table below
Step 6. Check by multiplying.		$(y + 7)(3y + 1)$ $3y^2 + 22y + 7$ ✓

$3y^2 + 22y + 7$	
Possible factors	**Product**
$(y + 1)(3y + 7)$	$3y^2 + 10y + 7$
$(y + 7)(3y + 1)$	$3y^2 + 22y + 7$

Factor trinomials of the form ax2 + bx + c using trial and error.

Write the trinomial in descending order of degrees as needed. Factor any GCF. Find all the factor pairs of the first term. Find all the factor pairs of the third term. Test all the possible combinations of the factors until the correct product is found. Check by multiplying.

Remember, when the middle term is negative and the last term is positive, the signs in the binomials must both be negative.

Factor completely using trial and error: $6b2-13b+5$.

The trinomial is already in descending order: [redacted]

Find the factors of the first term.

Find the factors of the last term. Consider the sig

Si m, 5, is positive its factors must both be positive or both be negative. The coefficient of the middle term is negative, so we use the negative factors.

Consider all the combinations of factors.

$6b^2 - 13b + 5$

Possible factors	Product
$(b-1)(6b-5)$	$6b^2 - 11b + 5$
$(b-5)(6b-1)$	$6b^2 - 31b + 5$
$(2b-1)(3b-5)$	$6b^2 - 13b + 5$*
$(2b-5)(3b-1)$	$6b^2 - 17b + 5$

The correct factors are those whose productis the original trinomial.$(2b-1)(3b-5)$Check by

multiplying:(2b − 1)(3b − 5)6b2 − 10b − 3b + 56b2 − 13b + 5✓

When we factor an expression, we always look for a greatest common factor first. If the expression does not have a greatest common factor, there cannot be one in its factors either. This may help us eliminate some of the possible factor combinations.

Factor completely using trial and error: 18x2 − 37xy + 15y2.

The trinomial is already in descending order.	$18x^2 - 37xy + 15y^2$
Find the factors of the first term.	$18x^2 - 37xy + 15y^2$ $1x \cdot 18x$ $2x \cdot 9x$ $3x \cdot 6x$
Find the factors of the	

las $18x^2 - 37xy + 15y^2$ the
sig

$1x \cdot 18x$	$-1, -15$
$2x \cdot 9x$	$-3, -5$
$3x \cdot 6x$	

Si... and
the coefficient of the
middle
term is negative, we
use the negative
factors.

Consider all the combinations of factors.

$18x^2 - 37xy + 15y^2$	
Possible factors	Product
$(x - 1y)(18x - 15y)$	Not an option
$(x - 15y)(18x - 1y)$	$18x^2 - 271xy + 15y^2$
$(x - 3y)(18x - 5y)$	$18x^2 - 59xy + 15y^2$
$(x - 5y)(18x - 3y)$	Not an option
$(2x - 1y)(9x - 15y)$	Not an option
$(2x - 15y)(9x - 1y)$	$18x^2 - 137xy + 15y^2$
$(2x - 3y)(9x - 5y)$	$18x^2 - 37xy + 15y^2$*
$(2x - 5y)(9x - 3y)$	Not an option
$(3x - 1y)(6x - 15y)$	Not an option
$(3x - 15y)(6x - 1y)$	Not an option
$(3x - 3y)(6x - 5y)$	Not an option

If the trinomial has no common factors, then neither factor can contain a common factor. That means this combination is not an option.

The correct factors are those whose product isthe original trinomial.$(2x - 3y)(9x - 5y)$Check by multiplying:$(2x - 3y)(9x - 5y)$ $18x2 - 10xy - 27xy + 15y2$ $18x2 - 37xy + 15y2$✓

Don’t forget to look for a GCF first and remember if

the leading coefficient is negative, so is the GCF.

Factor completely using trial and error: $-10y^4-55y^3-60y^2$.

$-10y^4 - 55y^3 - 60y^2$

Notice the greatest common factor, so factor it first.

$-5y^2(2y^2 + 11y + 12)$

Factor the trinomial.

$$-5y^2\left(\begin{array}{ccc} 2y^2 + 11y + 12 \\ y \cdot 2y \qquad\qquad 1 \cdot 12 \\ \qquad\qquad\quad 2 \cdot 6 \\ \qquad\qquad\quad 3 \cdot 4 \end{array}\right)$$

Consider all the combinations.

$2y^2 + 11y + 12$	
Possible factors	**Product**
$(y + 1)(2y + 12)$	Not an option
$(y + 12)(2y + 1)$	$2y^2 + 25y + 12$
$(y + 2)(2y + 6)$	Not an option
$(y + 6)(2y + 2)$	Not an option
$(y + 3)(2y + 4)$	Not an option
$(y + 4)(2y + 3)$	$2y^2 + 11y + 12$*

If the trinomial has no common factors, then neither factor can contain a common factor. That means this combination is not an option.

The correct factors are those whose productis the original trinomial. Remember to includethe factor −5y2. −5y2(y+4)(2y +3)Check by multiplying: −5y2(y+4)(2y +3) −5y2(2y2+8y+3y +12) −10y4−55y3−60y2✓

Factor Difference of Squares

Do you remember the Difference of Squares from the last module? A difference of squares is a perfect square subtracted from a perfect square. Recall that a difference of squares can be rewritten as factors containing the same terms but opposite signs because the middle terms cancel each other out when the two factors are multiplied.

Differences of Squares

A difference of squares can be rewritten as two factors containing the same terms but opposite signs.

$$a^2 - b^2 = (a+b)(a-b)$$

Step 1. Does the binomial fit the pattern? $a^2 - b^2$ Is this a difference? ___ $-$ ___ Are the first and last terms perfect squares? Step 2. Write them as squares. $(a)^2 - (b)^2$ Step 3. Write the product of conjugates. $(a-b)(a+b)$ Step 4. Check by multiplying.

Factor: $144x^2 - 49y^2$.
$144x^2 - 49y^2$ Is this a difference of squares? Yes. $(12x)^2 - (7y)^2$ Factor as the product of conjugates. $(12x - 7y)(12x + 7y)$ Check by multiplying. $(12x - 7y)(12x + 7y)$ $144x^2 - 49y^2$ ✓

Factoring a Difference of Squares

Factor $9x^2-25$.
Notice that $9x^2$ and 25 are perfect squares because $9x^2=(3x)^2$ and $25=5^2$. The polynomial represents a difference of squares and can be rewritten as $(3x+5)(3x-5)$.

As always, you should look for a common factor first whenever you have an expression to factor. Sometimes a common factor may “disguise” the difference of squares and you won’t recognize the perfect squares until you factor the GCF.

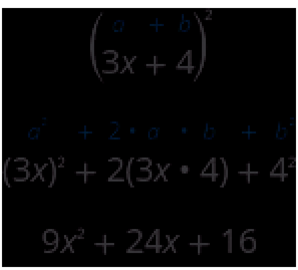
Factor: $48x^4y^2-243y^2$.
$48x^4y^2-243y^2$ Is there a GCF? Yes, $3y^2$—factor it out! $3y^2(16x^4-81)$ Is the binomial a difference of squares? Yes. $3y^2((4x^2)^2-(9)^2)$ Factor as a product of conjugates. $3y^2(4x^2-9)(4x^2+9)$ Notice the first binomial is also a difference of squares! $3y^2((2x)^2-(3)^2)(4x^2+9)$ Factor it as the product of conjugates. $3y^2(2x-3)(2x+3)(4x^2+9)$

The last factor, the sum of squares, cannot be factored.

Check by multiplying: $3y^2(2x-3)(2x+3)$
$(4x^2+9)3y^2(4x^2-9)$
$(4x^2+9)3y^2(16x^4-81)48x^4y^2-243y^2$✓

Factoring a Perfect Square Trinomial

Some trinomials are perfect squares. They result from multiplying a binomial times itself. We squared a binomial using the Binomial Squares pattern in a previous chapter.



Perfect Square Trinomials

$a^2 + 2ab + b^2 = (a+b)^2$ and $a^2 - 2ab + b^2 = (a-b)^2$

We can use this equation to factor any perfect square trinomial.

Step 1.Does the trinomial fit the pattern?a2+2ab +b2a2−2ab+b2 Is the first term a perfect square?(a)2(a)2 Write it as a square. Is the last term a perfect square?(a)2(b)2(a)2(b)2 Write it as a square. Check the middle term. Is it2ab?(a)2↘2·a·b↙(b)2(a)2↘2·a·b↙(b)2 Step 2.Write the square of the binomial.(a+b)2(a−b)2 Step 3.Check by multiplying.

Factor: $9x^2+12x+4$.

Step 1. Does the trinomial fit the perfect square trinomials pattern, $a^2+2ab+b^2$?		
• Is the first term a perfect square? Write it as a square, a^2.	Is $9x^2$ a perfect square? Yes—write it as $(3x)^2$.	$9x^2 + 12x + 4$ $(3x)^2$
• Is the last term a perfect square? Write it as a square, b^2.	Is 4 a perfect square? Yes—write it as $(2)^2$.	$(3x)^2 \qquad (2)^2$
• Check the middle term. Is it $2ab$?	Is $12x$ twice the product of $3x$ and 2? Does it match? Yes, so we have a perfect square trinomial!	$(3x)^2 \qquad (2)^2$ $2(3x)(2)$ $12x$

Step 2. Write the square of the binomial.		$9x^2+12x+4$ $a^2 + 2 \cdot a \cdot b + b^2$ $(3x)^2 + 2 \cdot 3x \cdot \quad + \quad ^2$ $(a+b)^2$ $(3x+\quad)^2$

Step 3. Check.	$(3x+2)^2$ $(3x)^2+2\cdot 3x\cdot 2+2^2$ $9x^2+12x+4$ ✓

Factoring a Perfect Square Trinomial

Factor $25x^2+20x+4$.

Notice that $25x^2$ and 4 are perfect squares because $25x^2=(5x)^2$ and $4=2^2$. Then check to see if the middle term is twice the product of $5x$ and 2. The middle term is, indeed, twice the product: $2(5x)(2)=20x$. Therefore, the trinomial is a perfect square trinomial and can be written as $(5x+2)^2$.

Factor: $81y^2-72y+16$.

The first and last terms are squares. See if the middle term fits the pattern of a perfect square trinomial. The middle term is negative, so the binomial square would be $(a-b)^2$.

[redacted]

Are the first and last terms perfect squares?

[redacted]

Check the middle term.

[redacted]

Does it match (a − b)2?
Yes.

[redacted]

Write as the square of a binomial.

[redacted]

Check by multiplying:

(9y − 4)2(9y)2 − 2·9y·4 + 4281y2 − 72y + 16✓

Factoring the Sum and Difference of Cubes

Now, we will look at two new special products: the sum and difference of cubes. Although the sum of squares cannot be factored, the sum of cubes can be factored into a binomial and a trinomial.

We can use the acronym SOAP to remember the signs when factoring the sum or difference of cubes. The first letter of each word relates to the signs: **S**ame **O**pposite **A**lways **P**ositive. For example, consider the following example.

$x^3 - 2^3 = (x-2)(x^2 + 2x + 4)$

The sign of the first 2 is the *same* as the sign between $x^3 - 2^3$. The sign of the $2x$ term is *opposite* the sign between $x^3 - 2^3$. And the sign of the last term, 4, is *always positive*.

Sum and Difference of Cubes
We can factor the sum of two cubes as
$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
We can factor the difference of two cubes as
$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

We'll check the first pattern and leave the second to you.

[redacted]

Distribute.

[redacted]

Multiply.

[redacted]

Combine like terms.

[redacted]

Given a sum of cubes or difference of cubes, factor it.

1. Confirm that the first and last term are cubes, $a^3 + b^3$ or $a^3 - b^3$.
2. For a sum of cubes, write the factored form as $(a+b)(a^2 - ab + b^2)$. For a difference of cubes, write the factored form as $(a-b)(a^2 + ab + b^2)$.

Factor: $x^3 + 64$.

Step 1. Does the binomial fit the sum or difference of cubes pattern?		$x^3 + 64$
• Is it a sum or difference? • Are the first and last terms perfect cubes?	This is a sum. Yes.	$x^3 + 64$
Step 2. Write the terms as cubes.	Write them as x^3 and 4^3.	$\underset{x^3 + 4^3}{a^3 + b^3}$
Step 3. Use either the sum or difference of cubes pattern.	This is a sum of cubes.	$\left(\underset{x+4}{a+b}\right)\left(\underset{x^2 - 4x + 4^2}{a^2 - ab + b^2}\right)$
Step 4. Simplify inside the parentheses.	It is already simplified.	$(x+4)(x^2 - 4x + 16)$
Step 5. Check by multiplying the factors.		$\begin{array}{r} x^2 - 4x + 16 \\ x + 4 \\ \hline 4x^2 - 16x + 64 \\ x^3 - 4x^2 + 16x \quad\quad \\ \hline x^3 \quad\quad\quad\quad + 64 \end{array}$ ✓

Factoring a Difference of Cubes

Factor $8x^3 - 125$.

Notice that $8x^3$ and 125 are cubes because

$8x^3 = (2x)^3$ and $125 = 5^3$. Write the difference of cubes as $(2x-5)(4x^2+10x+25)$.

Analysis

Just as with the sum of cubes, we will not be able to further factor the trinomial portion.

General Strategy to Factor Polynomials

You have now become acquainted with all the methods of factoring that you will need in this course. The following chart summarizes all the factoring methods we have covered, and outlines a strategy you should use when factoring polynomials.

General Strategy for Factoring Polynomials

GCF

Binomial

- **Difference of Squares**
  $a^2 - b^2 = (a - b)(a + b)$
- **Sum of Squares**
  Sums of squares do not factor.
- **Sum of Cubes**
  $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
- **Difference of Cubes**
  $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

Trinomial

- $x^2 + bx + c$
  $(x \quad)(x \quad)$
- $ax^2 + bx + c$
  - **'a' and 'c' squares**
    $(a + b)^2 = a^2 + 2ab + b^2$
    $(a - b)^2 = a^2 - 2ab + b^2$
  - **'ac' method**

More than 3 terms

- grouping

Use a general strategy for factoring polynomials.

Is there a greatest common factor?
Factor it out. Is the polynomial a binomial, trinomial, or are there more than three terms?
If it is a binomial: If it has more than three terms:

- Is it a sum?
  Of squares? Sums of squares do not factor.
  Of cubes? Use the sum of cubes pattern.
- Is it a difference?
  Of squares? Factor as the Difference of Squares.
  Of cubes? Use the difference of cubes pattern.

- Is it of the form x2 + bx + c? Undo FOIL.
- Is it of the form ax2 + bx + c?
  If *a* and *c* are squares, check if it fits the trinomial square pattern.
  Use the trial and error or "ac" method.

- Use the grouping method.

Check.
Is it factored completely?
Do the factors multiply back to the original polynomial?

Remember, a polynomial is completely factored if, other than monomials, its factors are prime!

Factor completely: $7x3-21x2-70x$.
$7x3-21x2-70x$Is there a GCF? Yes,$7x$.Factor out the GCF.$7x(x2-3x-10)$In the parentheses, is it a binomial, trinomial,or are there more terms?Trinomial with leading coefficient 1.“Undo” FOIL.$7x(x)(x)7x(x+2)(x-5)$Is the expression factored completely? Yes.Neither binomial can be factored.Check your answer.Multiply.$7x(x+2)(x-5)7x(x2-5x+2x-10)7x(x2-3x-10)7x3-21x2-70x$✓

Factor completely: $24y^2 - 150$.

$24y^2 - 150$ Is there a GCF? Yes, 6. Factor out the GCF. $6(4y^2 - 25)$ In the parentheses, is it a binomial, trinomial or are there more than three terms? Binomial. Is it a sum? No. Is it a difference? Of squares or cubes? Yes, squares. $6((2y)^2 - (5)^2)$ Write as a product of conjugates. $6(2y - 5)(2y + 5)$ Is the expression factored completely? Neither binomial can be factored. Check: Multiply. $6(2y - 5)(2y + 5)$ $6(4y^2 - 25)$ $24y^2 - 150$ ✓

The next example can be factored using several methods. Recognizing the trinomial squares pattern will make your work easier.

Factor completely: $4a^2 - 12ab + 9b^2$.

$4a^2 - 12ab + 9b^2$ Is there a GCF? No. Is it a binomial, trinomial, or are there more terms? Trinomial with $a \neq 1$. But the first term is a

perfect square.Is the last term a perfect square? Yes.$(2a)^2-12ab+(3b)^2$Does it fit the pattern,$a^2-2ab+b^2$?Yes.$(2a)^2 \searrow -12ab + -2(2a)(3b) \swarrow (3b)^2$Write it as a square.$(2a-3b)^2$Is the expression factored completely? Yes.The binomial cannot be factored.Check your answer.Multiply.$(2a-3b)^2$$(2a)^2-2\cdot2a\cdot3b+(3b)^2$$4a^2-12ab+9b^2$✓

When we have factored a polynomial with four terms, most often we separated it into two groups of two terms. Remember that we can also separate it into a trinomial and then one term.

Factor completely: $9x^2-12xy+4y^2-49$.
$9x^2-12xy+4y^2-49$Is there a GCF? No.With more than 3 terms, use grouping. Last 2 termshave no GCF. Try grouping first 3 terms.$9x^2-12xy+4y^2-49$Factor the trinomial with$a\neq1$.But the first term is aperfect square.Is the last term of the trinomial a perfect square? Yes.$(3x)^2-12xy+(2y)^2-49$Does the trinomial fit the pattern,$a^2-2ab+b^2$?Yes.$(3x)^2 \searrow -12xy +$

−2(3x)(2y)↙(2y)2−49Write the trinomial as a square.(3x−2y)2−49Is this binomial a sum or difference? Of squares orcubes? Write it as a difference of squares.(3x−2y)2−72Write it as a product of conjugates.((3x−2y)−7)((3x−2y)+7)(3x−2y−7)(3x−2y+7)Is the expression factored completely? Yes.Check your answer.Multiply.(3x−2y−7)(3x−2y+7)9x2−6xy−21x−6xy+4y2+14y+21x−14y−499x2−12xy+4y2−49✓

Factoring Expressions with Fractional or Negative Exponents

Expressions with fractional or negative exponents can be factored by pulling out a GCF. Look for the variable or exponent that is common to each term of the expression and pull out that variable or exponent raised to the lowest power. These expressions follow the same factoring rules as those with integer exponents. For instance, $2x^{\frac{1}{4}}+5x^{\frac{3}{4}}$ can be factored by pulling out $x^{\frac{1}{4}}$ and being rewritten as $x^{\frac{1}{4}}(2+5x^{\frac{1}{2}})$.

Factoring an Expression with Fractional or Negative Exponents

Factor $3x(x+2)^{-\frac{1}{3}} + 4(x+2)^{\frac{2}{3}}$.

Factor out the term with the lowest value of the exponent. In this case, that would be $(x+2)^{-\frac{1}{3}}$.

$(x+2)^{-\frac{1}{3}}(3x+4(x+2))$

Factor out the GCF. $(x+2)^{-\frac{1}{3}}(3x+4x+8)$

Simplify. $(x+2)^{-\frac{1}{3}}(7x+8)$

Factor $2(5a-1)^{\frac{3}{4}} + 7a(5a-1)^{-\frac{1}{4}}$.

$(5a-1)^{-\frac{1}{4}}(17a-2)$

Access these online resources for additional instruction and practice with factoring polynomials.

- Identify GCF
- Factor Trinomials when a Equals 1
- Factor Trinomials when a is not equal to 1

- Factor Sum or Difference of Cubes

Key Equations

difference of squares	$a^2 - b^2 = (a+b)(a-b)$
perfect square trinomial	$a^2 + 2ab + b^2 = (a+b)^2$
sum of cubes	$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
difference of cubes	$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

Write the factors as two binomials with first terms *x*. $x^2 + bx + c$ $(x\ \ )(x\ \ )$ Find two numbers *m* and *n* that

- multiply to c, $m \cdot n = c$
- add to b, $m + n = b$

Use *m* and *n* as the last terms of the factors. $(x+m)(x+n)$ Check by multiplying the factors.

- The greatest common factor, or GCF, can be factored out of a polynomial. Checking for a GCF should be the first step in any factoring problem. See [link].
- Trinomials with leading coefficient 1 can be factored by finding numbers that have a product of the third term and a sum of the second term. See [link].
- Trinomials can be factored using a process called factoring by grouping. See [link].
- Perfect square trinomials and the difference of squares are special products and can be factored using equations. See [link] and [link].
- The sum of cubes and the difference of cubes can be factored using equations. See [link] and [link].
- Polynomials containing fractional and negative exponents can be factored by pulling out a GCF. See [link].

Section Exercises

Algebraic

For the following exercises, find the greatest common factor.

$$49mb^2 - 35m^2ba + 77ma^2$$

$7m$

$200p^3m^3 - 30p^2m^3 + 40m^3$

$10m^3$

For the following exercises, factor by grouping.

$2a^2 + 9a - 18$

$(2a-3)(a+6)$

$6n^2 - 19n - 11$

$(3n-11)(2n+1)$

For the following exercises, factor the polynomial.

$10h^2 - 9h - 9$

$(5h+3)(2h-3)$

$9d^2 - 73d + 8$

$(9d-1)(d-8)$

$16x^2-100$

$(4x+10)(4x-10)$

$144b^2-25c^2$

$(12b+5c)(12b-5c)$

$49n^2+168n+144$

$(7n+12)^2$

$25p^2-120m+144$

$(5p-12)^2$

For the following exercises, factor the polynomials.

x^3+216

$(x+6)(x^2-6x+36)$

$64x^3-125$

$(4x-5)(16x^2+20x+25)$

$125r^3+1{,}728s^3$

$(5r+12s)(25r^2-60rs+144s^2)$

$3c(2c+3)^{-\frac{1}{4}}-5(2c+3)^{\frac{3}{4}}$

$(2c+3)^{-\frac{1}{4}}(-7c-15)$

$5z(2z-9)^{-\frac{3}{2}}+11(2z-9)^{-\frac{1}{2}}$

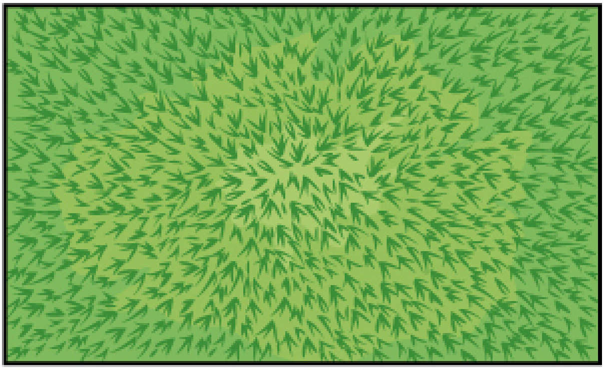
$(2z-9)^{-\frac{3}{2}}(27z-99)$

Real-World Applications

For the following exercises, consider this scenario:

Charlotte has appointed a chairperson to lead a city beautification project. The first act is to install

statues and fountains in one of the city’s parks. The park is a rectangle with an area of $98x^2+105x-27$ m^2, as shown in the figure below. The length and width of the park are perfect factors of the area.



$l \times w = 98x^2 + 105x - 27$

Factor by grouping to find the length and width of the park.

$(14x-3)(7x+9)$

At the northwest corner of the park, the city is going to install a fountain. The area of the base of the fountain is $9x^2-25\ m^2$. Factor the area to find the lengths of the sides of the fountain.

$(3x+5)(3x-5)$

Glossary

factor by grouping
: a method for factoring a trinomial in the form $ax^2 + bx + c$ by dividing the *x* term into the sum of two terms, factoring each portion of the expression separately, and then factoring out the GCF of the entire expression

greatest common factor
: the largest polynomial that divides evenly into each polynomial

Rational Expressions (P6)
In this section students will:

- Simplify rational expressions.
- Multiply rational expressions.
- Divide rational expressions.
- Add and subtract rational expressions.
- Simplify complex rational expressions.

This Module supports section **P6 of Mat 1023**.
In case you missed something in class, or just want to review a specific topic covered in this Module, here is a list of topics covered:

We previously reviewed the properties of fractions

and their operations. We introduced rational numbers, which are just fractions where the numerators and denominators are integers. In this chapter, we will work with fractions whose numerators and denominators are polynomials. We call this kind of expression a **rational expression**.

> Rational Expression
> A rational expression is an expression of the form pq, where *p* and *q* are polynomials and $q \neq 0$.

Here are some examples of rational expressions:
$-24565x12y4x+1x2-94x2+3x-12x-8$

Notice that the first rational expression listed above, -2456, is just a fraction. Since a constant is a polynomial with degree zero, the ratio of two constants is a rational expression, **provided the denominator is not zero.**

We will do the same operations with rational expressions that we did with fractions. We will simplify, add, subtract, multiply, divide and use them in applications.

Determine the Values for Which a

Rational Expression is Undefined

The **domain** of an expression is the set of numbers for which the expression is defined. If the denominator is zero, the rational expression is undefined. The numerator of a rational expression may be 0—but not the denominator.

When we work with a numerical fraction, it is easy to avoid dividing by zero because we can see the number in the denominator. In order to avoid dividing by zero in a rational expression, we must not allow values of the variable that will make the denominator be zero.

So before we begin any operation with a rational expression, we examine it first to find the values that would make the denominator zero. That way, when we solve a rational equation for example, we will know whether the algebraic solutions we find are allowed or not.

Determine the values for which a rational expression is undefined.

Set the denominator equal to zero. Solve the equation.

Determine the value for which each rational expression is undefined:

ⓐ 8a2b3c ⓑ 4b − 32b + 5 ⓒ x + 4x2 + 5x + 6.

The expression will be undefined when the denominator is zero.

ⓐ

8a2b3c Set the denominator equal to zero and solvefor the variable.3c = 0 c = 0 8a2b3cis undefined forc = 0.

ⓑ

4b − 32b + 5 Set the denominator equal to zero and solvefor the variable.2b + 5 = 02b = − 5b = − 52 4b − 32b + 5is undefined forb = − 52.

ⓒ

x + 4x2 + 5x + 6 Set the denominator equal to zero and solvefor the variable.x2 + 5x + 6 = 0(x + 2)(x + 3) = 0x + 2 = 0orx + 3 = 0x = − 2orx = − 3 x + 4x2 + 5x + 6is undefined forx = − 2orx = − 3.

Simplify Rational Expressions

A fraction is considered simplified if there are no common factors, other than 1, in its numerator and denominator. Similarly, a **simplified rational expression** has no common factors, other than 1, in its numerator and denominator.

Simplified Rational Expression

A rational expression is considered simplified if there are no common factors in its numerator and denominator.

For example,

$x+2x+3$is simplified because there are no common factors of$x+2$and$x+3$. $2x3x$is not simplified because$x$is a common factor of$2x$and$3x$.

We use the Equivalent Fractions Property to simplify numerical fractions. We restate it here as we will also use it to simplify rational expressions.

Equivalent Fractions Property

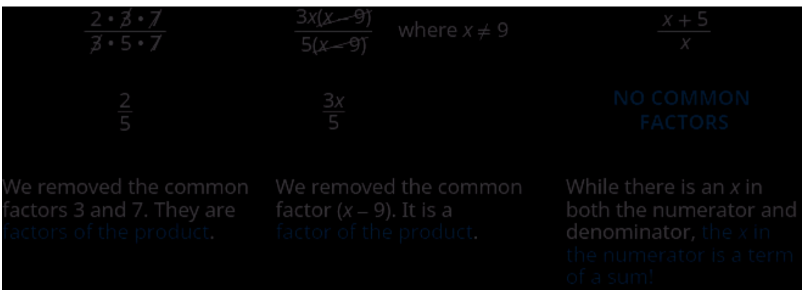
If *a*, *b*, and *c* are numbers where $b \neq 0, c \neq 0$,

then$ab = a \cdot cb \cdot c$and$a \cdot cb \cdot c = ab$.

Notice that in the Equivalent Fractions Property, the values that would make the denominators zero are specifically disallowed. We see $b \neq 0, c \neq 0$ clearly stated.

To simplify rational expressions, we first write the numerator and denominator in factored form. Then we remove the common factors using the Equivalent Fractions Property.

Be very careful as you remove common factors. Factors are multiplied to make a product. You can remove a factor from a product. You cannot remove a term from a sum.



Removing the *x*’s from $x+5x$ would be like cancelling the 2’s in the fraction $2+52$!

How to Simplify a Rational Expression

Simplify: $x2+5x+6x2+8x+12$.

Step 1. Factor the numerator and denominator completely.	Factor $x^2 + 5x + 6$ and $x^2 + 8x + 12$	$\frac{x^2 + 5x + 6}{x^2 + 8x + 12}$ $\frac{(x + 2)(x + 3)}{(x + 2)(x + 6)}$

Step 2. Simplify by dividing out common factors.	Remove the common factor $x + 2$ from the numerator and the denominator.	$\frac{\cancel{(x + 2)}(x + 3)}{\cancel{(x + 2)}(x + 6)}$ $\frac{(x + 3)}{(x + 6)}$ $x \neq -2 \quad x \neq -6$

Simplify a rational expression

Factor the numerator and denominator completely.
Simplify by dividing out common factors.

Usually, we leave the simplified rational expression in factored form. This way, it is easy to check that we have removed *all* the common factors.

Simplify: y2 + y − 42y2 − 36.

Solution

$y2+y-42y2-36$Factor the numerator and denominator.$(y+7)(y-6)(y+6)(y-6)$Remove the common factor$y-6$fromthe numerator and the denominator.$(y+7)(y-6)(y+6)(y-6)y+7y+6$

Now we will see how to simplify a rational expression whose numerator and denominator have opposite factors. We previously introduced opposite notation: the opposite of a is $-a$ and $-a=-1\cdot a$.

The numerical fraction, say $7-7$ simplifies to -1. We also recognize that the numerator and denominator are opposites.

The fraction $a-a$, whose numerator and denominator are opposites also simplifies to -1.
Let's look at the expression$b-a$.$b-a$ Rewrite.$-a+b$ Factor out–1.$-1(a-b)$

This tells us that $b-a$ is the opposite of $a-b$.

In general, we could write the opposite of $a-b$ as $b-a$. So the rational expression $a-bb-a$ simplifies to -1.

Opposites in a Rational Expression

The opposite of $a-b$ is $b-a$.

$a-bb-a=-1 \quad a\neq b$

An expression and its opposite divide to -1.

We will use this property to simplify rational expressions that contain opposites in their numerators and denominators. Be careful not to treat $a+b$ and $b+a$ as opposites. Recall that in addition, order doesn't matter so $a+b=b+a$. So if $a\neq -b$, then $a+bb+a=1$.

Simplify: $x2-4x-3264-x2$.

[redacted]

Factor the numerator and the denominator.

[redacted]

Recognize the factors that are opposites.

[redacted]

Simplify.

[redacted]

Simplify: $x2+x-21-x2$.

$-x+2x+1$

Multiply Rational Expressions

To multiply rational expressions, we do just what we did with numerical fractions. We multiply the numerators and multiply the denominators. Then, if there are any common factors, we remove them to simplify the result.

Multiplication of Rational Expressions

If *p*, *q*, *r*, and *s* are polynomials where $q \neq 0, s \neq 0$, then

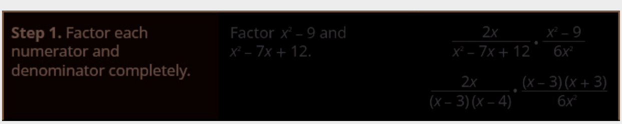
$$\frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs}$$

Factor each numerator and denominator completely. Multiply the numerators and denominators. Simplify by dividing out common factors.

Remember, throughout this chapter, we will assume that all numerical values that would make the denominator be zero are excluded. We will not write the restrictions for each rational expression, but keep in mind that the denominator can never be zero. So in this next example, $x \neq 0, x \neq 3$, and $x \neq 4$.

How to Multiply Rational Expressions

Simplify: $\frac{2x}{x^2 - 7x + 12} \cdot \frac{x^2 - 9}{6x^2}$.



Step 1. Factor each numerator and denominator completely.	Factor $x^2 - 9$ and $x^2 - 7x + 12$.	$\frac{2x}{x^2 - 7x + 12} \cdot \frac{x^2 - 9}{6x^2}$ $\frac{2x}{(x-3)(x-4)} \cdot \frac{(x-3)(x+3)}{6x^2}$

Step 2. Multiply the numerators and denominators.	Multiply the numerators and denominators. It is helpful to write the monomials first.	$\frac{2x(x-3)(x+3)}{6x^2(x-3)(x-4)}$
Step 3. Simplify by dividing out common factors.	Divide out the common factors.	$\frac{\cancel{2}\,\cancel{x}\,\cancel{(x-3)}(x+3)}{\cancel{2}\cdot 3\cdot \cancel{x}\cdot x\,\cancel{(x-3)}(x-4)}$
	Leave the denominator in factored form.	$\frac{(x+3)}{3x(x-4)}$

Simplify: 5xx2 + 5x + 6·x2 − 410x.

x − 22(x + 3)

The following is an alternate method, used in the College Algebra textbook, where you divide out the common factors prior to multiplying the numerators and denominators.

Multiply: 3a2 − 8a − 3a2 − 25·a2 + 10a + 253a2 − 14a − 5.

Factor numerators and Denominators	$\frac{(3a+1)(a-3)}{(a-5)(a+5)} \cdot \frac{(a+5)(a+5)}{(3a+1)(a-5)}$
Divide Common Factors	$\frac{\cancel{(3a+1)}(a-3)}{(a-5)\cancel{(a+5)}} \cdot \frac{\cancel{(a+5)}(a+5)}{\cancel{(3a+1)}(a-5)}$
Simplify	$\frac{(a-3)}{(a-5)} \cdot \frac{(a+5)}{(a-5)}$
Multiply remaining Factors in numerators and denominators	$\frac{(a-3)(a+5)}{(a-5)^2}$ $a \neq 5, -5, -\frac{1}{3}$

Divide Rational Expressions

Just like we did for numerical fractions, to divide a rational expression by another, multiply the first expression by the reciprocal of the second. Using this approach, we would rewrite 1 x ÷ x 2 3 as the product 1 x · 3 x 2 . Once the division expression has been rewritten as a multiplication expression, we can multiply as we did before.

Division of Rational Expressions

If *p*, *q*, *r*, and *s* are polynomials where $q \neq 0, r \neq 0, s \neq 0$, then

$pq \div rs = pq \cdot sr$

Rewrite the division as the product of the first

rational expression and the reciprocal of the second. Factor the numerators and denominators completely. Multiply the numerators and denominators together. Simplify by dividing out common factors.

Once we rewrite the division as multiplication of the first expression by the reciprocal of the second, we then factor everything and look for common factors.

How to Divide Rational Expressions

Divide: $\frac{p^3+q^3}{2p^2+2pq+2q^2} \div \frac{p^2-q^2}{6}$.

Step 1. Rewrite the division as the product of the first rational expression and the reciprocal of the second.	"Flip" the second fraction and change the division sign to multiplication.	$\frac{p^3+q^3}{2p^2+2pq+2q^2} \div \frac{p^2-q^2}{6}$ $\frac{p^3+q^3}{2p^2+2pq+2q^2} \cdot \frac{6}{p^2-q^2}$
Step 2. Factor the numerators and denominators completely.	Factor the numerators and denominators.	$\frac{(p+q)(p^2-pq+q^2)}{2(p^2+pq+q^2)} \cdot \frac{2 \cdot 3}{(p-q)(p+q)}$
Step 3. Multiply the numerators and denominators.	Multiply the numerators and multiply the denominators.	$\frac{(p+q)(p^2-pq+q^2)2 \cdot 3}{2(p^2+pq+q^2)(p-q)(p+q)}$

Step 4. Simplify by dividing out common factors.	Divide out the common factors.	$\frac{\cancel{(p+q)}(p^2-pq+q^2)\cancel{2}\cdot 3}{\cancel{2}(p^2+pq+q^2)(p-q)\cancel{(p+q)}}$ $\frac{3(p^2-pq+q^2)}{(p-q)(p^2+pq+q^2)}$

The following is an alternate method, used in the College Algebra textbook, where you divide out the common factors prior to multiplying the numerators and denominators.

Dividing Rational Expressions

Divide the rational expressions and express the quotient in simplest form:

$\frac{2x^2+x-6}{x^2-1} \div \frac{x^2-4}{x^2+2x+1}$

Rewrite the division as the product of the reciprocal.	$\frac{2x^2+x+6}{x^2-1}\cdot\frac{x^2+2x+1}{x^2-4}$
Factor numerators and Denominators	$\frac{(2x-3)(x+2)}{(x+1)(x-1)}\cdot\frac{(x+1)(x+1)}{(x+2)(x-2)}$
Divide Common Factors	$\frac{(2x-3)\cancel{(x+2)}}{\cancel{(x+1)}(x-1)}\cdot\frac{\cancel{(x+1)}(x+1)}{\cancel{(x+2)}(x-2)}$
Simplify	$\frac{(2x-3)}{(x-1)}\cdot\frac{(x+1)}{(x-2)}$
Multiply remaining Factors in numerators and denominators	$\frac{(2x-3)(x+1)}{(x-1)(x-2)}$ $x \neq 2, -2, -1, 1$

Add and subtract Rational Expressions

What is the first step you take when you add numerical fractions? You check if they have a common denominator. If they do, you add the numerators and place the sum over the common denominator. If they do not have a common denominator, you find one before you add.

It is the same with rational expressions. To add rational expressions, they must have a common denominator. When the denominators are the same, you add the numerators and place the sum over the common denominator.

Rational Expression Addition and Subtraction
If *p*, *q*, and *r* are polynomials where $r \neq 0$, then
$pr+qr=p+qrandpr-qr=p-qr$

How To:

- To add or subtract rational expressions with a common denominator, add or subtract the numerators and place the result over the common denominator.

- We always simplify rational expressions. Be sure to factor, if possible, after you subtract the numerators so you can identify any common factors.
- Remember, too, we do not allow values that would make the denominator zero.

Add: $11x+28x+4+x2x+4$.
Since the denominator is $x+4$, we must exclude the value $x=-4$. $11x+28x+4+x2x+4, x\neq-4$ The fractions have a common denominator,so add the numerators and place the sumover the common denominator.$11x+28+x2x+4$ Write the degrees in descending order.$x2+11x+28x+4$ Factor the numerator.$(x+4)(x+7)x+4$ Simplify by removing common factors.$(x+4)(x+7)x+4$ Simplify.$x+7$ The expression simplifies to $x+7$ but the original expression had a denominator of $x+4$ so $x\neq-4$.

To subtract rational expressions, they must also have a common denominator. When the denominators are the same, you subtract the numerators and place the difference over the common denominator. Be careful of the signs when you subtract a binomial or trinomial.

Subtract: $5x^2-7x+3x^2-3x+18-4x^2+x-9x^2-3x+18$.
$5x^2-7x+3x^2-3x+18-4x^2+x-9x^2-3x+18$ Subtract the numerators and place thedifference over the common denominator.$5x^2-7x+3-(4x^2+x-9)x^2-3x+18$ Distribute the sign in the numerator.$5x^2-7x+3-4x^2-x+9x^2-3x-18$ Combine like terms.$x^2-8x+12x^2-3x-18$ Factor the numerator and the denominator.$(x-2)(x-6)(x+3)(x-6)$ Simplify by removing common factors.$(x-2)(x-6)(x+3)(x-6)$ $(x-2)(x+3)$

Add, Subtract with Unlike Denominators

Adding and subtracting rational expressions works

just like adding and subtracting numerical fractions. To add fractions, we need to find a common denominator. Let’s look at an example of fraction addition.

$5\ 24 + 1\ 40 = 25\ 120 + 3\ 120 = 28\ 120 = 7\ 30$

We have to rewrite the fractions so they share a common denominator before we are able to add. We must do the same thing when adding or subtracting rational expressions.

The easiest common denominator to use will be the **least common denominator**, or LCD. The LCD is the smallest multiple that the denominators have in common. To find the LCD of two rational expressions, we factor the expressions and multiply all of the distinct factors. For instance, if the factored denominators were $(x+3)(x+4)$ and $(x+4)(x+5)$, then the LCD would be $(x+3)(x+4)(x+5)$.

Find the least common denominator of rational expressions.

Factor each denominator completely. List the factors of each denominator. Match factors vertically when possible, as this will make it easier to avoid duplication. Bring down the columns by including all factors, but do not include common

factors twice. Write the LCD as the product of the factors.

Remember, we always exclude values that would make the denominator zero. What values of x should we exclude in this next example?

ⓐ Find the LCD for the expressions 8x2 − 2x − 3,3xx2 + 4x + 3 and ⓑ rewrite them as equivalent rational expressions with the lowest common denominator.

ⓐ

Find the LCD for 8x2 − 2x − 3,3xx2 + 4x + 3.		
Factor each denominator completely, lining up		

co[redacted]

Bri[redacted]
columns.

Write the LCD as the product of the factors.

[redacted]

ⓑ

[redacted]

Factor each denominator.

[redacted]

Multiply each denominator by the ‘m[redacted]
LCD factor and multiply each numerator by the same factor.

Simplify the numerators.

[redacted]

ⓐ Find the LCD for the expressions $3xx2-3x$–$10,5x2+3x+2$ ⓑ rewrite them as equivalent rational expressions with the lowest common denominator.

ⓐ $(x+2)(x-5)(x+1)$
ⓑ $3x2+3x(x+2)(x-5)(x+1)$,
$5x-25(x+2)(x-5)(x+1)$

Now we have all the steps we need to add or subtract rational expressions with unlike denominators. Here's an example:

Determine if the expressions have a common denominator.

- **Yes** – go to step 2.
- **No** – Rewrite each rational expression with the LCD.
    - Find the LCD.

- Rewrite each rational expression as an equivalent rational expression with the LCD.

Add or subtract the rational expressions. Simplify, if possible.

Add: $3x - 3 + 2x - 2$.

Step 1. Determine if the expressions have a common denominator. • Yes–go to step 2. • No–Rewrite each rational expression with the LCD. ◦ Find the LCD. ◦ Rewrite each rational expression as an equivalent rational expression with the LCD.	No.	
	Find the LCD of $(x-3)$ and $(x-2)$.	$x-3 : (x-3)$ $x-2 : (x-2)$ LCD : $(x-3)(x-2)$
		$\frac{3}{x-3} + \frac{2}{x-2}$
	Change into equivalent rational expressions with the LCD, $(x-3)$ and $(x-2)$.	$\frac{3(x-2)}{(x-3)(x-2)} + \frac{2(x-3)}{(x-2)(x-3)}$
	Keep the denominators factored!	$\frac{3x-6}{(x-3)(x-2)} + \frac{2x-6}{(x-2)(x-3)}$

Step 2. Add or subtract the rational expressions.	Add the numerators and place the sum over the common denominator.	$\frac{3x-6+2x-6}{(x-3)(x-2)}$

Step 3. Simplify, if possible.	Because $5x - 12$ cannot be factored, the answer is simplified.	$\frac{5x-12}{(x-3)(x-2)}$

Subtract: $-3n-9n2+n-6-n+32-n$.

$$\frac{-3n-9}{n^2+n-6}-\frac{n+3}{2-n}$$

Factor the denominator.

$$\frac{-3n-9}{(n-2)(n+3)}-\frac{n+3}{2-n}$$

Since $n-2$ and $2-n$ are opposites, we will mu[illegible] rational expression by $-1-1$.

$$\frac{-3n-9}{(n-2)(n+3)}-\frac{(-1)(n+3)}{(-1)(2-n)}$$

$$\frac{-3n-9}{(n-2)(n+3)}-\frac{(-1)(n+3)}{(n-2)}$$

Simplify. Remember, a $-(-b)=a+b$.

$$\frac{-3n-9}{(n-2)(n+3)}+\frac{(n+3)}{(n-2)}$$

Do the rational expressions have a common denominator? No.

Find the LCD. $n^2 + n - 6 = (n - 2)(n + 3)$ $n - 2 = (n - 2)$ ______________ $\text{LCD} = (n - 2)(n + 3)$

Rewrite each rational expression as an equivalent rational expression with the LCD.

$$\frac{-3n - 9}{(n - 2)(n + 3)} + \frac{(n + 3)(n + 3)}{(n - 2)(n + 3)}$$

Simplify the numerators.

$$\frac{-3n - 9}{(n - 2)(n + 3)} + \frac{n^2 + 6n + 9}{(n - 2)(n + 3)}$$

Add the rational expressions.

$$\frac{-3n - 9 + n^2 + 6n + 9}{(n - 2)(n + 3)}$$

Simplify the numerator.

$$\frac{n^2 + 3n}{(n - 2)(n + 3)}$$

Factor the numerator to look for common factors.

$$\frac{n\cancel{(n + 3)}}{(n - 2)\cancel{(n + 3)}}$$

Simplify.

$$\frac{n}{(n - 2)}$$

Complex Rational Expressions

Complex fractions are fractions in which the numerator or denominator contains a fraction. We previously simplified complex fractions like these:
$3458x2xy6$

> A **complex rational expression** is a rational expression in which the numerator and/or the denominator contains a rational expression.
>
> **Examples**
>
> $4y-38y2-91x+1yxy-yx2x+64x-6-4x2-36$

We will use **two methods** to simplify complex rational expressions. The following is the first type:

For Example

$6x2-7x+24x-82x2-8x+3x2-5x+6$

We noted that fraction bars tell us to divide, so rewrote it as the division problem:
$(6x2-7x+24x-8) \div (2x2-8x+3x2-5x+6)$.

Then, we multiplied the first rational expression by the reciprocal of the second, just like we do when we divide two fractions.

Simplify the complex rational expression by writing it as division: 6x−43x2−16.

6x−43x2−16 Rewrite the complex fraction as division.6x−4÷3x2−16 Rewrite as the product of first times thereciprocal of the second.6x−4·x2−163 Factor.3·2x−4·(x−4)(x+4)3 Multiply.3·2(x−4)(x+4)3(x−4) Remove common factors.3·2(x−4)(x+4)3(x−4) Simplify.2(x+4)

Are there any value(s) of *x* that should not be allowed? The original complex rational expression had denominators of x−4 and x2−16. This expression would be undefined if x=4 or x=−4.

Simplify the numerator and denominator. Rewrite the complex rational expression as a division problem. Divide the expressions.

Simplify the complex rational expression by writing it as division: n−4nn+51n+5+1n

-5.

$$\frac{n-\frac{4n}{n+5}}{\frac{1}{n+5}+\frac{1}{n-5}}$$

Simplify the numerator and denominator.

Fi
de
nu
denominator.

$$\frac{\frac{n(n+5)}{1(n+5)}-\frac{4n}{n+5}}{\frac{1(n-5)}{(n+5)(n-5)}+\frac{1(n+5)}{(n-5)(n+5)}}$$

Simplify the numerators.

$$\frac{\frac{n^2+5n}{n+5}-\frac{4n}{n+5}}{\frac{n-5}{(n+5)(n-5)}+\frac{n+5}{(n-5)(n+5)}}$$

Subtract the rational expressions in the
nu
ad
de

$$\frac{\frac{n^2+5n-4n}{n+5}}{\frac{n-5+n+5}{(n+5)(n-5)}}$$

Simplify. (We now have one rational expression over

on[redacted]	
Rewrite as fraction division.	
[redacted]	
Multiply the first times the reciprocal of the se[redacted]	
Factor any expressions if possible.	
[redacted]	
Remove common factors.	
[redacted]	
Simplify.	
[redacted]	

As we did early, we can use that strategy here to simplify complex rational expressions. We will multiply the numerator and denominator by the LCD of all the rational expressions.

Find the LCD of all fractions in the complex rational expression. Multiply the numerator and denominator by the LCD. Simplify the expression.

Simplify the complex rational expression by using the LCD: 2x+64x−6−4x2−36.

$$\frac{\frac{2}{x+6}}{\frac{4}{x-6}-\frac{4}{x^2-36}}$$

Find the LCD of all fractions in the complex rational expression. The LCD is x2−36=(x+6)(x−6). Multiply the numerator and denominator by the

$$\frac{(x+6)(x-6)\frac{2}{x+6}}{(x+6)(x-6)\left(\frac{4}{x-6}-\frac{4}{(x+6)(x-6)}\right)}$$

Simplify the

expression.

Distribute in the denominator.

[redacted]

Simplify.

[redacted]

Simplify.

[redacted]

To simplify the denominator, dis[redacted]

and combine like terms.

Factor the denominator.

[redacted]

Remove common factors.

[redacted]

Simplify.

[redacted]

Notice that there are no more factors common to the numerator and denominator.

Simplify the complex rational expression by using the LCD: 4m2 − 7m + 123m − 3 − 2m − 4.

$$\frac{\frac{4}{m^2-7m+12}}{\frac{3}{m-3}-\frac{2}{m-4}}$$

Find the LCD of all fractions in the complex rational expression.

The LCD is (m − 3)(m − 4).

Multiply the numerator and de

LC

$$\frac{(m-3)(m-4)\frac{4}{(m-3)(m-4)}}{(m-3)(m-4)\left(\frac{3}{m-3}-\frac{2}{m-4}\right)}$$

Simplify.

[redacted]

Simplify.

[redacted]

Distribute.

[redacted]

Combine like terms.

[redacted]

Key Concepts

- **Determine the values for which a rational expression is undefined.**

  Set the denominator equal to zero. Solve the equation.

- **Equivalent Fractions Property**
  If *a*, *b*, and *c* are numbers where $b \neq 0, c \neq 0$, then $ab = a \cdot cb \cdot c$ and $a \cdot cb \cdot c = ab$.
- **How to simplify a rational expression.**

Factor the numerator and denominator completely. Simplify by dividing out common factors.

- **Opposites in a Rational Expression**

  The opposite of $a-b$ is $b-a$.

  $a-bb-a=-1a\neq b$

  An expression and its opposite divide to -1.
- **Multiplication of Rational Expressions**

  If *p*, *q*, *r*, and *s* are polynomials where $q\neq 0, s\neq 0$, then

  $pq\cdot rs=prqs$
- **How to multiply rational expressions.**

  Factor each numerator and denominator completely. Multiply the numerators and denominators. Simplify by dividing out common factors.

- **Division of Rational Expressions**

  If *p*, *q*, *r*, and *s* are polynomials where $q\neq 0, r\neq 0, s\neq 0$, then

  $pq\div rs=pq\cdot sr$
- **How to divide rational expressions.**

  Rewrite the division as the product of the first rational expression and the reciprocal of the second. Factor the numerators and denominators completely. Multiply the numerators and denominators together.

Simplify by dividing out common factors.

Practice Makes Perfect

Determine the Values for Which a Rational Expression is Undefined

In the following exercises, determine the values for which the rational expression is undefined.

ⓐ $2x2z$
ⓑ $4p-16p-5$
ⓒ $n-3n2+2n-8$

ⓐ $z=0$ ⓑ $p=56$
ⓒ $n=-4, n=2$

Simplify Rational Expressions

In the following exercises, simplify each rational expression.

$8m3n12mn2$

$2m23n$

$x2+4x-5x2-2x+1$

$x+5x-1$

Multiply Rational Expressions

In the following exercises, multiply the rational expressions.

$5x2y412xy3\cdot 6x220y2$

$x38y$

$c2-10c+25c2-25\cdot c2+10c+253c2-14c-5$

$c+53c+1$

Divide Rational Expressions

In the following exercises, divide the rational expressions.

$v-511-v\div v2-25v-11$

$-1v+5$

$3s2s2 - 16 \div s3 + 4s2 + 16ss3 - 64$

$3ss + 4$

$2a2 - a - 215a + 20a2 + 7a + 12a2 + 8a + 16$

$2a - 75$

In the following exercises, simplify each complex rational expression.

$2aa + 44a2a2 - 16$

$a - 42a$

$2x + 53x - 5 + 1x2 - 25$

$2x - 103x + 16$

$2 + 1p - 35p - 3$

$2p - 55$

Glossary

rational expression

A rational expression is an expression of the form pq, where *p* and *q* are polynomials and $q \neq 0$.

simplified rational expression

A simplified rational expression has no common factors, other than 1, in its numerator and denominator.

rational function

A rational function is a function of the form $R(x) = p(x)q(x)$ where p(x) and q(x) are polynomial functions and q(x) is not zero.

complex rational expression

A complex rational expression is a rational expression in which the numerator and/or denominator contains a rational expression.

Foundation Chapter Review (P1-6)
Review problems for Foundations.

Chapter Review Exercises

Real Numbers: Algebra Essentials

For the following exercises, perform the given operations.

$(5-3\cdot 2)^2-6$

-5

$64\div(2\cdot 8)+14\div 7$

$2\cdot 5^2+6\div 2$

53

For the following exercises, solve the equation.

$5x+9=-11$

$2y + \ 4\ 2\ = 64$

$y = 24$

For the following exercises, simplify the expression.

$9(\ y + 2\ ) \div 3 \cdot 2 + 1$

$3m(\ 4 + 7\ ) - m$

$32m$

For the following exercises, identify the number as rational, irrational, whole, or natural. Choose the most descriptive answer.

11

0

---

whole

$5\ 6$

11

---

irrational

Exponents and Scientific Notation

For the following exercises, simplify the expression.

$2^2 \cdot 2^4$

$\frac{4^5}{4^3}$

16

$(a^2 b^3)^4$

$\frac{6a^2 \cdot a^0}{2a^{-4}}$

---

a^6

$\frac{(xy)^4}{y^3} \cdot \frac{2}{x^5}$

$\frac{4^{-2} x^3 y^{-3}}{2x^0}$

---

x 3 32 y 3

$(2x^2y)^{-2}$

$(16a^3b^2)(4ab^{-1})^{-2}$

---

a

Write the number in standard notation: 2.1314×10^{-6}

Write the number in scientific notation: 16,340,000

1.634×10^7

Radicals and Rational Expressions

For the following exercises, find the principal square root.

121

196

14 361 75
5 3 162 32 25
4 2 5 80 81 49 1250
7 2 50 2 4 + 2

4 3 +6 3

---

10 3

12 5 −13 5

−243 5

−3

250 3 −8 3

Polynomials

For the following exercises, perform the given operations and simplify.

(3 x 3 +2x−1)+(4 x 2 −2x+7)

---

3 x 3 +4 x 2 +6

(2y+1)−(2 y 2 −2y−5)

(2 x 2 +3x−6)+(3 x 2 −4x+9)

$5x^2 - x + 3$

$(6a^2 + 3a + 10) - (6a^2 - 3a + 5)$

$(k+3)(k-6)$

$k^2 - 3k - 18$

$(2h+1)(3h-2)$

$(x+1)(x^2+1)$

$x^3 + x^2 + x + 1$

$(m-2)(m^2 + 2m - 3)$

$(a+2b)(3a-b)$

$3a^2 + 5ab - 2b^2$

$(x+y)(x-y)$

Factoring Polynomials

For the following exercises, find the greatest common factor.

$81p+9pq-27p^2q^2$

$9p$

$12x^2y+4xy^2-18xy$

$88a^3b+4a^2b-144a^2$

$4a^2$

For the following exercises, factor the polynomial.

$2x^2-9x-18$

$8a^2+30a-27$

$(4a-3)(2a+9)$

$d^2-5d-66$

$x^2 + 10x + 25$

$(x+5)^2$

$y^2 - 6y + 9$

$4h^2 - 12hk + 9k^2$

$(2h-3k)^2$

$361x^2 - 121$

$p^3 + 216$

$(p+6)(p^2 - 6p + 36)$

$8x^3 - 125$

$64q^3 - 27p^3$

$(4q-3p)(16q^2 + 12pq + 9p^2)$

$$4x(x-1)^{-\frac{1}{4}}+3(x-1)^{\frac{3}{4}}$$

$$3p(p+3)^{\frac{1}{3}}-8(p+3)^{\frac{4}{3}}$$

$$(p+3)^{\frac{1}{3}}(-5p-24)$$

$$4r(2r-1)^{-\frac{2}{3}}-5(2r-1)^{\frac{1}{3}}$$

Rational Expressions

For the following exercises, simplify the expression.

$$\frac{x^2-x-12}{x^2-8x+16}$$

$$\frac{x+3}{x-4}$$

$$\frac{4y^2-25}{4y^2-20y+25}$$

$$\frac{2a^2-a-3}{2a^2-6a-8}\cdot\frac{5a^2-19a-4}{10a^2-13a-3}$$

$$\frac{1}{2}$$

$$\frac{d-4}{d^2-9}\cdot\frac{d-3}{d^2-16}$$

m 2 +5m+6 2 m 2 −5m−3 ÷ 2 m 2 +3m −9 4 m 2 −4m−3

m+2 m−3

4 d 2 −7d−2 6 d 2 −17d+10 ÷ 8 d 2 +6d +1 6 d 2 +7d−10

10 x + 6 y

6x+10y xy

12 a 2 +2a+1 − 3 a 2 −1

1 d + 2 c 6c+12d dc

1 6

3 x − 7 y 2 x

Chapter Practice Test

For the following exercises, identify the number as rational, irrational, whole, or natural. Choose the most descriptive answer.

-13

---

rational

2

For the following exercises, evaluate the equations.

$2(x+3)-12=18$

$x=12$

$y\ (3+3)\ 2\ -26=10$

Write the number in standard notation: $3.1415 \times 10\ 6$

3,141,500

Write the number in scientific notation:

$0.000000212.$

For the following exercises, simplify the expression.

$-2\cdot(2+3\cdot 2)2 \ +144$

16

$4(x+3)-(6x+2)$

$3\ 5\cdot 3\ \ -3$

9

$(2\ 3)\ 3$

$8\ x\ 3\ (2x)\ 2$

$2x$

$(16\ y\ 0)2\ y\ \ -2$

441

21

490

9x 16

---

3 x 4

121 b 2 1 + b

6 24 + 7 54 − 12 6

---

21 6

−8 3 625 4

(13 q 3 + 2 q 2 − 3) − (6 q 2 + 5q − 3)

---

13 q 3 − 4 q 2 − 5q

(6 p 2 + 2p + 1) + (9 p 2 − 1)

$(n-2)(n^2-4n+4)$

$n^3-6n^2+12n-8$

$(a-2b)(2a+b)$

For the following exercises, factor the polynomial.

$16x^2-81$

$(4x+9)(4x-9)$

$y^2+12y+36$

$27c^3-1331$

$(3c-11)(9c^2+33c+121)$

$3x(x-6)^{-\frac{1}{4}}+2(x-6)^{\frac{3}{4}}$

For the following exercises, simplify the expression.

$\frac{2z^2+7z+3}{z^2-9}\cdot\frac{4z^2-15z+9}{4z^2-1}$

$4z-3\ 2z-1$

$x\ y\ +\ 2\ x$

$a\ 2b\ -\ 2b\ 9a\ 3a-2b\ 6a$

$3a+2b\ 3b$

Intro to Graphing (1.1)
By the end of this section, you will be able to:

- Plot points in a rectangular coordinate system
- Graph a linear equation by plotting points
- Find the x- and y-intercepts
- Identify other forms of a linear equation

This Module supports section **1.1 of Mat 1023**.
In case you missed something in class, or just want to review a specific topic covered in this Module, here is a list of topics covered:

1. Plot Points on a Coordinate System [link]
2. Graph Equations by Plotting Points[link]
3. Find x and y Intercepts[link]
4. Key Concepts [link]

Plot Points on a Rectangular Coordinate System

Just like maps use a grid system to identify locations, a grid system is used in algebra to show a relationship between two variables in a rectangular

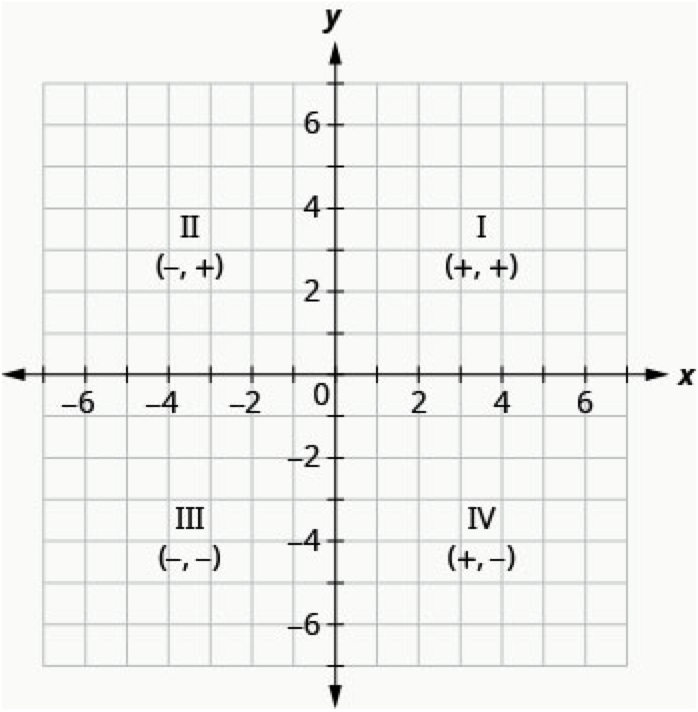
coordinate system. The rectangular coordinate system is also called the *xy*-plane or the “coordinate plane.”

The rectangular coordinate system is formed by two intersecting number lines, one horizontal and one vertical. The horizontal number line is called the *x*-axis. The vertical number line is called the *y*-axis. These axes divide a plane into four regions, called quadrants. The quadrants are identified by Roman numerals, beginning on the upper right and proceeding counterclockwise. See [link].

The signs of the *x*-coordinate and *y*-coordinate affect the location of the points. You may have noticed some patterns as you graphed the points in the previous example. We can summarize sign patterns of the quadrants in this way:

Quadrant I	Quadrant II	Quadrant III	Quadrant IV
(x,y)	(x,y)	(x,y)	(x,y)
$(+,+)$	$(-,+)$	$(-,-)$	$(+,-)$

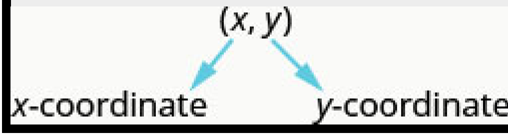
Quadrants



In the rectangular coordinate system, every point is represented by an **ordered pair**. The first number in the ordered pair is the *x*-coordinate of the point, and the second number is the *y*-coordinate of the point. The phrase “ordered pair” means that the order is important.

Ordered Pair

An **ordered pair**, (x,y) gives the coordinates of a point in a rectangular coordinate system. The first number is the *x*-coordinate. The second number is the *y*-coordinate.

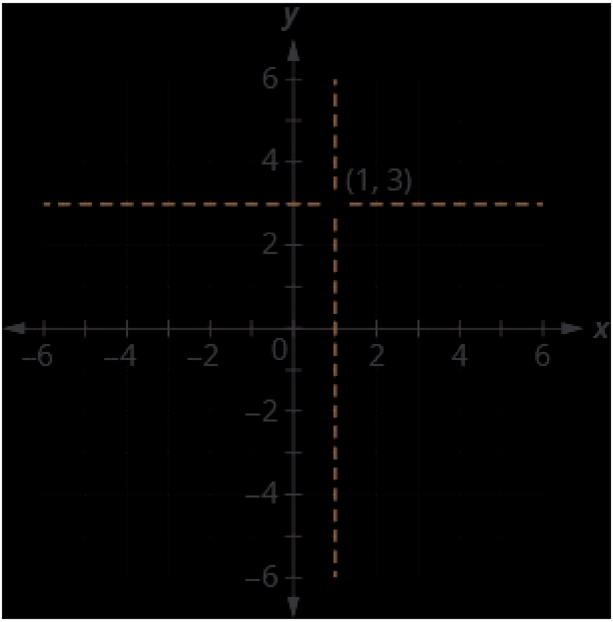


What is the ordered pair of the point where the axes cross? At that point both coordinates are zero, so its ordered pair is $(0,0)$. The point $(0,0)$ has a special name. It is called the **origin**.

The Origin

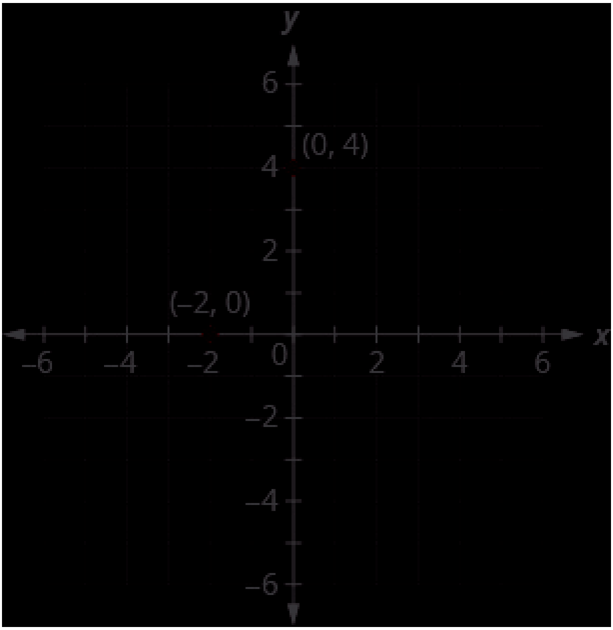
The point $(0,0)$ is called the **origin**. It is the point where the *x*-axis and *y*-axis intersect.

We use the coordinates to locate a point on the *xy*-plane. Let's plot the point $(1,3)$ as an example. First, locate 1 on the *x*-axis and lightly sketch a vertical line through $x=1$. Then, locate 3 on the *y*-axis and sketch a horizontal line through $y=3$. Now, find the point where these two lines meet—that is the point with coordinates $(1,3)$. See [link].



Notice that the vertical line through $x=1$ and the horizontal line through $y=3$ are not part of the graph. We just used them to help us locate the point $(1,3)$.

When one of the coordinate is zero, the point lies on one of the axes. In [link] the point $(0,4)$ is on the *y*-axis and the point $(-2,0)$ is on the *x*-axis.



Points on the Axes

Points with a *y*-coordinate equal to 0 are on the *x*-axis, and have coordinates (a,0).

Points with an *x*-coordinate equal to 0 are on the *y*-axis, and have coordinates (0,b).

Plot each point in the rectangular coordinate system and identify the quadrant in which the point is located:

ⓐ $(-5,4)$ ⓑ $(-3,-4)$ ⓒ $(2,-3)$ ⓓ $(0,-1)$

ⓔ $(3,52)$.

The first number of the coordinate pair is the *x*-coordinate, and the second number is the *y*-coordinate. To plot each point, sketch a vertical line through the *x*-coordinate and a horizontal line through the *y*-coordinate. Their intersection is the point.

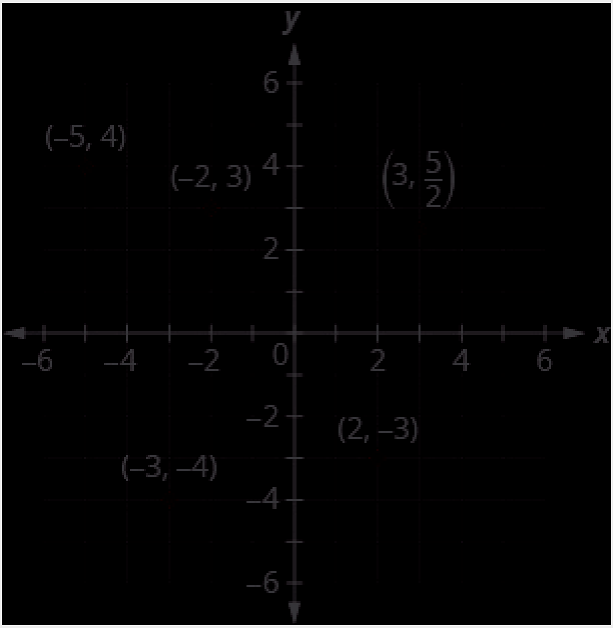
ⓐ Since $x=-5$, the point is to the left of the *y*-axis. Also, since $y=4$, the point is above the *x*-axis. The point $(-5,4)$ is in Quadrant II.

ⓑ Since $x=-3$, the point is to the left of the *y*-axis. Also, since $y=-4$, the point is below the *x*-axis. The point $(-3,-4)$ is in Quadrant III.

ⓒ Since $x=2$, the point is to the right of the *y*-axis. Since $y=-3$, the point is below the *x*-axis. The point $(2,-3)$ is in Quadrant IV.

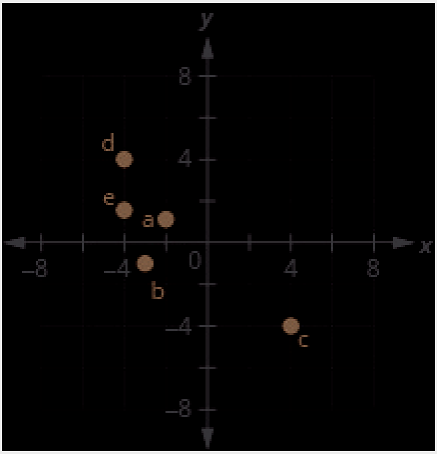
ⓓ Since $x=0$, the point whose coordinates are $(0,-1)$ is on the *y*-axis.

ⓔ Since $x=3$, the point is to the right of the *y*-axis. Since $y=52$, the point is above the *x*-axis. (It may be helpful to write 52 as a mixed number or decimal.) The point $(3,52)$ is in Quadrant I.



Plot each point in a rectangular coordinate system and identify the quadrant in which the point is located:

ⓐ $(-2,1)$ ⓑ $(-3,-1)$ ⓒ $(4,-4)$ ⓓ $(-4,4)$
ⓔ $(-4,32)$



Up to now, all the equations you have solved were equations with just one variable. In almost every case, when you solved the equation you got exactly one solution. But equations can have more than one variable. Equations with two variables may be of the form $Ax+By=C$. An equation of this form is called a **linear equation** in two variables.

Linear Equation
An equation of the form $Ax+By=C$, where A and B are not both zero, is called a **linear equation** in two variables.

Here is an example of a linear equation in two variables, *x* and *y*.

[redacted]

The equation $y = -3x + 5$ is also a linear equation. But it does not appear to be in the form $Ax + By = C$. We can use the Addition Property of Equality and rewrite it in $Ax + By = C$ form.

$y = -3x + 5$ Add to both sides.$y + 3x = -3x + 5 + 3x$ Simplify.$y + 3x = 5$ Use the Commutative Property to put it in $Ax + By = C$form.$3x + y = 5$

By rewriting $y = -3x + 5$ as $3x + y = 5$, we can easily see that it is a linear equation in two variables because it is of the form $Ax + By = C$. When an equation is in the form $Ax + By = C$, we say it is in **standard form of a linear equation**.

Standard Form of Linear Equation

A linear equation is in **standard form** when it is written $Ax + By = C$.

Most people prefer to have *A*, *B*, and *C* be integers and $A \geq 0$ when writing a linear equation in

standard form, although it is not strictly necessary.

Linear equations have infinitely many solutions. For every number that is substituted for *x* there is a corresponding *y* value. This pair of values is a **solution** to the linear equation and is represented by the ordered pair (x,y). When we substitute these values of *x* and *y* into the equation, the result is a true statement, because the value on the left side is equal to the value on the right side.

> Solution of a Linear Equation in Two Variables
> An ordered pair (x,y) is a **solution** of the linear equation$Ax+By=C$, if the equation is a true statement when the *x*- and *y*-values of the ordered pair are substituted into the equation.

Linear equations have infinitely many solutions. We can plot these solutions in the rectangular coordinate system. The points will line up perfectly in a straight line. We connect the points with a straight line to get the graph of the equation. We put arrows on the ends of each side of the line to indicate that the line continues in both directions.

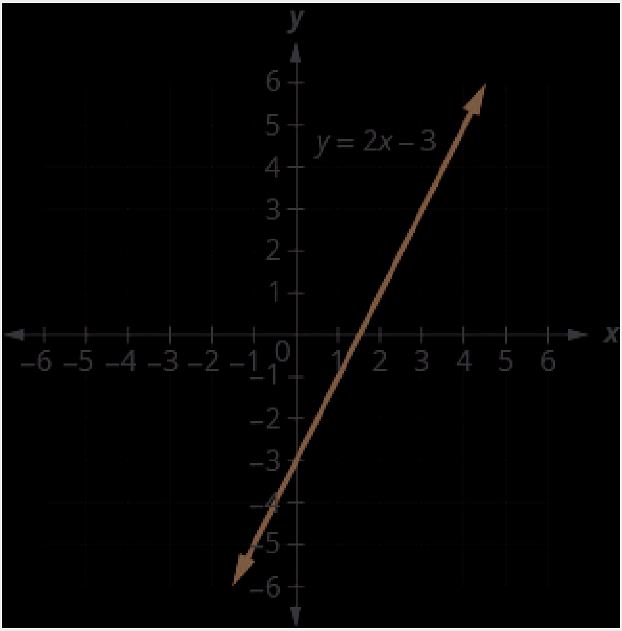
A graph is a visual representation of all the solutions of the equation. It is an example of the saying, “A

picture is worth a thousand words." The line shows you *all* the solutions to that equation. Every point on the line is a solution of the equation. And, every solution of this equation is on this line. This line is called the graph of the equation. Points *not* on the line are not solutions!

The graph of a linear equation $Ax+By=C$ is a straight line.

- Every point on the line is a solution of the equation.
- Every solution of this equation is a point on this line.

The graph of $y=2x-3$ is shown.



For each ordered pair, decide:

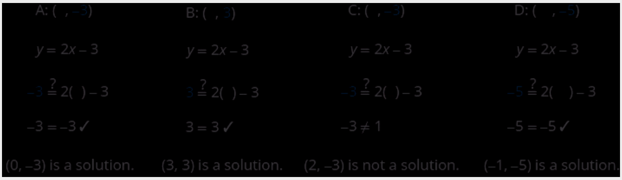
ⓐ Is the ordered pair a solution to the equation?

ⓑ Is the point on the line?

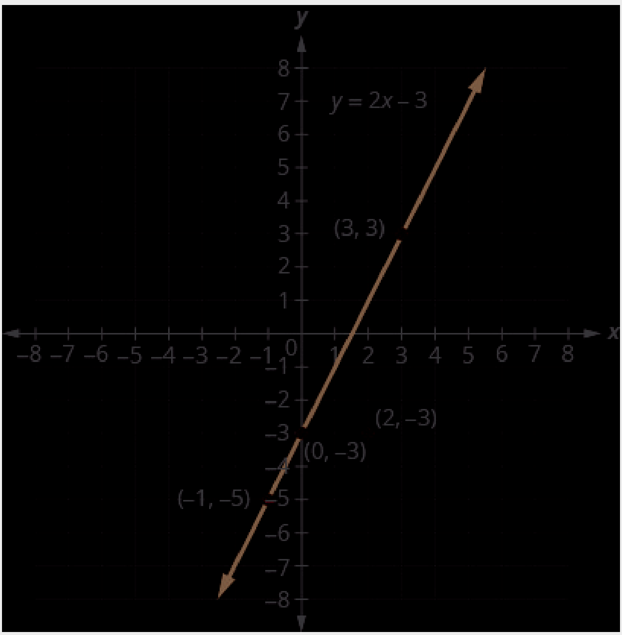
A: $(0,-3)$ B: $(3,3)$ C: $(2,-3)$ D: $(-1,-5)$

Substitute the *x*- and *y*-values into the equation to check if the ordered pair is a solution to the equation.

ⓐ



ⓑ Plot the points $(0,-3),(3,3),(2,-3)$, and $(-1,-5)$.



The points $(0,3),(3,-3)$, and $(-1,-5)$ are on the line $y=2x-3$, and the point $(2,-3)$ is not on the line.

The points that are solutions to $y=2x-3$ are on the line, but the point that is not a solution

is not on the line.

Graph an Equation by Plotting Points

There are several methods that can be used to graph a linear equation. The first method we will use is called plotting points, or the Point-Plotting Method. We find three points whose coordinates are solutions to the equation and then plot them in a rectangular coordinate system. By connecting these points in a line, we have the graph of the linear equation.

How to Graph a Linear Equation by Plotting Points

Graph the equation $y = 2x + 1$ by plotting points.

Step 1. Find three points whose coordinates are solutions to the equation.

You can choose any values for x or y.

In this case, since y is isolated on the left side of the equation, it is easier to choose values for x.

$y = 2x + 1$

$x =$
$y = 2x + 1$
$y = 2 \cdot \quad + 1$
$y = 0 + 1$
$y = 1$

$x =$
$y = 2x + 1$
$y = 2 \cdot \quad + 1$
$y = 2 + 1$
$y = 3$

$x =$
$y = 2x + 1$
$y = 2(\quad) + 1$
$y = -4 + 1$
$y = -3$

Organize the solutions in a table.

Put the three solutions in a table.

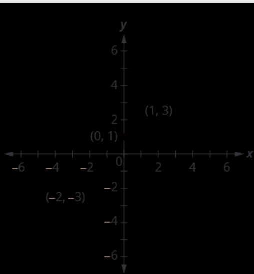
$y = 2x + 1$		
x	y	(x, y)
0	1	(0, 1)
1	3	(1, 3)
−2	−3	(−2, −3)

Step 2. Plot the points in a rectangular coordinate system.

Plot: (0, 1), (1, 3), (−2, −3).

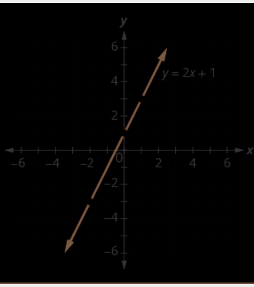
Check that the points line up. If they do not, carefully check your work!

Do the points line up? Yes, the points line up.



Step 3. Draw the line through the three points. Extend the line to fill the grid and put arrows on both ends of the line.

This line is the graph of $y = 2x + 1$.



Graph a linear equation by plotting points.

Find three points whose coordinates are solutions to the equation. Organize them in a table. Plot the points in a rectangular coordinate system. Check that the points line up. If they do not, carefully check your work. Draw the line through the three points. Extend the line to fill the grid and put arrows on both ends of the line.

It is true that it only takes two points to determine a line, but it is a good habit to use three points. If you only plot two points and one of them is incorrect, you can still draw a line but it will not represent the solutions to the equation. It will be the wrong line.

If you use three points, and one is incorrect, the points will not line up. This tells you something is wrong and you need to check your work. Look at the difference between these illustrations.

(a) (b)

When an equation includes a fraction as the coefficient of x, we can still substitute any numbers for *x*. But the arithmetic is easier if we make “good” choices for the values of *x*. This way we will avoid fractional answers, which are hard to graph precisely.

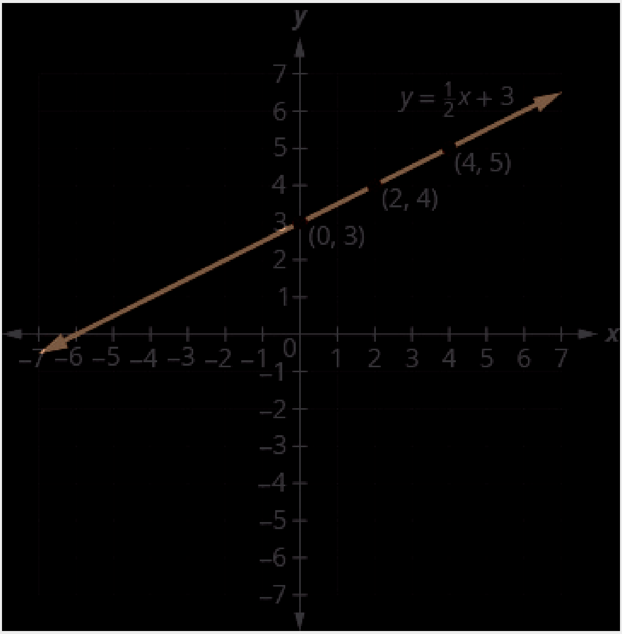
Graph the equation: $y=12x+3$.
Find three points that are solutions to the equation. Since this equation has the fraction 12 as a coefficient of *x*, we will choose values of *x* carefully. We will use zero as one choice and multiples of 2 for the other choices. Why are multiples of two a good choice for values of *x*? By choosing multiples of 2 the multiplication by 12 simplifies to a whole number

$$\begin{array}{lll} x = & x = & x = \\ y = \frac{1}{2}x + 3 & y = \frac{1}{2}x + 3 & y = \frac{1}{2}x + 3 \\ y = \frac{1}{2}(\ ) + 3 & y = \frac{1}{2}(\ ) + 3 & y = \frac{1}{2}(\ ) + 3 \\ y = 0 + 3 & y = 1 + 3 & y = 2 + 3 \\ y = 3 & y = 4 & y = 5 \end{array}$$

The points are shown in [link].

$y = 12x + 3$		
x	y	(x, y)
0	3	(0,3)
2	4	(2,4)
4	5	(4,5)

Plot the points, check that they line up, and draw the line.

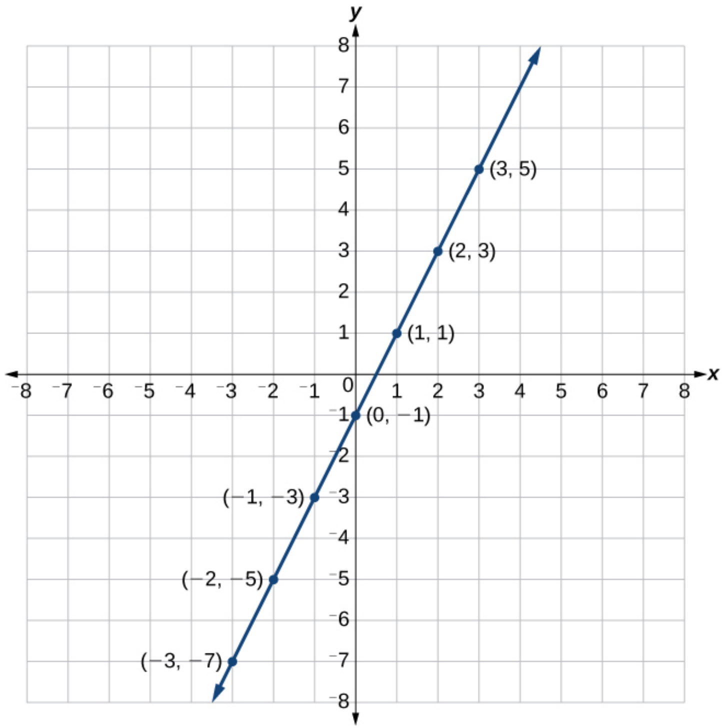


What if the graph is NOT Linear?

Suppose we want to graph the equation $y = 2x - 1$. We can begin by substituting a value for *x* into the equation and determining the resulting value of *y*. Each pair of *x*- and *y*-values is an ordered pair that can be plotted. [link] lists values of *x* from –3 to 3 and the resulting values for *y*.

x	$y=2x-1$	(x,y)
-3	$y=2(-3)-1=-7$	$(-3,-7)$
-2	$y=2(-2)-1=-5$	$(-2,-5)$
-1	$y=2(-1)-1=-3$	$(-1,-3)$
0	$y=2(0)-1=-1$	$(0,-1)$
1	$y=2(1)-1=1$	$(1,1)$
2	$y=2(2)-1=3$	$(2,3)$
3	$y=2(3)-1=5$	$(3,5)$

We can plot the points in the table. The points for this particular equation form a line, so we can connect them. See [link]. This is not true for all equations.



Note that the x-values chosen are arbitrary, regardless of the type of equation we are graphing. Of course, some situations may require particular values of x to be plotted in order to see a particular result. Otherwise, it is logical to choose values that can be calculated easily, and it is always a good idea to choose values that are both negative and positive. There is no rule dictating how many points to plot, although we need at least two to graph a line. Keep in mind, however, that the more points we plot, the more accurately we can sketch the graph.

Given an equation, graph by plotting points.

1. Make a table with one column labeled *x*, a second column labeled with the equation, and a third column listing the resulting ordered pairs.
2. Enter *x*-values down the first column using positive and negative values. Selecting the *x*-values in numerical order will make the graphing simpler.
3. Select *x*-values that will yield *y*-values with little effort, preferably ones that can be calculated mentally.
4. Plot the ordered pairs.
5. Connect the points if they form a line.

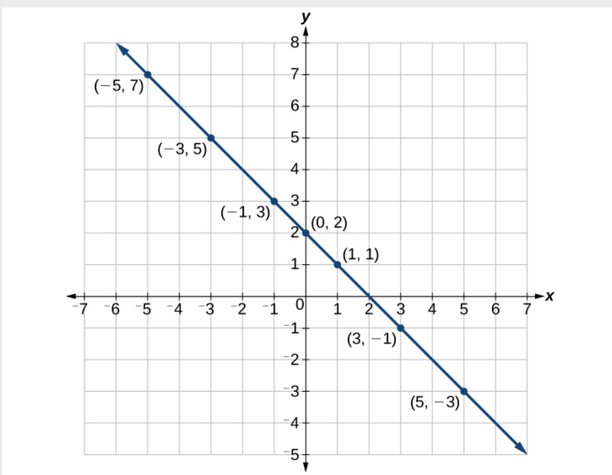
Graphing an Equation in Two Variables by Plotting Points

Graph the equation $y=-x+2$ by plotting points.

First, we construct a table similar to [link]. Choose *x* values and calculate *y*.

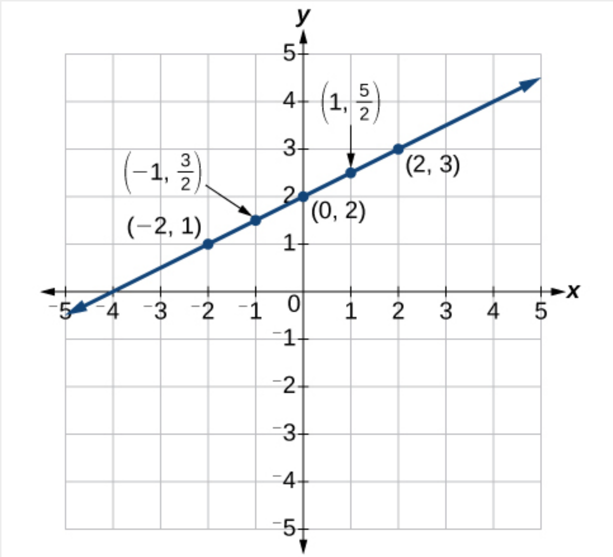
x	$y = -x + 2$	(x, y)
-5	$y = -(-5) + 2 = 7$	$(-5, 7)$
-3	$y = -(-3) + 2 = 5$	$(-3, 5)$
-1	$y = -(-1) + 2 = 3$	$(-1, 3)$
0	$y = -(0) + 2 = 2$	$(0, 2)$
1	$y = -(1) + 2 = 1$	$(1, 1)$
3	$y = -(3) + 2 = -1$	$(3, -1)$
5	$y = -(5) + 2 = -3$	$(5, -3)$

Now, plot the points. Connect them if they form a line. See [link]



Construct a table and graph the equation by plotting points: $y=\frac{1}{2}x+2$.

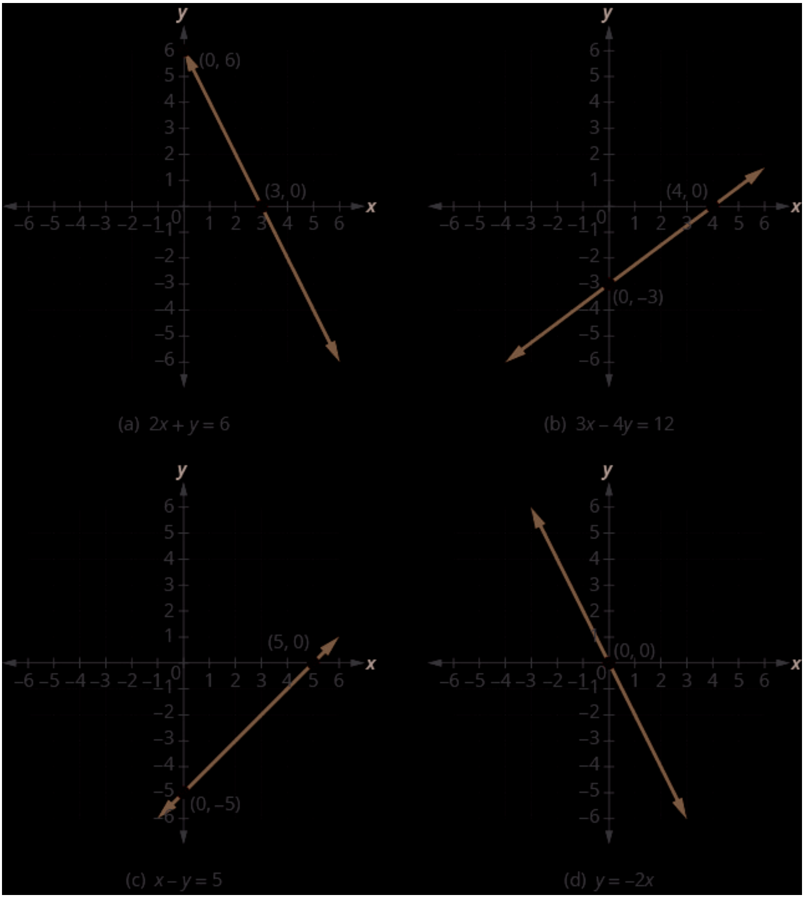
x	$y=\frac{1}{2}x+2$	(x,y)
-2	$y=\frac{1}{2}(-2)+2=1$	$(-2,1)$
-1	$y=\frac{1}{2}(-1)+2=\frac{3}{2}$	$\left(-1,\frac{3}{2}\right)$
0	$y=\frac{1}{2}(0)+2=2$	$(0,2)$
1	$y=\frac{1}{2}(1)+2=\frac{5}{2}$	$\left(1,\frac{5}{2}\right)$
2	$y=\frac{1}{2}(2)+2=3$	$(2,3)$



Find *x*- and *y*-intercepts

The **intercepts** of a graph are points at which the graph crosses the axes. The ***x*-intercept** is the point at which the graph crosses the *x*-axis. At this point, the *y*-coordinate is zero. The ***y*-intercept** is the point at which the graph crosses the *y*-axis. At this point, the *x*-coordinate is zero.

Let's look at the graphs of the lines.



First, notice where each of these lines crosses the *x*-axis. See [link].

Now, let’s look at the points where these lines cross the *y*-axis.

Figure	The line crosses the x-axis at:	Ordered pair for this point	The line crosses the y-axis at:	Ordered pair for this point
Figure (a)	3	(3,0)	6	(0,6)
Figure (b)	4	(4,0)	-3	$(0,-3)$
Figure (c)	5	(5,0)	-5	(0,5)
Figure (d)	0	(0,0)	0	(0,0)
General Figure	a	(a,0)	b	(0,b)

Do you see a pattern?

For each line, the y-coordinate of the point where the line crosses the x-axis is zero. The point where the line crosses the x-axis has the form (a,0) and is called the *x-intercept* of the line. The x-intercept occurs when y is zero.

In each line, the x-coordinate of the point where the line crosses the y-axis is zero. The point where the line crosses the y-axis has the form (0,b) and is called the *y-intercept* of the line. The y-intercept occurs when x is zero.

x-intercept and y-intercept of a Line

The x-intercept is the point (a,0) where the line crosses the x-axis.

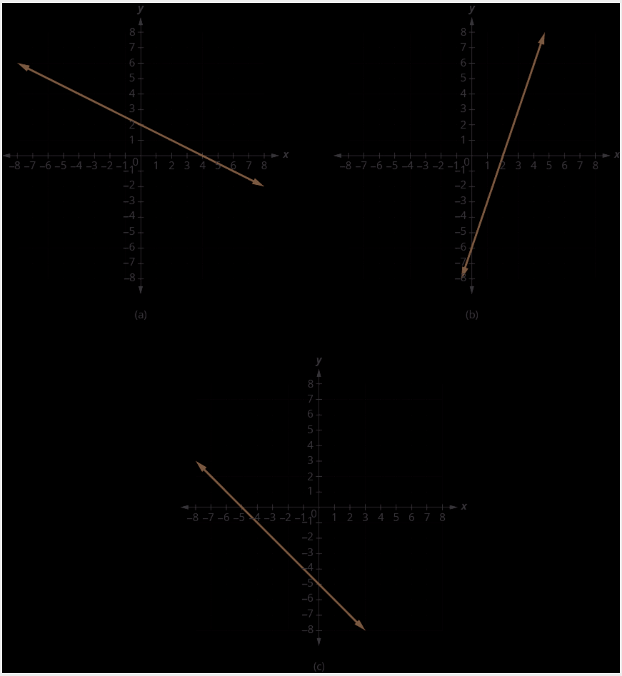
The y-intercept is the point (0,b) where the line

crosses the y-axis.

- The x-intercept occurs when y is zero.
- The y-intercept occurs when x is zero.

x	y
a	0
0	b

Find the x- and y-intercepts on each graph shown.



ⓐ The graph crosses the x-axis at the point (4,0). The x-intercept is (4,0).
The graph crosses the y-axis at the point (0,2). The y-intercept is (0,2).

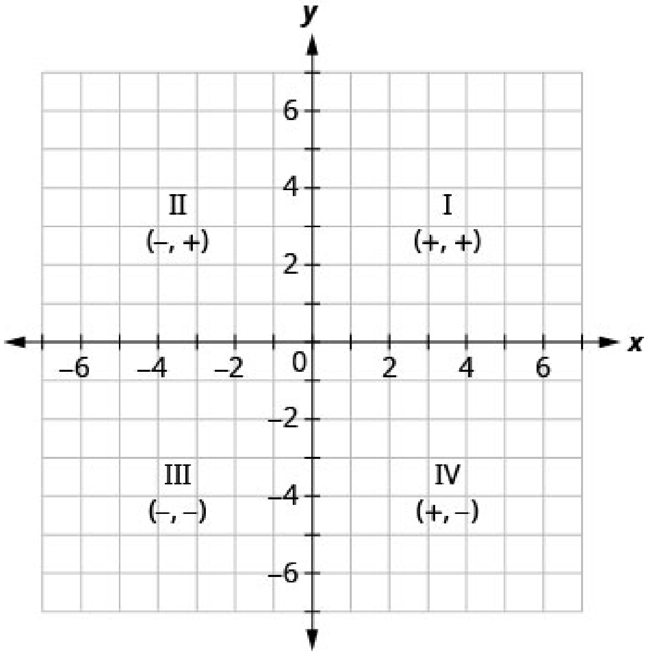
ⓑ The graph crosses the x-axis at the point (2,0). The x-intercept is (2,0).
The graph crosses the y-axis at the point $(0, -6)$. The y-intercept is $(0, -6)$.

ⓒ The graph crosses the x-axis at the point $(-5,0)$. The x-intercept is $(-5,0)$.
The graph crosses the y-axis at the point $(0, -5)$. The y-intercept is $(0, -5)$.

Key Concepts

- **Points on the Axes**
    - Points with a y-coordinate equal to 0 are on the x-axis, and have coordinates (a,0).
    - Points with an x-coordinate equal to 0 are on the y-axis, and have coordinates (0,b).
- **Quadrant**
  Quadrant IQuadrant IIQuadrant IIIQuadrant IV

(x,y)(x,y)(x,y)(x,y) $(+,+)(-,+)(-,-)(+,-)$



- **Graph of a Linear Equation:** The graph of a linear equation $Ax+By=C$ is a straight line. Every point on the line is a solution of the equation.
  Every solution of this equation is a point on this line.
- **How to graph a linear equation by plotting points.**

  Find three points whose coordinates are solutions to the equation. Organize them in a table. Plot the points in a rectangular coordinate system. Check that the points line up. If they do not, carefully check your work.

Draw the line through the three points. Extend the line to fill the grid and put arrows on both ends of the line.

- ***x*-intercept and *y*-intercept of a Line**
  - The *x*-intercept is the point (a,0) where the line crosses the *x*-axis.
  - The *y*-intercept is the point (0,b) where the line crosses the *y*-axis.

- The *x*-intercept occurs when *y* is zero.
- The *y*-intercept occurs when *x* is zero.

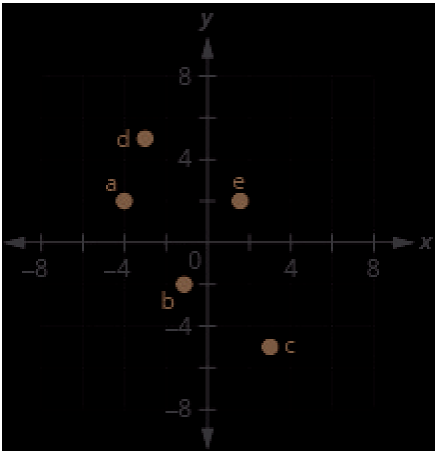
x	y
a	0
0	b

Practice Makes Perfect

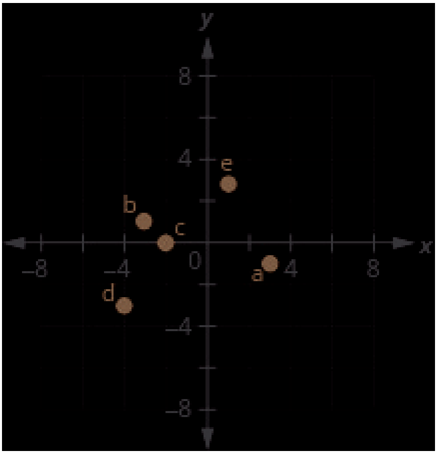
Plot Points in a Rectangular Coordinate System

In the following exercises, plot each point in a rectangular coordinate system and identify the quadrant in which the point is located.

ⓐ $(-4,2)$ ⓑ $(-1,-2)$ ⓒ $(3,-5)$ ⓓ $(-3,0)$ ⓔ $(53,2)$



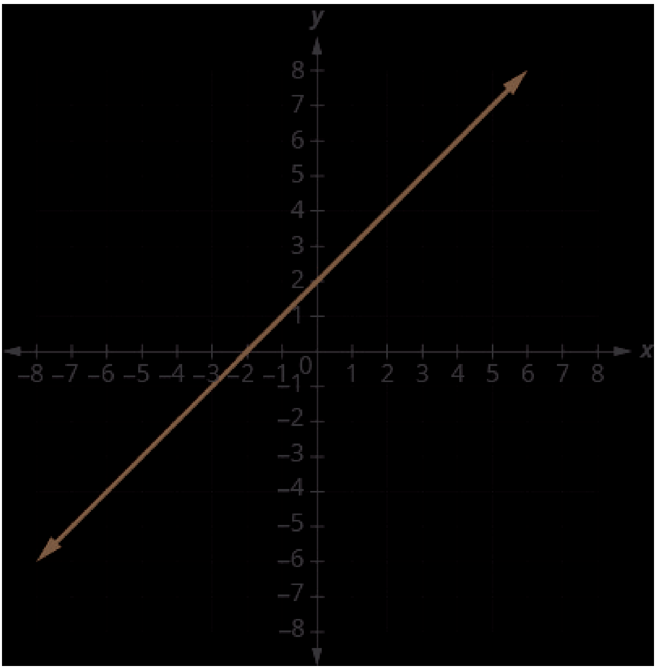
ⓐ $(3,-1)$ ⓑ $(-3,1)$ ⓒ $(-2,0)$ ⓓ $(-4,-3)$
ⓔ $(1,\frac{14}{5})$



In the following exercises, for each ordered pair, decide

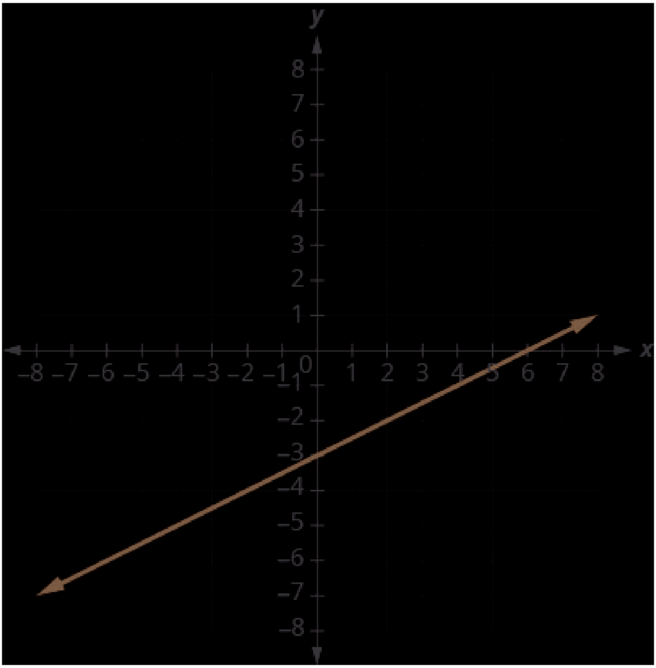
ⓐ is the ordered pair a solution to the equation? ⓑ is the point on the line?

$y = x + 2;$
A: $(0,2)$; B: $(1,2)$; C: $(-1,1)$; D: $(-3,-1)$.



ⓐ A: yes, B: no, C: yes, D: yes ⓑ A: yes, B: no, C: yes, D: yes

$y = 12x - 3;$
A: $(0,-3)$; B: $(2,-2)$; C: $(-2,-4)$; D: $(4,1)$

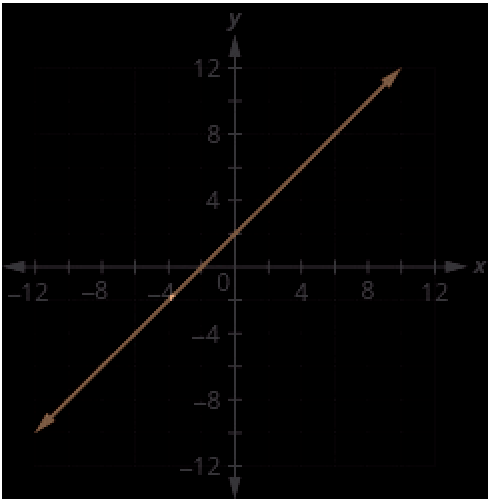


ⓐ A: yes, B: yes, C: yes, D: no ⓑ A: yes, B: yes, C: yes, D: no

Graph a Linear Equation by Plotting Points

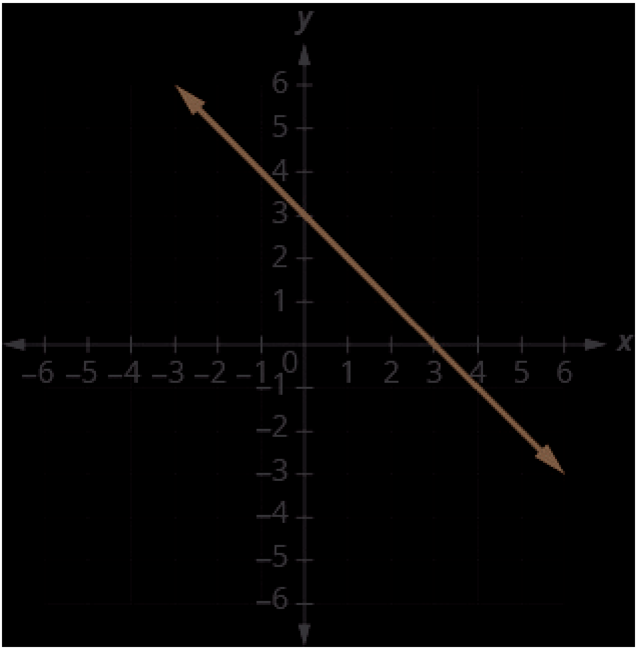
In the following exercises, graph by plotting points.

$y = x + 2$, given $x =$ -2, 0, 2

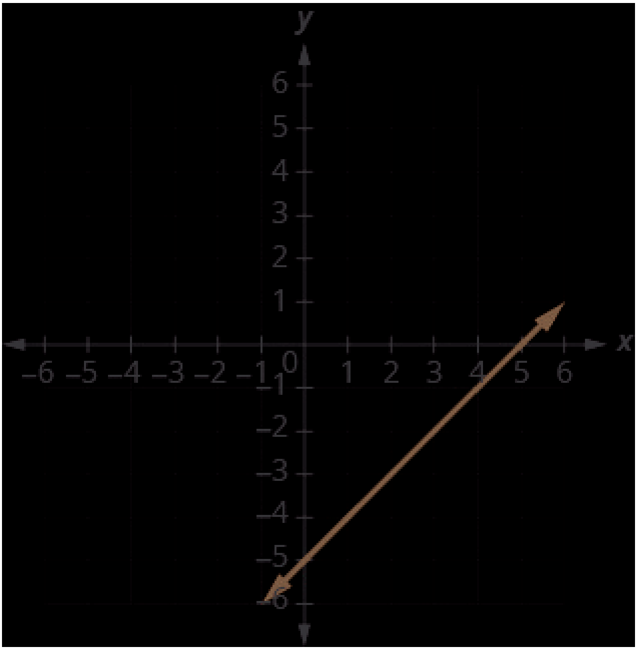


Find *x*- and *y*-Intercepts

In the following exercises, find the *x*- and *y*-intercepts on each graph.



(3,0),(0,3)



$(5,0), (0,-5)$

Glossary

horizontal line
: A horizontal line is the graph of an equation of the form $y = b$. The line passes through the *y*-axis at $(0,b)$.

intercepts of a line
: The points where a line crosses the *x*-axis and the *y*-axis are called the intercepts of the line.

linear equation

An equation of the form $Ax+By=C$, where *A* and *B* are not both zero, is called a linear equation in two variables.

ordered pair
: An ordered pair, (x,y) gives the coordinates of a point in a rectangular coordinate system. The first number is the *x*-coordinate. The second number is the *y*-coordinate.

origin
: The point $(0,0)$ is called the origin. It is the point where the *x*-axis and *y*-axis intersect.

solution of a linear equation in two variables
: An ordered pair (x,y) is a solution of the linear equation$Ax+By=C$, if the equation is a true statement when the *x*- and *y*-values of the ordered pair are substituted into the equation.

standard form of a linear equation
: A linear equation is in standard form when it is written $Ax+By=C$.

vertical line
: A vertical line is the graph of an equation of the form $x=a$. The line passes through the *x*-axis at $(a,0)$.

Solve Linear and Rational Equations (1.2)
By the end of this section, you will be able to:

- Solve linear equations using a general strategy
- Classify equations
- Solve equations with fraction or decimal coefficients

This Module supports section **1.2 of Mat 1023**.
In case you missed something in class, or just want to review a specific topic covered in this Module, here is a list of topics covered:

1. Solve Equations with a General Strategy [link]
2. Solve Equations with Fractions [link]
3. Solve Rational Equations [link]
4. Classify Equations [link]
5. Key Concepts [link]

Solve Linear Equations Using a General Strategy

Solving an equation is like discovering the answer to a puzzle. The purpose in solving an equation is to

find the value or values of the variable that makes it a true statement. Any value of the variable that makes the equation true is called a **solution** to the equation. It is the answer to the puzzle!

Solution of an Equation

A **solution** of an equation is a value of a variable that makes a true statement when substituted into the equation.

The **solution set** consists of all values that make the equation true.

To determine whether a number is a solution to an equation, we substitute the value for the variable in the equation. If the resulting equation is a true statement, then the number is a solution of the equation.

Determine Whether a Number is a Solution to an Equation.

Substitute the number for the variable in the equation. Simplify the expressions on both sides of the equation. Determine whether the resulting equation is true.

- If it is true, the number is a solution.
- If it is not true, the number is not a solution.

Determine whether the values are solutions to the equation: $5y+3=10y-4$.

ⓐ $y=35$ ⓑ $y=75$

Since a solution to an equation is a value of the variable that makes the equation true, begin by substituting the value of the solution for the variable.

ⓐ

[redacted]

[redacted]

Multiply.

[redacted]

Simplify.

[redacted]

Since $y = 35$ does not result in a true equation, $y = 35$ is not a solution to the equation $5y + 3 = 10y - 4$.

ⓑ

[redacted]

[redacted]

Multiply.

[redacted]

Simplify.

[redacted]

Since $y = 75$ results in a true equation, $y = 75$ is a solution to the equation $5y + 3 = 10y - 4$.

There are many types of equations that we will learn to solve. In this section we will focus on a **linear equation**.

Linear Equation

A **linear equation** is an equation in one variable that can be written, where *a* and *b* are real numbers and $a \neq 0$, as:

$ax + b = 0$

To solve a linear equation it is a good idea to have an overall strategy that can be used to solve any linear equation. In the next example, we will give the steps of a general strategy for solving any linear equation. Simplifying each side of the equation as much as possible first makes the rest of the steps easier.

Simplify each side of the equation as much as possible.

Use the Distributive Property to remove any parentheses.

Combine like terms. Collect all the variable terms on one side of the equation.

Use the Addition or Subtraction Property of

Equality. Collect all the constant terms on the other side of the equation.
Use the Addition or Subtraction Property of Equality. Make the coefficient of the variable term equal to 1.
Use the Multiplication or Division Property of Equality.
State the solution to the equation. Check the solution.
Substitute the solution into the original equation to make sure the result is a true statement.

How to Solve a Linear Equation Using a General Strategy

Solve: $7(n-3)-8=-15.$

Step 1. Simplify each side of the equation as much as possible.	Use the Distributive Property. Notice that each side of the equation is now simplified as much as possible.	$7(n-3)-8=-15$ $7n-21-8=-15$ $7n-29=-15$

Step 2. Collect all variable terms on one side of the equation.	Nothing to do—all n' s are on the left side.	

Step 3. Collect constant terms on the other side of the equation.	To get constants only on the right, add 29 to each side. Simplify.	$7n-29+29=-15+29$ $7n=14$

Step 4. Make the coefficient of the variable term equal to 1.	Divide each side by 7. Simplify.	$\frac{7n}{7} = \frac{14}{7}$ $n = 2$

Step 5. Check the solution.	 Let $n = 2$ Subtract.	**Check:** $7(n-3) - 8 = -15$ $7(2-3) - 8 \stackrel{?}{=} -15$ $7(-1) - 8 \stackrel{?}{=} -15$ $-7 - 8 \stackrel{?}{=} -15$ $-15 = -15$ ✓

Solve: $23(3m-6)=5-m$.

	$\frac{2}{3}(3m-6) = 5-m$
Distribute.	$2m - 4 = 5 - m$
Add m to both sides to get the variables only	$2m + m - 4 = 5 - m + m$

on the left.
Simplify.

[redacted]

Add 4 to both sides to get co[redacted] on the right.
Simplify.

[redacted]

Divide both sides by three.

[redacted]

Simplify.

[redacted]

Check:

[redacted]

Let m = 3.

[redacted]

[redacted]

We can solve equations by getting all the variable terms to either side of the equal sign. By collecting the variable terms on the side where the coefficient of the variable is larger, we avoid working with some negatives. This will be a good strategy when we solve inequalities later in this chapter. It also helps us prevent errors with negatives.

Solve: $4(x-1)-2=5(2x+3)+6.$

[redacted]

Distribute.

[redacted]

Combine

lik [redacted]
terms.
Subtract 4x from ea [redacted] to get the variables only on the right since $10 > 4$. Simplify.

[redacted]

Subtract 21 from ea [redacted] to get the constants on left. Simplify.

[redacted]

Divide both sid [redacted] 6. Simplify.

[redacted]

Check:

[redacted]

Let $x =$
$-92.$

[redacted]

[redacted]

[redacted]

[redacted]

[redacted]

Solve: $6(p-3)-7=5(4p+3)-12.$

$p=-2$

Solve: $10[3-8(2s-5)]=15(40-5s)$.

[redacted]

Simplify from the innermost parentheses first.

[redacted]

Combine like terms in the brackets.

[redacted]

Distribute.

[redacted]

Add 160s to both sides to get the variables to the right.

[redacted]

Simplify.

[redacted]

Subtract 600 from both sides

to [redacted] constants to the left.

Simplify.

[redacted]

Divide both sides by 85.

[redacted]

Simplify.

[redacted]

Check:

[redacted]

Let $s = -2$.

[redacted]

Solve: $12[1-5(4z-1)]=3(24+11z)$.

$z=0$

Solve Equations with Fractions

We could use the General Strategy to solve the next example. This method would work fine, but many students do not feel very confident when they see all those fractions. So, we are going to show an alternate method to solve equations with fractions. This alternate method eliminates the fractions.

We will apply the Multiplication Property of Equality and multiply both sides of an equation by the least common denominator (LCD) of all the fractions in the equation. The result of this operation will be a new equation, equivalent to the first, but without fractions. This process is called *clearing* the equation of fractions.

Find the least common denominator (LCD) of *all* the fractions and decimals (in fraction form) in the equation. Multiply both sides of the equation by that LCD. This clears the fractions and decimals. Solve using the General Strategy for Solving Linear Equations.

How to Solve Equations with Fraction or Decimal Coefficients

Solve: 112x + 56 = 34.

Step 1. Find the least common denominator of *all* the fractions and decimals in the equation.	What is the LCD of $\frac{1}{12}$, $\frac{5}{6}$, and $\frac{3}{4}$?	$\frac{1}{12}x + \frac{5}{6} = \frac{3}{4}$ LCD = 12

Step 2. Multiply both sides of the equation by that LCD. This clears the fractions and decimals.	Multiply both sides of the equation by the LCD, 12.	$12\left(\frac{1}{12}x + \frac{5}{6}\right) = 12\left(\frac{3}{4}\right)$
	Use the Distributive Property.	$12 \cdot \frac{1}{12}x + 12 \cdot \frac{5}{6} = 12 \cdot \frac{3}{4}$
	Simplify—and notice, no more fractions!	$x + 10 = 9$

Step 3. Solve using the General Strategy for Solving Linear Equations.	To isolate the variable term, subtract 10. Simplify.	$x+10-10=9-10$ $x=-1$ **Check:** $\frac{1}{12}x+\frac{5}{6}=\frac{3}{4}$ $\frac{1}{12}(-1)+\frac{5}{6}\stackrel{?}{=}\frac{3}{4}$ $-\frac{1}{12}+\frac{5}{6}\stackrel{?}{=}\frac{3}{4}$ $-\frac{1}{12}+\frac{10}{12}\stackrel{?}{=}\frac{9}{12}$ $\frac{9}{12}=\frac{9}{12}$ ✓

Solve: $14x+12=58$.

$x=12$

Notice in the previous example, once we cleared the equation of fractions, the equation was like those we solved earlier in this chapter. We changed the problem to one we already knew how to solve. We then used the General Strategy for Solving Linear Equations.

Solve: $12(y-5)=14(y-1)$.

[redacted]

Distribute.

[redacted]

Simplify.

[redacted]

Multiply by the LCD, four.

[redacted]

Distribute.

[redacted]

Simplify.

[redacted]

Collect the variables to the left.

[redacted]

Simplify.

[redacted]

Collect the constants to the right.

[redacted]

Simplify.

[redacted]

An alternate way to solve this equation is to clear the fractions without distributing first. If you multiply the factors correctly, this method will be easier.

[redacted]

Multiply by the LCD, 4.

[redacted]

Multiply four times the fractions.

[redacted]

Distribute.

[redacted]

Collect the variables to the left.

[redacted]

Simplify.

[redacted]

Collect the constants to [redacted]

Simplify.

[redacted]

Check:

[redacted]

Let $y = 9$.

[redacted]

Finish the check on your own.

When you multiply both sides of an equation by the LCD of the fractions, make sure you multiply each term by the LCD—even if it does not contain a fraction.

Solve: $4q + 32 + 6 = 3q + 54$

[redacted]

Multiply both sides by the LC [redacted]

Distribute.

[redacted]

Simplify.

[redacted]

[redacted]

[redacted]

Collect the variables to the left.

[redacted]

Simplify.

[redacted]

Collect the constants to the right.

[redacted]

Simplify.

[redacted]

Divide both

sid[redacted]

Simplify.

[redacted]

Check:

[redacted]

Let $q = -5$.

[redacted]

Finish the check on your own.

Solve Rational Equations

After defining the terms ‘expression’ and ‘equation’ earlier, we have used them throughout this book. We have *simplified* many kinds of *expressions* and *solved* many kinds of *equations*..

A **rational equation** is an equation that contains

at least one rational expression where the variable appears in at least one of the denominators.

You must make sure to know the difference between rational expressions and rational equations. The equation contains an equal sign.

Rational ExpressionRational Equation 18x+12 y +6y2−36 1n−3+1n+4 18x+12=14 y +6y2−36=y+1 1n−3+1n+4=15n2+n−12

We have already solved linear equations that contained fractions. We found the LCD of all the fractions in the equation and then multiplied both sides of the equation by the LCD to “clear” the fractions.

However, since the original equation may have a variable in a denominator, we must be careful that we don’t end up with a solution that would make a denominator equal to zero. So before we begin solving a rational equation, we examine it first to find the values that would make any denominators zero. That way, when we solve a rational equation we will know if there are any algebraic solutions we must discard.

Note any value of the variable that would make

any denominator zero. Find the least common denominator of *all* denominators in the equation. Clear the fractions by multiplying both sides of the equation by the LCD. Solve the resulting equation. Check:

- If any values found in Step 1 are algebraic solutions, discard them.
- Check any remaining solutions in the original equation.

How to Solve a Rational Equation

Solve: $1x+13=56$.

Step 1. Note any value of the variable that would make any denominator zero.	If $x=0$, then $\frac{1}{x}$ is undefined. So we'll write $x \neq 0$ next to the equation.	$\frac{1}{x}+\frac{1}{3}=\frac{5}{6}, x \neq 0$

Step 2. Find the least common denominator of *all* denominators in the equation.	Find the LCD of $\frac{1}{x}$, $\frac{1}{3}$, and $\frac{5}{6}$.	The LCD is $6x$.

Step 3. Clear the fractions by multiplying both sides of the equation by the LCD.	Multiply both sides of the equation by the LCD, $6x$.	$6x \cdot \left(\frac{1}{x}+\frac{1}{3}\right)=6x \cdot \left(\frac{5}{6}\right)$
	Use the Distributive Property.	$6x \cdot \frac{1}{x}+6x \cdot \frac{1}{3}=6x \cdot \left(\frac{5}{6}\right)$
	Simplify – and notice, no more fractions!	$6+2x=5x$

Step 4. Solve the resulting equation.	Simplify.	$6=3x$ $2=x$

Step 5. Check. • If any values found in Step 1 are algebraic solutions, discard them. • Check any remaining solutions in the original equation.	We did not get 0 as an algebraic solution. We substitute $x=2$ into the original equation.	$\frac{1}{x}+\frac{1}{3}=\frac{5}{6}$ $\frac{1}{2}+\frac{1}{3}\overset{?}{=}\frac{5}{6}$ $\frac{3}{6}+\frac{2}{6}\overset{?}{=}\frac{5}{6}$ $\frac{5}{6}=\frac{5}{6}\checkmark$ The solution is $x=2$.

How to Solve a Rational Equation using the Zero Product Property

Solve: $1-5y=-6y2.$

[redacted]

Note any value of the variable that would ma[redacted] any denominator zero.

Find the least common denominator of all denominators in the equation. The LCD is *y*2.

Clear the fractions by multiplying both sides of [redacted] the equation by the LCD.

Distribute.

[redacted]

Multiply.

[redacted]

Solve the resulting equation. First wr[redacted] equation in standard form.

Factor.

[redacted]

Use the Zero Product Property.

[redacted]

Solve.

[redacted]

Check.
We did not get 0 as an algebraic solution.

[redacted]

The solution is $y = 2$, $y = 3$.

A common mistake made when solving rational equations involves finding the LCD when one of the denominators is a binomial—two terms added or

subtracted—such as $(x+1)$. Always consider a binomial as an individual factor—the terms cannot be separated. For example, suppose a problem has three terms and the denominators are x, $x-1$, and $3x-3$. First, factor all denominators. We then have x, $(x-1)$, and $3(x-1)$ as the denominators. (Note the parentheses placed around the second denominator.) Only the last two denominators have a common factor of $(x-1)$. The x in the first denominator is separate from the x in the $(x-1)$ denominators. An effective way to remember this is to write factored and binomial denominators in parentheses, and consider each parentheses as a separate unit or a separate factor. The LCD in this instance is found by multiplying together the x, one factor of $(x-1)$, and the 3. Thus, the LCD is the following:

$$x(x-1)3=3x(x-1)$$

So, both sides of the equation would be multiplied by $3x(x-1)$. Leave the LCD in factored form, as this makes it easier to see how each denominator in the problem cancels out. Another example is a problem with two denominators, such as x and x^2+2x. Once the second denominator is factored as $x^2+2x=x(x+2)$, there is a common factor of *x* in both denominators and the LCD is $x(x+2)$. In the next example, the last denominators is a difference of squares. Remember to factor it first to find the LCD.

Solve: $2x+2+4x-2=x-1x2-4$.

[redacted]

Note any value of the variable
th[redacted]
denominator
zero.

Find the least common
denominator of all
denominators
in the equation.
The LCD is $(x+2)(x-2)$.

Clear the fractions by
multiplying
bo[redacted]
equation by the
LCD.

Distribute.

[redacted]

Remove common factors.

[redacted]

Simplify.

[redacted]

Distribute.

[redacted]

Solve.

[redacted]

[redacted]

[redacted]

Check:
We did not get 2 or -2 as algebraic solutions.

[redacted]

The solution is $x = -1$.

Solve: $5y+3+2y-3=5y2-9$.

$y=2$

In the next example, the first denominator is a trinomial. Remember to factor it first to find the LCD. Refer back to Module 5 to see an example of factoring trinomials.

Solve: $43x2 - 10x + 3 + 33x2 + 2x - 1 = 2x2 - 2x - 3$.

[redacted]

Factor all the denominators, so we ca [redacted] the variable that would make any denominator zero.

$x \neq -1, x \neq 13, x \neq 3$

Find the least common denominator. The LCD is $(3x - 1)(x + 1)(x - 3)$.

Clear the fractions.

[redacted]

Simplify.

[redacted]

Distribute.

[redacted]

Simplify.

[redacted]

The only algebraic solution was $x=3$, but we said that $x=3$ would make a denominator equal to zero. The algebraic solution is an extraneous solution.

There is no solution to this equation.

Classify Equations

Whether or not an equation is true depends on the value of the variable. The equation $7x+8=-13$ is true when we replace the variable, x, with the value -3, but not true when we replace x with any other value. An equation like this is called a **conditional equation**.

Conditional Equation
An equation that is true for one or more values of the variable and false for all other values of the variable is a **conditional equation**.

For example, if we are to solve the equation $5x + 2 = 3x - 6$, we have the following:

$5x + 2 \;=\; 3x - 6 \; 2x \;=\; -8 \; x \;=\; -4$

The solution set consists of one number: $\{ -4 \}$. It is the only solution and, therefore, we have solved a conditional equation.

Now let's consider the equation $7y + 14 = 7(y + 2)$. Do you recognize that the left side and the right side are equivalent? Let's see what happens when we solve for *y*.

Solve:

[redacted]

Distribute.

[redacted]

Subtract 7y to each side to get the y’s to one side.

[redacted]

Simplify—the *y*’s are eliminated.

[redacted]

But $14=14$ is true.

This means that the equation $7y+14=7(y+2)$ is true for any value of *y*. We say the solution to the equation is all of the real numbers. An equation that is true for any value of the variable is called an **identity**.

> Identity
>
> An equation that is true for any value of the variable is called an **identity**.
> The solution of an identity is all real numbers.

What happens when we solve the equation $-8z=-8z+9$?

[redacted]

Add 8z to both sides to leave the constant alone on [redacted]

Simplify—the z’s are eliminated.

[redacted]

But $0 \neq 9$.

Solving the equation $-8z = -8z + 9$ led to the false statement $0 = 9$. The equation $-8z = -8z + 9$ will not be true for any value of *z*. It has no solution. An equation that has no solution, or that is false for all values of the variable, is called a **Inconsistent**.

> An equation that is false for all values of the variable is called a **Inconsistent**.
> An Inconsistent Equation has no solution.

For example, if we are to solve $5x - 15 = 5(x - 4)$, we have the following:

$5x - 15 = 5x - 20$ $5x - 15 - 5x = 5x - 20 - 5x$ Subtract 5x from both sides. $-15 \neq -20$ False statement

Indeed, $-15 \neq -20$. There is no solution because this is an inconsistent equation.

The next few examples will ask us to classify an equation as conditional, an identity, or as inconsistent.

Classify the equation as a conditional equation, an identity, or inconsistent and then state the solution: $6(2n-1)+3=2n-8+5(2n+1)$.

[redacted]

Distribute.

[redacted]

Combine like terms.

[redacted]

Subtract 12n from each side to get the *n*’s to

on[redacted]
Simplify.

[redacted]

This is a true statement.

The equation is an identity.
The solution is all real numbers.

Classify the equation as a conditional equation, an identity, or a inconsistent and then state the solution: $8+3(a-4)=0$.

[redacted]

Distribute.

[redacted]

Combine like terms.

[redacted]

Add 4 to both sides.

[redacted]

Simplify.

[redacted]

Divide.

[redacted]

Simplify.

[redacted]

The equation is true when $a = 43$. This is a conditional equation. The solution is $a = 43$.

Classify the equation as a conditional equation, an identity, or inconsistent and then state the solution: $5m + 3(9 + 3m) = 2(7m - 11)$.

[redacted]

Distribute.

[redacted]

Combine like terms.

[redacted]

Subtract 14m from both sides.

[redacted]

Simplify.

[redacted]

But $27 \neq -22$.

The equation is a ~~inconsistent.~~
It has no solution.

We summarize the methods for classifying equations in the table.

Type of	**What happens**	**Solution**

equation	**when you solve it?**	
Conditional Equation	True for one or more values of the variables and false for all other values	One or more values
Identity	True for any value of the variable	All real numbers
Inconsistent	False for all values of the variable	No solution

Key Concepts

- **How to determine whether a number is a solution to an equation**

  Substitute the number in for the variable in the equation. Simplify the expressions on both sides of the equation. Determine whether the resulting equation is true.
  If it is true, the number is a solution.
  If it is not true, the number is not a solution.

- **How to Solve Linear Equations Using a General Strategy**

Simplify each side of the equation as much as possible.
Use the Distributive Property to remove any parentheses.
Combine like terms. Collect all the variable terms on one side of the equation.
Use the Addition or Subtraction Property of Equality. Collect all the constant terms on the other side of the equation.
Use the Addition or Subtraction Property of Equality. Make the coefficient of the variable term equal to 1.
Use the Multiplication or Division Property of Equality.
State the solution to the equation. Check the solution.
Substitute the solution into the original equation to make sure the result is a true statement.

- **How to Solve Equations with Fraction or Decimal Coefficients**

  Find the least common denominator (LCD) of *all* the fractions and decimals (in fraction form) in the equation. Multiply both sides of the equation by that LCD. This clears the fractions and decimals. Solve using the General Strategy for Solving Linear Equations.

Practice Makes Perfect

Solve Equations Using the General Strategy

In the following exercises, determine whether the given values are solutions to the equation.

$6y+10=12y$

ⓐ $y=53$
ⓑ $y=-12$

ⓐ yes ⓑ no

$4x+9=8x$

ⓐ $x=-78$
ⓑ $x=94$

In the following exercises, solve each linear equation.

$15(y-9)=-60$

$y=5$

$3(10-2x)+54=0$

$x = 14$

$23(9c - 3) = 22$

$c = 4$

$15(15c + 10) = c + 7$

$c = 52$

$4(p - 4) - (p + 7) = 5(p - 3)$

$p = -4$

$3[-9 + 8(4h - 3)] = 2(5 - 12h) - 19$

$h = 34$

$5[2(m + 4) + 8(m - 7)] = 2[3(5 + m) - (21 - 3m)]$

$m = 6$

Classify Equations

In the following exercises, classify each equation as a conditional equation, an identity, or an inconsistent and then state the solution.

$18(5j-1)+29=47$

---

conditional equation;$j=25$

$7v+42=11(3v+8)-2(13v-1)$

---

inconsistent ; no solution

$9(14d+9)+4d=13(10d+6)+3$

---

identity; all real numbers

Solve Equations with Fraction or Decimal Coefficients

In the following exercises, solve each equation with fraction coefficients.

$14x-12=-34$

$x = -1$

$12a + 38 = 34$

$a = 34$

$2 = 13x - 12x + 23x$

$x = 4$

$14(p - 7) = 13(p + 5)$

$p = -41$

$4n + 84 = n3$

$n = -3$

$3x + 42 + 1 = 5x + 108$

$x = -2$

Explain why you should simplify both sides of an equation as much as possible before collecting the variable terms to one side and the constant terms to the other side.

Glossary

conditional equation
: An equation that is true for one or more values of the variable and false for all other values of the variable is a conditional equation.

contradiction
: An equation that is false for all values of the variable is called a contradiction. A contradiction has no solution.

identity
: An equation that is true for any value of the variable is called an Identity. The solution of an identity is all real numbers.

linear equation
: A linear equation is an equation in one variable that can be written, where a and b are real numbers and $a \neq 0$, as $ax + b = 0$.

solution of an equation
: A solution of an equation is a value of a variable that makes a true statement when

substituted into the equation.

Applications and Modeling (1.3)
By the end of this section, you will be able to:

- Use a problem solving strategy for word problems
- Solve number word problems
- Solve percent applications
- Solve simple interest applications

This Module supports section **1.3 of Mat 1023**.
In case you missed something in class, or just want to review a specific topic covered in this Module, here is a list of topics covered:

1. Use a Problem Solving Strategy [link]
2. Solve Percent Applications [link]
3. Solve Simple Interest Applications [link]
4. Geometry Applications [link]
5. Solve Formulas for One Variable[link]
6. Key Concepts [link]

Have you ever had any negative experiences in the past with word problems? When we feel we have no control, and continue repeating negative thoughts, we set up barriers to success. Realize that your negative experiences with word problems are in

your past. To move forward you need to calm your fears and change your negative feelings.

Use a Problem Solving Strategy for Word Problems

Now that we can solve equations, we are ready to apply our new skills to word problems. We will develop a strategy we can use to solve any word problem successfully.

Use a Problem Solving Strategy for word problems.

Read the problem. Make sure all the words and ideas are understood. **Identify** what you are looking for. **Name** what you are looking for. Choose a variable to represent that quantity. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebra equation. **Solve** the equation using proper algebra techniques. **Check** the answer in the problem to make sure it makes sense. **Answer** the question with a complete sentence.

When dealing with real-world applications, there are certain expressions that we can translate directly into math. [link] lists some common verbal expressions and their equivalent mathematical expressions.

Verbal	Translation to Math Operations
One number exceeds another by *a*	$x,\ x+a$
Twice a number	$2x$
One number is *a* more than another number	$x,\ x+a$
One number is *a* less than twice another number	$x,\ 2x-a$
The product of a number and *a*, decreased by *b*	$ax-b$
The quotient of a number and the number plus *a* is three times the number	$\frac{x}{x+a}=3x$
The product of three times a number and the number decreased by *b* is *c*	$3x(x-b)=c$

Normal yearly snowfall at the local ski resort is 12 inches more than twice the amount it received last season. The normal yearly snowfall is 62 inches. What was the snowfall last season at the ski resort?

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	What was the snowfall last season?
Step 3. Name what we are looking for and choose a variable to represent it.	Let $s =$ the snowfall last season.
Step 4. Translate. Restate the problem in on[e sentence with all] the important information.	The normal snowfall was twelve more than twice the amount last year
Translate into an equation.	$62 = 2s + 12$
Step 5. Solve the equation.	$62 = 2s + 12$
Subtract 12 from each	

side.

[redacted]

Simplify.

[redacted]

Divide each side by two.

[redacted]

Simplify.

[redacted]

Step 6. Check: First, is our answer reasonable? Yes, having 25 inches of snow seems OK. The problem says the normal snowfall is twelve inches more than twice the number of last season. Twice 25 is 50 and 12 more than that is 62.

Step 7. Answer the question.	The snowfall last season was 25 inches.

A married couple together earns \$110,000 a year. The wife earns \$16,000 less than twice

what her husband earns. What does the husband earn?

Step 1. Readthe problem. Step 2. Identifywhat you are looking for. How much does the husband earn? Step 3. Name. Choose a variable to representLeth = the amount the husband earns.the amount the husband earns.The wife earns \$16,000 less than twice that.Step 4. Translate.Restate the problem in one sentencewith all the important information.Translate into an equation.2h − 16,000 = the amount the wife earnsTogether the husband and wife earn \$110,000. $h + 2h - 16{,}000 = 110{,}000$ Step 5. Solvethe equation. Combine like terms. Add 16,000 to both sides and simplify. Divide each side by three. $h + 2h - 16{,}000 = 110{,}000$ $3h - 16{,}000 = 110{,}000$ $3h = 126{,}000$ $h = 42{,}000$ \$42,000 amount husband earns $2h - 16{,}000$ amount wife earns $2(42{,}000) - 16{,}000$ $84{,}000 - 16{,}000$ $68{,}000$ Step 6. Check:If the wife earns \$68,000 and the husbandearns \$42,000, is that \$110,000? Yes! Step 7. Answerthe question.The husband earns \$42,000 a year.

Access this online resource for additional instruction and practice with using a problem

solving strategy.

- Begining Arithmetic Problems

Solve Percent Applications

How many cents are in one dollar? There are 100 cents in a dollar. How many years are in a century? There are 100 years in a century. Does this give you a clue about what the word “percent” means? It is really two words, “per cent,” and means per one hundred. A **percent** is a ratio whose denominator is 100. We use the percent symbol %, to show percent.

A percent is a ratio whose denominator is 100.

Similarly, 25% means a ratio of 25100,3% means a ratio of 3100 and 100% means a ratio of 100100. In words, "one hundred percent" means the total 100% is 100100, and since 100100=1, we see that 100% means 1 whole.

There are several methods to solve percent

equations. In algebra, it is easiest if we just translate English sentences into algebraic equations and then solve the equations. Be sure to change the given percent to a decimal before you use it in the equation.

To Review: To convert a percent number to a decimal number, we move the decimal point two places to the left and remove the % sign. (Sometimes the decimal point does not appear in the percent number, but just like we can think of the integer 6 as 6.0, we can think of 6% as 6.0%.) Notice that we may need to add zeros in front of the number when moving the decimal to the left.

Write the percent as a ratio with the denominator 100. Convert the fraction to a decimal by dividing the numerator by the denominator.

Convert each percent to a decimal:

ⓐ135%
ⓑ12.5%

Solution

ⓐ		
	135%	
Write as a ratio with denominator 100.	$\frac{135}{100}$	
Change the fraction to a decimal by dividing the numerator by the denominator.	1.35	

ⓑ		
	12.5%	
Write as a ratio with denominator 100.	$\frac{12.5}{100}$	
Change the fraction to a decimal by dividing the numerator by the denominator.	0.125	

Translate and solve:

ⓐ What number is 45% of 84?

ⓑ 8.5% of what amount is $4.76?
ⓒ 168 is what percent of 112?

ⓐ

[redacted]

Translate into algebra. Let $n =$ the number.

[redacted]

Multiply.

[redacted]

37.8 is 45% of 84.

ⓑ

[redacted]

Translate. Let n = the amount.	[redacted]	
Multiply.	[redacted]	
Divide both sides by 0.085 and simplify.	[redacted]	
		8.5% of \$56 is \$4.76

ⓒ

We are asked to find percent, so we must have our	[redacted]	

result in percent form.		
Translate into algebra. Let $p =$ the percent.	[redacted]	
Multiply.	[redacted]	
Divide both sides by 112 and simplify.	[redacted]	
Convert to percent.	[redacted]	
		168 is 150% of 112.

Now that we have a problem solving strategy to refer to, and have practiced solving basic percent equations, we are ready to solve percent applications. Be sure to ask yourself if your final answer makes sense—since many of the applications we will solve involve everyday situations, you can

rely on your own experience.

The label on Audrey's yogurt said that one serving provided 12 grams of protein, which is 24% of the recommended daily amount. What is the total recommended daily amount of protein?

What are you asked to find?	What total amount of protein is recommended?
Choose a variable to represent it.	Let $a =$ total amount of protein.
Write a sentence that gives the information to find it.	12 g is 24% of the total amount
Translate into an equation.	$12 = 0.24 \cdot a$
Solve.	$50 = a$
Check: Does this make	

sense? Yes, 24% is about 14 of the total and 12 is about 14 of 50.	
Write a complete sentence to answer the question.	The amount of protein that is recommended is 50 g.

Veronica is planning to make muffins from a mix. The package says each muffin will be 240 calories and 60 calories will be from fat. What percent of the total calories is from fat?

What are you asked to find?	What percent of the total calories is fat?
Choose a variable to represent it.	Let p = percent of fat.
Write a sentence that gives the inf	What percent of 240 is 60?

Translate the sentence into an equation.	[redacted]
Multiply.	[redacted]
Divide both sides by 240.	[redacted]
Put in percent form.	[redacted]
Check: does this make sense? Yes, 25% is one-fourth; 60 is one-fourth of 240. So, 25% makes sense.	
Write a complete sentence to answer the question.	Of the total calories in each muffin, 25% is fat.

When you buy an item on sale, the original price has been discounted by some dollar amount. The **discount rate**, usually given as a percent, is used to determine the amount of the discount. To determine the **amount of discount**, we multiply the discount rate by the original price.

The price a retailer pays for an item is called the **original cost**. The retailer then adds a **mark-up** to the original cost to get the **list price**, the price he sells the item for. The mark-up is usually calculated as a percent of the original cost. To determine the amount of mark-up, multiply the mark-up rate by the original cost.

Discount
amount of discount = discount rate·original pricesale price = original amount–discount price
The sale price should always be less than the original price.

Mark-up
amount of mark-up = mark-up rate·original pricelist price = original cost + mark-up
The list price should always be more than the original cost.

Liam's art gallery bought a painting at an original cost of $750. Liam marked the price up 40%. Find ⓐ the amount of mark-up and

ⓑ the list price of the painting.

ⓐ

Identify what you are asked to find, and choose a variable to represent it.	What is the amount of mark-up? Let m = the amount of mark up.
Write a sentence that gives the information to find it.	The mark-up is 40% of the $750 original cost
Translate into an equation.	$m = 0.40 \times 750$
Solve the equation.	$m = 300$
Write a complete sentence.	The mark-up on the painting was $300.

ⓑ

Identify what you are asked to find, and choose a variable to represent it.	What is the list price? Let p = the list price.
Write a sentence that gives the information to find it.	The list price is original cost plus the mark-up
Translate into an equation.	$p = 750 + 300$
Solve the equation.	$p = 1{,}050$
Check.	Is the list price more than the original cost? Is \$1,050 more than \$750? Yes.
Write a complete sentence.	The list price of the painting was \$1,050.

Solve Simple Interest Applications

Interest is a part of our daily lives. From the interest earned on our savings to the interest we pay on a car loan or credit card debt, we all have some

experience with interest in our lives.

The amount of money you initially deposit into a bank is called the **principal**, *P*, and the bank pays you **interest**, *I*. When you take out a loan, you pay interest on the amount you borrow, also called the principal.

In either case, the interest is computed as a certain percent of the principal, called the **rate of interest**, *r*. The rate of interest is usually expressed as a percent per year, and is calculated by using the decimal equivalent of the percent. The variable *t*, (for time) represents the number of years the money is saved or borrowed.

Interest is calculated as simple interest or compound interest. Here we will use simple interest.

Simple Interest

If an amount of money, *P*, called the principal, is invested or borrowed for a period of *t* years at an annual interest rate *r*, the amount of interest, *I*, earned or paid is

$$I = \text{interest} I = Prt \text{where} P = \text{principal} r = \text{rate} t = \text{time}$$

Interest earned or paid according to this formula is called **simple interest**.

The formula we use to calculate interest is $I=Prt$. To use the formula we substitute in the values for variables that are given, and then solve for the unknown variable. It may be helpful to organize the information in a chart.

Areli invested a principal of \$950 in her bank account that earned simple interest at an interest rate of 3%. How much interest did she earn in five years?
$I=?P=\$950r=3\%t=5\text{years}$ Identify what you are asked to find, and choose aWhat is the simple interest?variable to represent it.Let$I=$interest.Write the formula.$I=Prt$Substitute in the given information.$I=(950)(0.03)(5)$Simplify.$I=142.5$Check.Is \$142.50 a reasonable amount of interest on \$950? Yes.Write a complete sentence.The interest is \$142.50.

There may be times when we know the amount of interest earned on a given principal over a certain

length of time, but we do not know the rate.

Hang borrowed $7,500 from her parents to pay her tuition. In five years, she paid them $1,500 interest in addition to the $7,500 she borrowed. What was the rate of simple interest?

$I = \$1500 P = \$7500 r = ? t = 5 years$

Identify what you are asked to find, and chooseWhat is the rate of simple interest?a variable to represent it.Write the formula.Substitute in the given information.Multiply.Divide.Change to percent form.Let$r$ = rate of interest.$I = Prt 1,500 = (7,500)r(5) 1,500 = 37,500r 0.04 =$ $(7,500)(0.04)(5) 1,500 = 1,500$✓Write a complete sentence.The rate of interest was 4%.

In the next example, we are asked to find the principal—the amount borrowed.

Sean's new car loan statement said he would pay $4,866,25 in interest from a simple interest rate of 8.5% over five years. How much did he borrow to buy his new car?
I = 4,866.25P = ?r = 8.5%t = 5years Identify what you are asked to find,What is the amount borrowed (the principal)?and choose a variable to represent it.Write the formula.Substitute in the given information.Multiply.Divide.LetP = principal borrowed.I = Prt4,866.25 = P(0.085)(5)4,866.25 = 0.425P11,450 = PCheck.I = Prt4,866.25 = ?(11,450)(0.085)(5)4,866.25 = 4,866.25✓Write a complete sentence.The principal was $11,450.

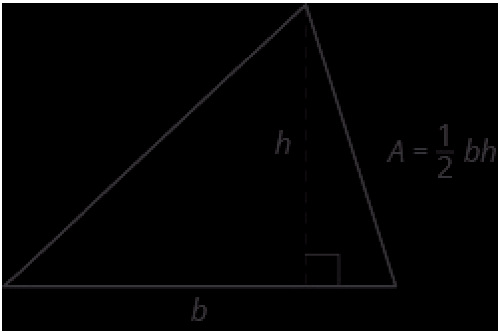
Geometry applications

In this objective we will use some common geometry formulas. We will adapt our problem solving strategy so that we can solve geometry applications. The geometry formula will name the variables and give us the equation to solve.

In addition, since these applications will all involve shapes of some sort, most people find it helpful to draw a figure and label it with the given information. We will include this in the first step of the problem solving strategy for geometry applications.

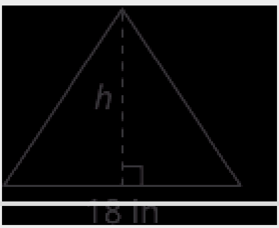
When we solve geometry applications, we often have to use some of the properties of the figures. We will review those properties as needed.

The next example involves the area of a triangle. The area of a triangle is one-half the base times the height. We can write this as A=12bh, where b = length of the base and h = height.



The area of a triangular painting is 126 square inches. The base is 18 inches. What is the height?

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	height of a triangle
Step 3. Name.	
Choose a variable to represent it.	Let $h =$ the height.
Draw the figure and label it with the given information.	Area $=$ 126 sq. in.



Step 4. Translate.	
Write the appropriate formula.	$A = 12bh$
Substitute in the given information.	$126 = 12 \cdot 18 \cdot h$
Step 5. Solve the equation.	$126 = 9h$
Divide both sides by 9.	$14 = h$
Step 6. Check.	

$A = 12bh 126 \overset{?}{=} 12 \cdot 18 \cdot 14 126 = 126 ✓$

Step 7. Answer the question.	The height of the triangle is 14 inches.

The area of a triangular church window is 90 square meters. The base of the window is 15 meters. What is the window's height?

The window's height is 12 meters.

In the next example, we will work with a right triangle. To solve for the measure of each angle, we need to use two triangle properties. In any triangle, the sum of the measures of the angles is $180°$. We can write this as a formula: $m\angle A + m\angle B + m\angle C = 180$. Also, since the triangle is a right triangle, we remember that a right triangle has one $90°$ angle.

Here, we will have to define one angle in terms of another. We will wait to draw the figure until we write expressions for all the angles we are looking for.

The measure of one angle of a right triangle is 40 degrees more than the measure of the smallest angle. Find the measures of all three angles.

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	the measures of all three angles
Step 3. Name. Choose a variable to represent it.	Let $a = 1\text{st angle}$. $a + 40 = 2\text{nd angle}$ $90 = 3\text{rd angle}$ (the right angle)
Draw the figure and label it with the given information.	

Step 4. Translate.	
Write the appropriate formula.	$m\angle A + m\angle B + m\angle C = 180$
Substitute into the formula.	$a + (a + 40) + 90 = 180$

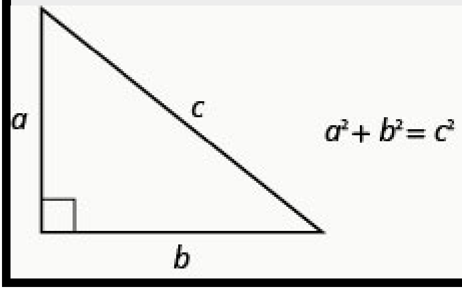
Step 5. Solve the equation.	$2a + 130 = 180$ $2a = 50$ $a = 25$ first angle $a + 40$ second angle $25 + 40$ 65 90 third angle
Step 6. Check. $25 + 65 + 90 = ?$ $180180 = 180✓$	
Step 7. Answer the question.	The three angles measure $25°, 65°$, and $90°$.

The next example uses another important geometry formula. The **Pythagorean Theorem** tells how the lengths of the three sides of a right triangle relate to each other. Writing the formula in every exercise and saying it aloud as you write it may help you memorize the Pythagorean Theorem.

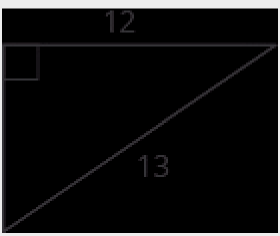
The Pythagorean Theorem

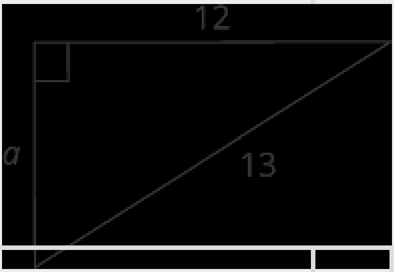
In any right triangle, where *a* and *b* are the lengths of the legs, and *c* is the length of the hypotenuse,

the sum of the squares of the lengths of the two legs equals the square of the length of the hypotenuse.



Use the Pythagorean Theorem to find the length of the other leg in

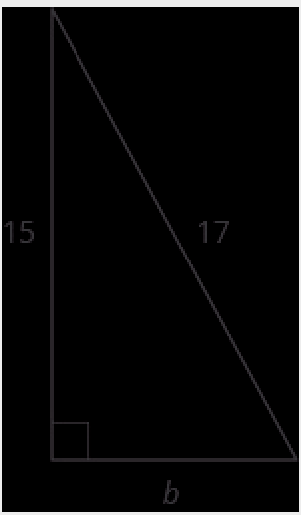


Step 1. Read the problem.	
Step 2. Identify what you are looking for.	the length of the leg of the triangle
Step 3. Name. Choose a variable to represent it. Label side a. 	Let a = the leg of the triangle.
Step 4. Translate. Write the appropriate formula. Substitute.	$a2+b2=c2a2+122=132$
Step 5. Solve the equation. Isolate the variable term. Use the definition of square root. Simplify.	$a2+144=169$ $a2=25a=25\ a=5$
Step 6. Check.	

$$5^2 + 12^2 \stackrel{?}{=} 13^2$$
$$25 + 144 \stackrel{?}{=} 169$$
$$169 = 169 \checkmark$$

Step 7. Answer the question.	The length of the leg is 5.

Use the Pythagorean Theorem to find the length of the leg in the figure.



The length of the leg is 8.

The next example is about the perimeter of a rectangle. Since the perimeter is just the distance around the rectangle, we find the sum of the lengths of its four sides—the sum of two lengths and two widths. We can write is as $P = 2L + 2W$ where L is the length and W is the width. To solve the example, we will need to define the length in terms of the width.

The length of a rectangle is six centimeters more than twice the width. The perimeter is 96 centimeters. Find the length and width.

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	the length and the width
Step 3. Name. Choose a variable to represent the width. The length is six more than twice the width.	Let $w =$ width. $2w + 6 =$ length

P = 96 cm

Step 4. Translate.
Write the appropriate formula.

$P = 2L + 2W$

Substitute in the given information.

$96 = 2(2w + 6) + 2w$

Step 5. Solve the equation.

$96 = 4w + 12 + 2w$

$96 = 6w + 12$

$84 = 6w$

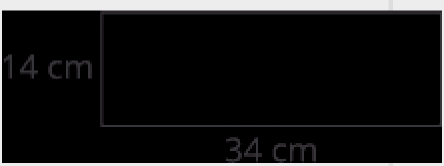
$14 = w$ (width)

$2w + 6$ (length)

$2(14) + 6$

34 The length is 34 cm.

Step 6. Check.



$P = 2L + 2W 96 = ?$

$2 \cdot 34 + 2 \cdot 14 96 = 96 \checkmark$

Step 7. Answer the question.	The length is 34 cm and the width is 14 cm.

The next example is about the perimeter of a triangle. Since the perimeter is just the distance around the triangle, we find the sum of the lengths of its three sides. We can write this as $P = a + b + c$, where *a*, *b*, and *c* are the lengths of the sides.

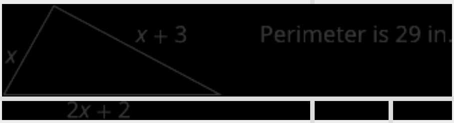
One side of a triangle is three inches more than the first side. The third side is two inches more than twice the first. The perimeter is 29 inches. Find the length of the three sides of the triangle.

Step 1. Read the	

problem.

Step 2. Identify what we are looking for. — the lengths of the three sides of a triangle

Step 3. Name. Choose a variable to represent the length of the first side. — Let x = length of 1st side. $x+3$ = length of 2nd side $2x+2$ = length of 3rd side



Step 4. Translate.
Write the appropriate formula.

$$P = a + b + c$$

Substitute in the given information.

$$29 = x + (x+3) + (2x+2)$$

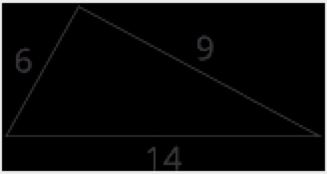
Step 5. Solve the equation.

$$29 = 4x + 5$$
$$24 = 4x$$
$$6 = x \quad \text{length of first side}$$

$$x + 3 \quad \text{length of second side}$$
$$6 + 3$$
$$9$$

$$2x + 2 \quad \text{length of second side}$$
$$2 \cdot 6 + 2$$
$$14$$

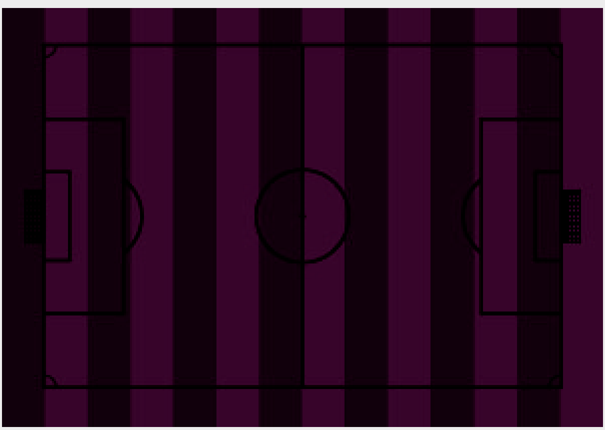
Step 6. Check.



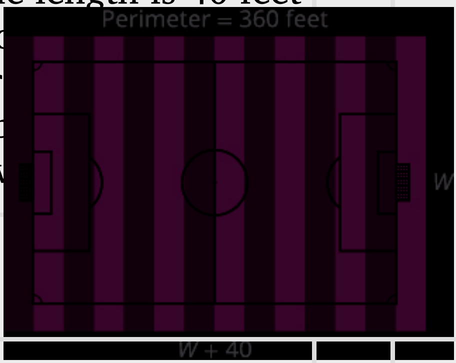
$29 \stackrel{?}{=} 6 + 9 + 14$
$29 = 29 \checkmark$

Step 7. Answer the question.	The lengths of the sides of the triangle are 6, 9, and 14 inches.

The perimeter of a rectangular soccer field is 360 feet. The length is 40 feet more than the width. Find the length and width.



Step 1. Read the problem.	
Step 2. Identify what we are looking for.	the length and width of the soccer field
Step 3. Name. Choose a variable to represent it. The length is 40 feet mo Dr lab giv	Let w = width. $w + 40 =$ length



Step 4. Translate. Write the appropriate formula and substitute.	$P = 2L + 2W$ $360 = 2(w + 40) + 2w$
Step 5. Solve the equation.	$360 = 2w + 80 + 2w$ $360 = 4w + 80$ $280 = 4w$ $70 = w$ the width of the field $w + 40$ the length of the field $70 + 40$ 110
Step 6. Check.	

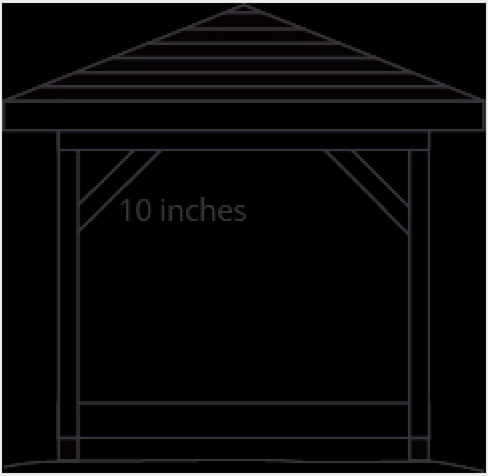
$P = 2L + 2W \quad 360 = ?$
$2(110) + 2(70) \quad 360 = 360 \checkmark$

Step 7. Answer the question.

The length of the soccer field is 110 feet and the width is 70 feet.

Applications of these geometric properties can be found in many everyday situations as shown in the next example.

Kelvin is building a gazebo and wants to brace each corner by placing a 10” piece of wood diagonally as shown.



How far from the corner should he fasten the wood if wants the distances from the corner to be equal? Approximate to the nearest tenth of an inch.

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	the distance from the corner that the bracket should be attached
Step 3. Name. Choose a variable to represent it.	Let $x =$ the distance from the corner.

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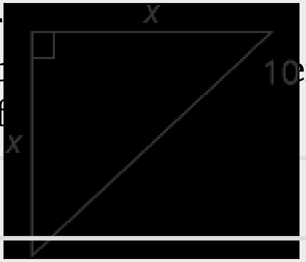


Step 4. Translate.

Write the appropriate formula and substitute.	$a2 + b2 = c2$ $x2 + x2 = 102$
Step 5. Solve the equation. Isolate the variable. Use the definition of square root. Simplify. Approximate to the nearest tenth.	$2x2 = 100 x2 = 50 x = 50 x \approx 7.1$

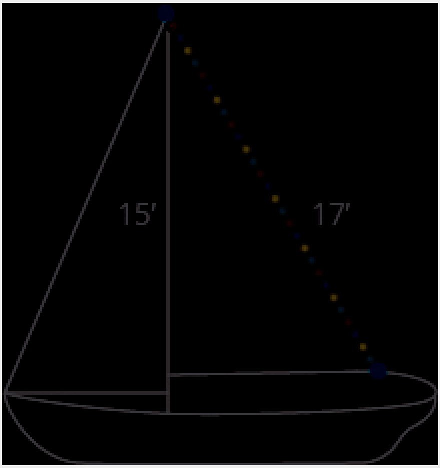
Step 6. Check.

$a2 + b2 = c2 (7.1)2 + (7.1)2 \approx 102$ Ycs.

Step 7. Answer the question.	Kelvin should fasten each piece of wood approximately 7.1” from the corner.

Randy wants to attach a 17-foot string of lights to the top of the 15 foot mast of his sailboat, as shown in the figure. How far from the base

of the mast should he attach the end of the light string?



He should attach the lights 8 feet from the base of the mast.

Access this online resource for additional instruction and practice with solving for a variable in literal equations.

- Solving Literal Equations

Solve Formulas for One Variable

We have all probably worked with some geometric formulas in our study of mathematics. Formulas are used in so many fields, it is important to recognize formulas and be able to manipulate them easily.

It is often helpful to solve a formula for a specific variable. If you need to put a formula in a spreadsheet, it is not unusual to have to solve it for a specific variable first. We isolate that variable on one side of the equals sign with a coefficient of one and all other variables and constants are on the other side of the equal sign.

Geometric formulas often need to be solved for another variable, too. The formula $V=13\pi r2h$ is used to find the volume of a right circular cone when given the radius of the base and height. In the next example, we will solve this formula for the height.

Solve the formula $V=13\pi r2h$ for *h*.

Write the formula.	[redacted]
Remove the fraction on the right.	[redacted]
Simplify.	[redacted]
Divide both sides by πr2.	[redacted]

We could now use this formula to find the height of a right circular cone when we know the volume and the radius of the base, by using the formula h = 3Vπr2.

In the sciences, we often need to change temperature from Fahrenheit to Celsius or vice versa. If you travel in a foreign country, you may want to change the Celsius temperature to the more familiar Fahrenheit temperature.

Solve the formula $C=59(F-32)$ for *F*.

Write the formula.	$C=\frac{5}{9}(F-32)$
Remove the fraction on the right.	$\frac{9}{5}C=\frac{9}{5}\cdot\frac{5}{9}(F-32)$
Simplify.	$\frac{9}{5}C=(F-32)$
Add 32 to both sides.	$\frac{9}{5}C+32=F$

We can now use the formula $F=95C+32$ to find the Fahrenheit temperature when we know the Celsius temperature.

The next example uses the formula for the surface area of a right cylinder.

Solve the formula $S = 2\pi r2 + 2\pi rh$ for *h*.

Write the formula.

[redacted]

Isolate the *h* term by subtracting $2\pi r2$ from each side.

[redacted]

Simplify.

[redacted]

Solve for *h* by dividing both sides by $2\pi r$.

[redacted]

Simplify.

[redacted]

Key Concepts

- **How To Use a Problem Solving Strategy for Word Problems**

  Read the problem. Make sure all the words and ideas are understood. **Identify** what you are looking for. **Name** what you are looking for. Choose a variable to represent that quantity. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebra equation. **Solve** the equation using proper algebra techniques.

  Check the answer in the problem to make sure it makes sense. **Answer** the question with a complete sentence.

- **How To Find Percent Change**

  Find the amount of change
  $\text{change} = \text{new amount} - \text{original amount}$ Find what percent the amount of change is of the original amount.
  change is what percent of the original amount?

- **Discount**

amount of discount = discount rate·original pricesale price = original amount − discount

- **Mark-up**
  amount of mark-up = mark-up rate·original costlist price = original cost + mark up
- **Simple Interest**
  If an amount of money, P, called the principal, is invested or borrowed for a period of t years at an annual interest rate r, the amount of interest, I, earned or paid is:
  I = interestI = PrtwhereP = principalr = ratet = time

Practice Makes Perfect

Philip pays \$1,620 in rent every month. This amount is \$120 more than twice what his brother Paul pays for rent. How much does Paul pay for rent?

\$750

Marc just bought an SUV for \$54,000. This is \$7,400 less than twice what his wife paid for her car last year. How much did his wife pay for her car?

Solve Percent Applications

In the following exercises, solve.

Geneva treated her parents to dinner at their favorite restaurant. The bill was $74.25. Geneva wants to leave 16% of the total bill as a tip. How much should the tip be?

$11.88

One serving of oatmeal has 8 grams of fiber, which is 33% of the recommended daily amount. What is the total recommended daily amount of fiber?

24.2 g

Emma gets paid $3,000 per month. She pays $750 a month for rent. What percent of her monthly pay goes to rent?

25%

In the following exercises, solve.

Tamanika received a raise in her hourly pay, from \$15.50 to \$17.36. Find the percent change.

12%

Annual student fees at the University of California rose from about \$4,000 in 2000 to about \$12,000 in 2010. Find the percent change.

200%

In the following exercises, find ⓐ the amount of discount and ⓑ the sale price.

Janelle bought a beach chair on sale at 60% off. The original price was \$44.95.

ⓐ \$26.97 ⓑ \$17.98

In the following exercises, find ⓐ the amount of discount and ⓑ the discount rate (Round to the nearest tenth of a percent if needed.)

Larry and Donna bought a sofa at the sale price

of $1,344. The original price of the sofa was $1,920.

ⓐ $576 ⓑ 30%

In the following exercises, find ⓐ the amount of the mark-up and ⓑ the list price.

Daria bought a bracelet at original cost $16 to sell in her handicraft store. She marked the price up 45%. What was the list price of the bracelet?

ⓐ $7.20 ⓑ $23.20

Solve Simple Interest Applications

In the following exercises, solve.

Casey deposited $1,450 in a bank account that earned simple interest at an interest rate of 4%. How much interest was earned in two years?

$116

Hilaria borrowed $8,000 from her grandfather

to pay for college. Five years later, she paid him back the $8,000, plus $1,200 interest. What was the rate of simple interest?

3%

In 10 years, a bank account that paid 5.25% simple interest earned $18,375 interest. What was the principal of the account?

$35,000

Joshua’s computer loan statement said he would pay $1,244.34 in simple interest for a three-year loan at 12.4%. How much did Joshua borrow to buy the computer?

$3345

Complex Number System (1.4)
By the end of this section, you will be able to:

- Evaluate the square root of a negative number
- Add and subtract complex numbers
- Multiply complex numbers
- Divide complex numbers
- Simplify powers of i

This Module supports section **1.4 of Mat 1023**.
In case you missed something in class, or just want to review a specific topic covered in this Module, here is a list of topics covered:

1. Evaluate Square Root of Negative Number [link]
2. Add, Subtract Complex Numbers [link]
3. Multiply Complex Numbers [link]
4. Divide Complex Numbers[link]
5. Negative Roots[link]
6. Key Concepts [link]

Evaluate the Square Root of a Negative Number

Whenever we have a situation where we have a square root of a negative number we say there is no real number that equals that square root. For example, to simplify -1, we are looking for a real number x so that $x^2 = –1$. Since all real numbers squared are positive numbers, there is no real number that equals –1 when squared.

Mathematicians have often expanded their numbers systems as needed. They added 0 to the counting numbers to get the whole numbers. When they needed negative balances, they added negative numbers to get the integers. When they needed the idea of parts of a whole they added fractions and got the rational numbers. Adding the irrational numbers allowed numbers like 5. All of these together gave us the real numbers and so far in your study of mathematics, that has been sufficient.

But now we will expand the real numbers to include the square roots of negative numbers. We start by defining the **imaginary unit** i as the number whose square is –1.

Imaginary Unit

The **imaginary unit** *i* is the number whose square is –1.

$i^2 = -1 \text{ or } i = -1$

We will use the imaginary unit to simplify the square roots of negative numbers.

Square Root of a Negative Number

If *b* is a positive real number, then

$-b = bi$

We will use this definition in the next example. Be careful that it is clear that the *i* is not under the radical. Sometimes you will see this written as $-b = ib$ to emphasize the *i* is not under the radical. But the $-b = bi$ is considered standard form.

Write each expression in terms of *i* and simplify if possible:

ⓐ -25 ⓑ -7 ⓒ -12.

ⓐ

-25 Use the definition of the square root ofnegative numbers.$25i$ Simplify.$5i$

ⓑ

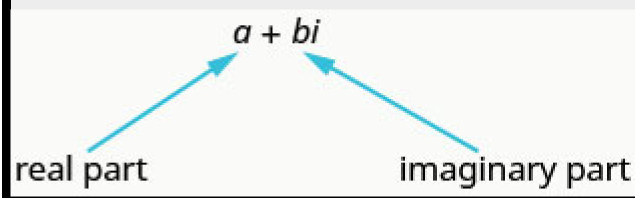
−7 Use the definition of the square root ofnegative numbers.7i Simplify.Be careful that it is clear thatiis not under theradical sign.

ⓒ
−12 Use the definition of the square root ofnegative numbers.12i Simplify12.23i

Now that we are familiar with the imaginary number *i*, we can expand the real numbers to include imaginary numbers. The **complex number system** includes the real numbers and the imaginary numbers. A **complex number** is of the form $a + bi$, where *a*, *b* are real numbers. We call *a* the real part and *b* the imaginary part.

Complex Number

A **complex number** is of the form $a + bi$, where *a* and *b* are real numbers.



A complex number is in **standard form** when written as $a+bi$, where *a* and *b* are real numbers.

If $b=0$, then $a+bi$ becomes $a+0\cdot i=a$, and is a real number.

If $b\neq 0$, then $a+bi$ is an imaginary number.

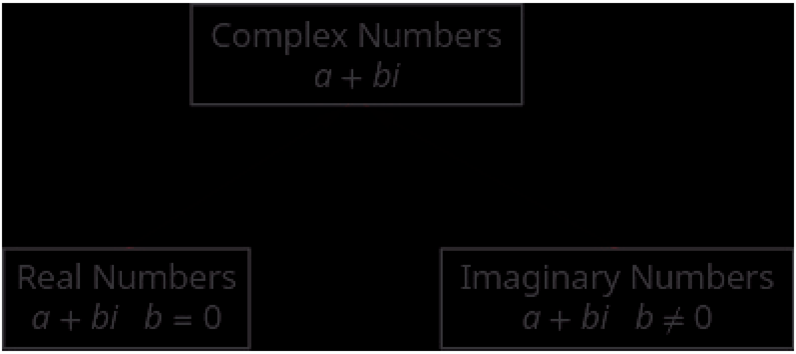
If $a=0$, then $a+bi$ becomes $0+bi=bi$, and is called a pure imaginary number.

We summarize this here.

	$a+bi$	
$b=0$	$a+0\cdot ia$	Real number
$b\neq 0$	$a+bi$	Imaginary number
$a=0$	$0+bibi$	Pure imaginary number

The standard form of a complex number is $a+bi$, so this explains why the preferred form is $-b=bi$ when $b>0$.

The diagram helps us visualize the complex number system. It is made up of both the real numbers and the imaginary numbers.



Expressing an Imaginary Number in Standard Form

Express -9 in standard form.

$$-9 = 9 \; -1 = 3i$$

In standard form, this is $0 + 3i$.

Add or Subtract Complex Numbers

We are now ready to perform the operations of addition, subtraction, multiplication and division on the complex numbers—just as we did with the real numbers. Check out Module 3 for a review of

operations with radicals.

Adding and subtracting complex numbers is much like adding or subtracting like terms. We add or subtract the real parts and then add or subtract the imaginary parts. Our final result should be in standard form.

Add: $-12+-27$.
$-12+-27$ Use the definition of the square root ofnegative numbers.$12i+27i$ Simplify the square roots.$23i+33i$ Add.$53i$

Remember to add both the real parts and the imaginary parts in this next example.

Adding complex numbers: $(a+bi)+(c+di)=(a+c)+(b+d)i$ Subtracting complex numbers: $(a+bi)-(c+di)=(a-c)+(b-d)i$

Given two complex numbers, find the sum or difference.

1. Identify the real and imaginary parts of each number.
2. Add or subtract the real parts.
3. Add or subtract the imaginary parts.

Adding and Subtracting Complex Numbers

Add or subtract as indicated.

1. $(3-4i)+(2+5i)$
2. $(-5+7i)-(-11+2i)$

We add the real parts and add the imaginary parts.

1. $(3-4i)+(2+5i) = 3-4i+2+5i = 3+2+(-4i)+5i = (3+2)+(-4+5)i = 5+i$
2. $(-5+7i)-(-11+2i) = -5+7i+11-2i = -5+11+7i-2i =$

$$(-5+11)+(7-2)i = 6+5i,$$

Simplify: ⓐ $(2+7i)+(4-2i)$ ⓑ $(8-4i)-(2-i)$.

ⓐ $6+5i$ ⓑ $6-3i$

Multiply Complex Numbers

Multiplying complex numbers is also much like multiplying expressions with coefficients and variables. There is only one special case we need to consider. We will look at that after we practice in the next two examples.

Lets begin by multiplying a complex number by a real number. We distribute the real number just as we would with a binomial. Consider, for example, $3(6+2i)$:

$$\begin{aligned} 3(6+2i) &= (3\cdot 6)+(3\cdot 2i) && \text{Distribute.} \\ &= 18+6i && \text{Simplify.} \end{aligned}$$

Multiply: $2i(7-5i)$.
$2i(7-5i)$ Distribute. $14i-10i2$ Simplify $i2$. $14i-10(-1)$ Multiply. $14i+10$ Write in standard form. $10+14i$

Now, let’s multiply two complex numbers. We can use either the distributive property or more specifically the FOIL method because we are dealing with binomials. Recall that FOIL is an acronym for multiplying First, Inner, Outer, and Last terms together. The difference with complex numbers is that when we get a squared term, i^2, it equals -1.

$(a+bi)(c+di) = ac+adi+bci+bd\,i^2 = ac+adi+bci-bd\;\; i^2=-1 \;\; = (ac-bd)+(ad+bc)i$ Group real terms and imaginary terms.

How To:

- Use the distributive property or the FOIL

method.
- Remember that $i^2 = -1$.
- Group together the real terms and the imaginary terms

Multiply: $(3+2i)(4-3i)$.

$(3+2i)(4-3i)$ Use FOIL. $12-9i+8i-6i^2$ Simplify i^2 and combine like terms. $12-i-6(-1)$ Multiply. $12-i+6$ Combine the real parts. $18-i$

Multiplying a Complex Number by a Complex Number

Multiply: $(4+3i)(2-5i)$.

$(4+3i)(2-5i) = 4(2)-4(5i)+3i(2)-(3i)(5i)$
$= 8-20i+6i-15(i^2) =$
$(8+15)+(-20+6)i = 23-14i$

In the next example, we could use FOIL or the

Product of Binomial Squares Pattern.

Multiply: $(3+2i)2$

[redacted]

Use the Product of Binomial Squares Pa [redacted] $+ b)2 = a2 + 2ab + b2$.

Simplify.

[redacted]

Simplify $i2$.

[redacted]

Simplify.

[redacted]

Divide Complex Numbers

Dividing two complex numbers is more complicated than adding, subtracting, or multiplying because we cannot divide by an imaginary number, meaning that any fraction must have a real-number denominator to write the answer in standard form $a + bi$. We need to find a term by which we can multiply the numerator and the denominator that will eliminate the imaginary portion of the denominator so that we end up with a real number as the denominator. This term is called the complex conjugate of the denominator, which is found by changing the sign of the imaginary part of the complex number. In other words, the complex conjugate of $a + bi$ is $a - bi$.

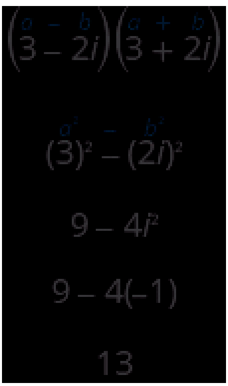
We first looked at conjugate pairs when we studied polynomials. We said that a pair of binomials that each have the same first term and the same last term, but one is a sum and one is a difference is called a *conjugate pair* and is of the form $(a - b), (a + b)$.

A **complex conjugate pair** is very similar. For a complex number of the form $a + bi$, its conjugate is $a - bi$. Notice they have the same first term and the same last term, but one is a sum and one is a difference.

Complex Conjugate Pair
A **complex conjugate pair** is of the form $a+bi$, $a-bi$.

From our study of polynomials, we know the product of conjugates is always of the form $(a-b)(a+b)=a^2-b^2$. The result is called a difference of squares. We can multiply a complex conjugate pair using this pattern.

Earlier for multiplying complex numbers we used FOIL. Now we will use the Product of Conjugates Pattern.



When we multiply complex conjugates, the product of the last terms will always have an i^2 which simplifies to -1.

$$(a-bi)(a+bi)$$
$$a^2-(bi)^2$$
$$a^2-b^2i^2$$
$$a^2-b^2(-1)$$
$$a^2+b^2$$

This leads us to the Product of Complex Conjugates

Pattern: $(a-bi)(a+bi)=a2+b2$

Complex Conjugates
If *a* and *b* are real numbers, then
$(a-bi)(a+bi)=a2+b2$

Note that **complex conjugates** have an opposite relationship: The complex conjugate of $a+bi$ is $a-bi$, and the complex conjugate of $a-bi$ is $a+bi$.

The **complex conjugate** of a complex number $a+bi$ is $a-bi$. It is found by changing the sign of the imaginary part of the complex number. The real part of the number is left unchanged.

- When a complex number is multiplied by its complex conjugate, the result is a real number.
- When a complex number is added to its complex conjugate, the result is a real number.

Dividing complex numbers is much like

rationalizing a denominator. We want our result to be in standard form with no imaginary numbers in the denominator.

Write both the numerator and denominator in standard form. Multiply the numerator and denominator by the complex conjugate of the denominator. Simplify and write the result in standard form.

How to Divide Complex Numbers

Divide: $4+3i3-4i$.

Step 1. Write both the numerator and denominator in standard form.	They are both in standard form.	$\frac{4+3i}{3-4i}$

Step 2. Multiply the numerator and denominator by the complex conjugate of the denominator.	The complex conjugate of $3-4i$ is $3+4i$.	$\frac{(4+3i)(3+4i)}{(3-4i)(3+4i)}$

Step 3. Simplify and write the result in standard form.	Use the pattern $(a-bi)(a+bi)=a^2+b^2$ in the denominator.	$\frac{12+16i+9i+12i^2}{9+16}$
	Combine like terms.	$\frac{12+25i-12}{25}$
	Simplify.	$\frac{25i}{25}$
	Write the result in standard form.	i

Divide, writing the answer in standard form: $-35+2i$.

$-35+2i$ Multiply the numerator and denominator by thecomplex conjugate of the denominator. $-3(5-2i)(5+2i)(5-2i)$ Multiply in the numerator and use the Product ofComplex Conjugates Pattern in the denominator. $-15+6i52+22$ Simplify. $-15+6i29$ Write in standard form. $-1529+629i$

Dividing Complex Numbers

Divide: $(2+5i)$ by $(4-i)$.

We begin by writing the problem as a fraction. $(2+5i)(4-i)$

Then we multiply the numerator and denominator by the complex conjugate of the denominator.

$(2+5i)\ (4-i)\ \cdot (4+i)\ (4+i)$

To multiply two complex numbers, we expand the product as we would with polynomials (using FOIL).

$(2+5i)\ (4-i) \cdot (4+i)\ (4+i) = 8+2i+20i+5$ $i\ 2$ $16+4i-4i-\ i\ 2 = 8+2i+20i+5(-1)$ $16+4i-4i-(-1)$ Because $i\ 2 = -1.$ $=$ $3+22i\ 17 = 3\ 17 + 22\ 17\ i$ Separate real and imaginary parts.

Note that this expresses the quotient in standard form.

Divide: $5+3i4i$.

$5+3i4i$ Write the denominator in standard form.$5+3i0+4i$ Multiply the numerator and denominator bythe complex conjugate of the denominator.$(5+3i)(0-4i)(0+4i)(0-4i)$ Simplify.$(5+3i)(-4i)(4i)(-4i)$ Multiply.$-20i$ $-12i2-16i2$ Simplify thei2.$-20i+1216$ Rewrite in standard form.$1216-2016i$ Simplify the fractions.$34-54i$

Negative Roots

Since the square root of a negative number is not a real number, we cannot use the Product Property for Radicals. In order to multiply square roots of negative numbers we should first write them as complex numbers, using $-b=bi$. This is one place students tend to make errors, so be careful when you see multiplying with a negative square root.

Principal Square Root of a Negative Number

$-b = i\, b$

Multiply: $-36\cdot-4$.

To multiply square roots of negative numbers, we first write them as complex numbers.

$-36\cdot-4$ Write as complex numbers using $-b=bi$. $36i\cdot 4i$ Simplify. $6i\cdot 2i$ Multiply. $12i2$ Simplify $i2$ and multiply. -12

Multiply: $-36 \cdot -81$.
-54

In the next example, each binomial has a square root of a negative number. Before multiplying, each square root of a negative number must be written as a complex number.

Multiply: $(3 - -12)(5 + -27)$.
To multiply square roots of negative numbers, we first write them as complex numbers. $(3 - -12)(5 + -27)$ Write as complex numbers using $-b = bi$.$(3 - 23i)(5 + 33i)$ Use FOIL.$15 + 93i - 103i - 6 \cdot 3i2$ Combine like terms and simplify$i2$.$15 - 3i - 6 \cdot (-3)$ Multiply and combine like terms.$33 - 3i$

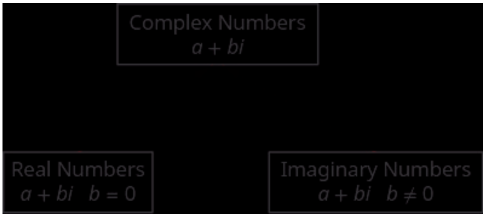
Multiply: $(4 - -12)(3 - -48)$.
$-12 - 223i$

Key Concepts

- **Square Root of a Negative Number**
  - If *b* is a positive real number, then $-b = bi$

	$a + bi$	
$b = 0$	$a + 0 \cdot ia$	Real number
$b \neq 0$	$a + bi$	Imaginary number
$a = 0$	$0 + bibi$	Pure imaginary number

  - A complex number is in **standard form** when written as $a + bi$, where *a*, *b* are real numbers.



- **Product of Complex Conjugates**
  - If *a*, *b* are real numbers, then $(a - bi)(a + bi) = a2 + b2$
- **How to Divide Complex Numbers**

  Write both the numerator and denominator in standard form. Multiply the numerator and denominator by the complex conjugate of the denominator. Simplify and write the result in standard form.

Section Exercises

Practice Makes Perfect

Evaluate the Square Root of a Negative Number

In the following exercises, write each expression in terms of *i* and simplify if possible.

ⓐ -16
ⓑ -11
ⓒ -8

ⓐ $4i$ ⓑ $11i$ ⓒ $22i$

Add or Subtract Complex Numbers In the following exercises, add or subtract.

$-75 + -48$

$93i$

$(1+3i)+(7+4i)$

$8+7i$

$(8-i)+(6+3i)$

$14+2i$

$(1-4i)-(3-6i)$

$-2+2i$

$(6+i)-(-2-4i)$

$8+5i$

$(5--36)+(2--49)$

$7-13i$

$(-7--50)-(-32--18)$

$25-22i$

Multiply Complex Numbers

In the following exercises, multiply.

$4i(5-3i)$

$12+20i$

$-6i(-3-2i)$

$-12+18i$

$(4+3i)(-5+6i)$

$-38++9i$

$(-3+3i)(-2-7i)$

$27+15i$

In the following exercises, multiply using the Product of Binomial Squares Pattern.

$(3+4i)2$

$-7+24i$

$(-2-3i)2$

$-5-12i$

In the following exercises, multiply.

$-25 \cdot -36$

30i

$(-2--27)(4--48)$

$-44+43i$

$(2+-8)(-4+-18)$

$-20-22i$

$(2-i)(2+i)$

5

In the following exercises, multiply using the Product of Complex Conjugates Pattern.

$(7-i)(7+i)$

50

$(9-2i)(9+2i)$

85

Divide Complex Numbers

In the following exercises, divide.

$2+i3-4i$

$225+1125i$

$-43-2i$

$-1213-813i$

$-2-3i4i$

$-34+12i$

Simplify Powers of *i*

In the following exercises, simplify.

$i66$

-1

i

Glossary

complex conjugate pair
: A complex conjugate pair is of the form $a + bi$, $a - bi$.

complex number
: A complex number is of the form $a + bi$, where a and b are real numbers. We call a the real part and b the imaginary part.

complex number system
: The complex number system is made up of both the real numbers and the imaginary numbers.

imaginary unit
: The imaginary unit i is the number whose square is –1. $i^2 = -1$ or $\mathrm{i} = -1$.

standard form
: A complex number is in standard form when written as $\mathrm{a} + \mathrm{bi}$, where a, b are real numbers.

Quadratic Equations (1.5)
By the end of this section, you will be able to:

- Use the Zero Product Property
- Solve quadratic equations by factoring
- Solve quadratic equations in quadratic form

This Module supports section **1.5 of Mat 1023**.
In case you missed something in class, or just want to review a specific topic covered in this Module, here is a list of topics covered:

1. Zero Product Property [link]
2. Solve Quadratic Equations by Factoring [link]
3. Solve Quadratic Equations in the form ax2=k using the Square Root Property [link]
4. Solve Quadratic Equations in the form $a(x - h)2 = k$ Using the Square Root Property [link]
5. Solve Quadratic Equations using Quadratic Formula [link]
6. Discriminant [link]
7. Pythagorean Theorem [link]
8. Key Concepts [link]

We have spent considerable time learning how to factor polynomials. We will now look at polynomial

equations and solve them using factoring, if possible.

A **polynomial equation** is an equation that contains a polynomial expression. The **degree of the polynomial equation** is the degree of the polynomial.

> Polynomial Equation
>
> A **polynomial equation** is an equation that contains a polynomial expression.
>
> The **degree of the polynomial equation** is the degree of the polynomial.

We have already solved polynomial equations of degree one. Polynomial equations of degree one are linear equations are of the form $ax+b=c$.

We are now going to solve polynomial equations of degree two. A polynomial equation of degree two is called a **quadratic equation**. Listed below are some examples of quadratic equations:

$$x^2+5x+6=0 \quad 3y^2+4y=10 \quad 64u^2-81=0 \quad n(n+1)=42$$

The last equation doesn't appear to have the variable squared, but when we simplify the

expression on the left we will get $n2+n$.

Quadratic Equation

An equation of the form $ax2+bx+c=0$ is called a quadratic equation.

$a,b,$ and c are real numbers and $a\neq 0$ (If $a=0$, then $0\cdot x2=0$ and we are left with no quadratic term.)

Quadratic equations differ from linear equations by including a quadratic term with the variable raised to the second power of the form $ax2$. We use different methods to solve quadratic equations than linear equations, because just adding, subtracting, multiplying, and dividing terms will not isolate the variable.

Use the Zero Product Property

We will first solve some quadratic equations by using the **Zero Product Property**. The Zero Product Property says that if the product of two quantities is zero, then at least one of the quantities is zero. The only way to get a product equal to zero is to multiply by zero itself.

Zero Product Property

If $a \cdot b = 0$, then either $a = 0$ or $b = 0$ or both.

We will now use the Zero Product Property, to solve a quadratic equation.

Use the Zero Product Property.

Set each factor equal to zero. Solve the linear equations. Check.

How to Solve a Quadratic Equation Using the Zero Product Property

Solve: $(5n - 2)(6n - 1) = 0$.

Step 1. Set each factor equal to zero.	The product equals zero, so at least one factor must equal zero.	$(5n - 2)(6n - 1) = 0$ $5n - 2 = 0$ or $6n - 1 = 0$
Step 2. Solve the linear equations.	Solve each equation.	$n = \frac{2}{5}$ $\quad n = \frac{1}{6}$

Step 3. Check.	Substitute each solution separately into the original equation.	$n = \frac{2}{5}$ $(5n - 2)(6n - 1) = 0$ $\left(5 \cdot \frac{2}{5} - 2\right)\left(6 \cdot \frac{2}{5} - 1\right) \stackrel{?}{=} 0$ $(2 - 2)\left(\frac{12}{5} - 1\right) \stackrel{?}{=} 0$ $0 \cdot \frac{7}{5} \stackrel{?}{=} 0$ $0 = 0 \checkmark$ $n =$ $(5n - 2)(6n - 1) = 0$ $\left(5 \cdot \quad - 2\right)\left(6 \cdot \quad - 1\right) \stackrel{?}{=} 0$ $\left(\frac{5}{6} - \frac{12}{6}\right)(1 - 1) \stackrel{?}{=} 0$ $\left(-\frac{7}{6}\right)(0) \stackrel{?}{=} 0$ $0 = 0 \checkmark$

Solve: $(3m - 2)(2m + 1) = 0.$

$m = 23, m = -12$

Solve Quadratic Equations by Factoring

The Zero Product Property works very nicely to solve quadratic equations. The quadratic equation must be factored, with zero isolated on one side. So we be sure to start with the quadratic equation in standard form, $ax2+bx+c=0$. Then we factor the expression on the left.

Solve a quadratic equation by factoring.

Write the quadratic equation in standard form, $ax2+bx+c=0$. Factor the quadratic expression. Use the Zero Product Property. Solve the linear equations. Check. Substitute each solution separately into the original equation.

How to Solve a Quadratic Equation by Factoring

Solve: $2y2=13y+45$.

Step 1. Write the quadratic equation in standard form, $ax^2 + bx + c = 0$.	Write the equation in standard form.	$2y^2 = 13y + 45$ $2y^2 - 13y - 45 = 0$

Step 2. Factor the quadratic expression.	Factor $2y^2 - 13y + 45$ $(2y + 5)(y - 9)$	$(2y + 5)(y - 9) = 0$
Step 3. Use the Zero Product Property.	Set each factor equal to zero. We have two linear equations.	$2y + 5 = 0 \quad y - 9 = 0$
Step 4. Solve the linear equations.		$y = -\frac{5}{2} \quad y = 9$
Step 5. Check. Substitute each solution separately into the original equation.	Substitute each solution separately into the original equation.	$y = -\frac{5}{2}$ $2y^2 = 13y + 45$ $2\left(-\frac{5}{2}\right)^2 \stackrel{?}{=} 13\left(-\frac{5}{2}\right) + 45$ $2\left(\frac{25}{4}\right) \stackrel{?}{=} \left(-\frac{65}{2}\right) + \frac{90}{2}$ $\frac{25}{2} = \frac{25}{2}$ ✓ $y =$ $2y^2 = 13y + 45$ $2(\ )^2 \stackrel{?}{=} 13(\ ) + 45$ $2(81) \stackrel{?}{=} 117 + 45$ $162 = 162$ ✓

Before we factor, we must make sure the quadratic equation is in standard form.

Solve: 3c2 = 10c − 8.

$c=2, c=43$

Solve: $2d2-5d=3$.
$d=3, d=-12$

Solving quadratic equations by factoring will make use of all the factoring techniques you have learned in earlier sections! Do you recognize the special product pattern in the next example?

Solve: $169q2=49$.
$169x2=49$Write the quadratic equation in standard form.$169x2-49=0$Factor. It is a difference of squares.$(13x-7)(13x+7)=0$Use the Zero Product Property to set each factor to0.Solve each equation.$13x-7=013x+7=013x=713x=-7x=713x=-713$

Check:

We leave the check up to you.

Solve: $36x2=121$.

$x=116,x=-116$

In the next example, the left side of the equation is factored, but the right side is not zero. In order to use the Zero Product Property, one side of the equation must be zero. We'll multiply the factors and then write the equation in standard form.

Solve: $(3x-8)(x-1)=3x$.

$(3x-8)(x-1)=3x$Multiply the binomials.$3x2-11x+8=3x$Write the quadratic equation in standard form.$3x2-14x+8=0$Factor the trinomial.$(3x-2)(x$

$-4)=0$Use the Zero Product Property to set each factor to 0.Solve each equation.$3x-2=0x-4=03x=2x=4x=23$Check your answers.The check is left to you.

Solve: $(2m+1)(m+3)=12m$.

$m=1,m=32$

In the next example, when we factor the quadratic equation we will get three factors. However the first factor is a constant. We know that factor cannot equal 0.

Solve: $3x2=12x+63$.

$3x2=12x+63$Write the quadratic equation in standard form.$3x2-12x-63=0$Factor the greatest common factor first.$3(x2-4x-21)=0$Factor the trinomial.$3(x-7)(x$

$+3)=0$ Use the Zero Product Property to set each factor to 0.Solve each equation.$3\neq 0x-7=0x+3=03\neq 0x=7x=-3$Check your answers.The check is left to you.

Solve Quadratic Equations of the form $ax2=k$ using the Square Root Property

We have already solved some quadratic equations by factoring. Let's review how we used factoring to solve the quadratic equation $x2 = 9$.

Put the equation in standard form.Factor the difference of squares.$x2=9x2-9=0(x-3)(x+3)=0$Use the Zero Product Property.Solve each equation.$x-3=0x-3=0x=3x=-3$

We can easily use factoring to find the solutions of similar equations, like $x2 = 16$ and $x2 = 25$, because 16 and 25 are perfect squares. In each case, we would get two solutions, $x=4, x=-4$ and $x=5, x=-5$.

But what happens when we have an equation like $x2 = 7$? Since 7 is not a perfect square, we cannot solve the equation by factoring.

Previously we learned that since 169 is the square of 13, we can also say that 13 is a *square root* of 169. Also, $(-13)^2 = 169$, so -13 is also a square root of 169. Therefore, both 13 and -13 are square roots of 169. So, every positive number has two square roots—one positive and one negative. We earlier defined the square root of a number in this way:

If $n^2 = m$, then n is a square root of m.

Since these equations are all of the form $x^2 = k$, the square root definition tells us the solutions are the two square roots of k. This leads to the **Square Root Property**.

Square Root Property
If $x^2 = k$, then
$x = k$ or $x = -k$ or $x = \pm k$.

Notice that the Square Root Property gives two solutions to an equation of the form $x^2 = k$, the principal square root of k and its opposite. We could also write the solution as $x = \pm k$. We read this as x equals positive or negative the square root of k.

Now we will solve the equation $x^2 = 9$ again, this time using the Square Root Property.

Use the Square Root Property.$x2 = 9x = \pm 9x = \pm 3$So$x = 3$or$x = -3$.

What happens when the constant is not a perfect square? Let's use the Square Root Property to solve the equation $x2 = 7$.

$x2 = 7$Use the Square Root Property.$x = 7, x = -7$

We cannot simplify 7, so we leave the answer as a radical.

Solve a quadratic equation using the square root property.

Isolate the quadratic term and make its coefficient one. Use Square Root Property. Simplify the radical. Check the solutions.

How to solve a Quadratic Equation of the form $ax2 = k$ Using the Square Root Property

Solve: $x2 - 50 = 0$.

Step 1. Isolate the quadratic term and make its coefficient one.	Add 50 to both sides to get x^2 by itself.	$x^2 - 50 = 0$ $x^2 = 50$

Step 2. Use Square Root Property.	Remember to write the $\pm$ symbol.	$x = \pm\sqrt{50}$
Step 3. Simplify the radical.		$x = \pm\sqrt{25} \cdot \sqrt{2}$ $x = \pm 5\sqrt{2}$
	Rewrite to show two solutions.	$x = 5\sqrt{2},\ x = -5\sqrt{2}$
Step 4. Check the solutions.	Substitute in $x = 5\sqrt{2}$ and $x = -5\sqrt{2}$	$x^2 - 50 = 0$ $(5\sqrt{2})^2 - 50 \stackrel{?}{=} 0$ $25 \cdot 2 - 50 \stackrel{?}{=} 0$ $0 = 0 \checkmark$ $x^2 - 50 = 0$ $(-5\sqrt{2})^2 - 50 \stackrel{?}{=} 0$ $25 \cdot 2 - 50 \stackrel{?}{=} 0$ $0 = 0 \checkmark$

Solve: $x2-48=0$.

$x=43, x=-43$

In order to use the Square Root Property, the coefficient of the variable term must equal one. In the next example, we must divide both sides of the

equation by the coefficient 3 before using the Square Root Property.

Solve: $3z^2=108$.

	$3z^2=108$
The quadratic term is isolated. Divide by 3 to make its coefficient 1.	$\frac{3z^2}{3}=\frac{108}{3}$
Simplify.	$z^2=36$
Use the Square Root Property.	$z=\pm\sqrt{36}$
Simplify the radical.	$z=\pm 6$
Rewrite to show two solutions.	$z=6, z=-6$
Check the solutions:	

$$\begin{array}{ll} 3z^2 = 108 & 3z^2 = 108 \\ 3(6)^2 \stackrel{?}{=} 108 & 3(-6)^2 \stackrel{?}{=} 108 \\ 3(36) \stackrel{?}{=} 108 & 3(36) \stackrel{?}{=} 108 \\ 108 = 108\ \checkmark & 108 = 108\ \checkmark \end{array}$$

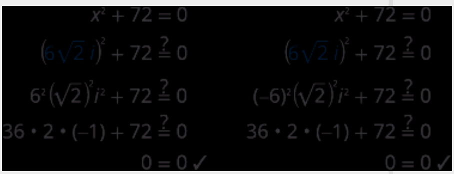
Solve: 2x2 = 98.

x = 7, x = −7

The Square Root Property states ‘If x2 = k,’ What will happen if k < 0? This will be the case in the next example.

Solve: x2 + 72 = 0.

	$x2+72=0$
Isolate the quadratic term.	$x2=-72$
Use the Square Root Property.	$x=\pm-72$
Simplify using complex numbers.	$x=\pm72i$
Simplify the radical.	$x=\pm62i$
Rewrite to show two solutions.	$x=62i, x=-62i$
Check the solutions:	



Solve: $c2+12=0$.

$c=23i, c=-23i$

Our method also works when fractions occur in the equation, we solve as any equation with fractions. In the next example, we first isolate the quadratic term, and then make the coefficient equal to one.

Solve: 23u2 + 5 = 17.

	23u2 + 5 = 17
Isolate the quadratic term.	[redacted]
Multiply by 32 to make the coefficient 1.	[redacted]
Simplify.	[redacted]
Use the Square Root Property.	[redacted]
Simplify the radical.	

$$u=\pm\sqrt{9\cdot 2}$$

Simplify.

$$u=\pm 3\sqrt{2}$$

Rewrite to show two solutions.

$$u=3\sqrt{2},\quad u=-3\sqrt{2}$$

Check:

$\frac{2}{3}u^2+5=17$	$\frac{2}{3}u^2+5=17$
$\frac{2}{3}\left(3\sqrt{2}\right)^2+5\stackrel{?}{=}17$	$\frac{2}{3}\left(-3\sqrt{2}\right)^2+5\stackrel{?}{=}17$
$\frac{2}{3}\cdot 18+5\stackrel{?}{=}17$	$\frac{2}{3}\cdot 18+5\stackrel{?}{=}17$
$12+5\stackrel{?}{=}17$	$12+5\stackrel{?}{=}17$
$17=17\checkmark$	$17=17\checkmark$

Solve: 34y2 − 3 = 18.

y = 27,y = − 27

The solutions to some equations may have fractions inside the radicals. When this happens, we must

rationalize the denominator.

Solve: $2x2 - 8 = 41$.

[redacted]

Isolate the quadratic term.

[redacted]

Divide by 2 to make the coefficient 1.

[redacted]

Simplify.

[redacted]

Use the Square Root Property.

[redacted]

Rewrite the radical as a fraction of square roots.

$$x = \pm \frac{\sqrt{49}}{\sqrt{2}}$$

Rationalize the denominator.

$$x = \pm \frac{\sqrt{49} \cdot \sqrt{2}}{\sqrt{2} \cdot \sqrt{2}}$$

Simplify.

$$x = \pm \frac{7\sqrt{2}}{2}$$

Rewrite to show two solutions.

$$x = \frac{7\sqrt{2}}{2}, \quad x = -\frac{7\sqrt{2}}{2}$$

Check:
We leave the check for you.

Solve: $3t2+6=70$.
$t=833, t=-833$

Solve Quadratic Equations of the Form $a(x - h)^2 = k$ Using the Square Root Property

We can use the Square Root Property to solve an equation of the form $a(x - h)2 = k$ as well. Notice that the quadratic term, x, in the original form $ax2 = k$ is replaced with $(x - h)$.

$ax^2 = k \qquad a(x - h)^2 = k$

The first step, like before, is to isolate the term that has the variable squared. In this case, a binomial is being squared. Once the binomial is isolated, by dividing each side by the coefficient of a, then the Square Root Property can be used on $(x - h)2$.

Solve: $4(y-7)2=48$.

	$4(y-7)2=48$
Divide both sides by the coefficient 4.	$(y-7)2=12$
Use the Square Root Property on the binomial	$y-7=\pm 12$
Simplify the radical.	$y-7=\pm 23$

Solve for y.	y = 7 ± 23
Rewrite to show two solutions.	y = 7 + 23, y = 7 − 23
Check:	

$$4(y-7)^2 = 48 \qquad 4(y-7)^2 = 48$$
$$4(7+2\sqrt{3}-7)^2 \stackrel{?}{=} 48 \qquad 4(7-2\sqrt{3}-7)^2 \stackrel{?}{=} 48$$
$$4(2\sqrt{3})^2 \stackrel{?}{=} 48 \qquad 4(-2\sqrt{3})^2 \stackrel{?}{=} 48$$
$$4(12) \stackrel{?}{=} 48 \qquad 4(12) \stackrel{?}{=} 48$$
$$48 = 48 \checkmark \qquad 48 = 48 \checkmark$$

Remember when we take the square root of a fraction, we can take the square root of the numerator and denominator separately.

Solve: (x − 13)2 = 59.

(x − 13)2 = 59 Use the Square Root Property.x − 13 = ±59 Rewrite the radical as a fraction of square roots.x − 13 = ±59 Simplify the radical.x − 13 = ±53 Solve forx.x = 13 ± 53 Rewrite to show two

solutions.x = 13 + 53,x = 13 − 53 Check: We leave the check for you.

Solve: (x − 12)2 = 54.
x = 12 + 52,x = 12 − 52

Sometimes the solutions are complex numbers.

Solve: (2x − 3)2 = − 12.
Use the Square Root Property.Simplify the radical.Add 3 to both sides.Divide both sides by 2.Rewrite in standard form.Simplify.(2x − 3)2 = − 122x − 3 = ± − 122x − 3 = ± 23i2x = 3 ± 23ix = 3 ± 23i2x = 32 ± 23i2x = 32 ± 3i Rewrite to show two solutions.x = 32 + 3i,x = 32 − 3i Check:We leave the check for you.

The left sides of the equations in the next two examples do not seem to be of the form $a(x-h)^2$. But they are perfect square trinomials, so we will factor to put them in the form we need.

Solve: $4n^2+4n+1=16$.

We notice the left side of the equation is a perfect square trinomial. We will factor it first.

	$4n^2+4n+1=16$
Factor the perfect square trinomial.	$(2n+1)^2=16$
Use the Square Root Property.	$2n+1=\pm\sqrt{16}$
Simplify the radical.	$2n+1=\pm 4$
Solve for n.	$2n=-1\pm 4$
Divide each side by 2.	$\frac{2n}{2}=\frac{-1\pm 4}{2}$
	$n=\frac{-1\pm 4}{2}$
Rewrite to show two solutions.	$n=\frac{-1+4}{2},\ n=\frac{-1-4}{2}$
Simplify each equation.	$n=\frac{3}{2},\ n=-\frac{5}{2}$

Check:

$$
\begin{array}{rcl}
4n^2+4n+1=16 & \quad & 4n^2+4n+1=16 \\
4\left(\frac{3}{2}\right)^2+4\left(\frac{3}{2}\right)+1 \stackrel{?}{=} 16 & & 4\left(-\frac{5}{2}\right)^2+4\left(-\frac{5}{2}\right)+1 \stackrel{?}{=} 16 \\
4\left(\frac{9}{4}\right)+4\left(\frac{3}{2}\right)+1 \stackrel{?}{=} 16 & & 4\left(\frac{25}{4}\right)+4\left(-\frac{5}{2}\right)+1 \stackrel{?}{=} 16 \\
9+6+1 \stackrel{?}{=} 16 & & 25-10+1 \stackrel{?}{=} 16 \\
16=16\checkmark & & 16=16\checkmark
\end{array}
$$

Solve: $16n2+40n+25=4$.

$n=-34, n=-74$

Access this online resource for additional instruction and practice with using the Square Root Property to solve quadratic equations.

- Solving Quadratic Equations: The Square Root Property
- Using the Square Root Property to Solve Quadratic Equations

Solve Quadratic Equations using Quadratic Formula

When we solved quadratic equations in the last section by completing the square, we took the same steps every time. By the end of the exercise set, you may have been wondering ‘isn’t there an easier way to do this?’ The answer is ‘yes’. Mathematicians look for patterns when they do things over and over in order to make their work easier. In this section we will derive and use a formula to find the solution of a quadratic equation.

We start with the standard form of a quadratic equation and solve it for *x* by completing the square.

[redacted]

Isolate the variable terms on one side.

[redacted]

Make the coefficient of x2

eq[redacted]
dividing by a.
Simplify.

[redacted]

To complete the square, find (12·ba)2 and add it to both sides of the equation.
(12ba)2=b24a2

[redacted]

The left side is a perfect square, factor it.

[redacted]

Find the common denominator of the right sid[redacted] fractions with the common denominator.
Simplify.

[redacted]

Combine to one fraction.

[redacted]

Use the square root property.

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

Simplify the radical.

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

Add $-\frac{b}{2a}$ to both sides of the equation.

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

Combine the terms on the right side.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This equation is the Quadratic Formula.

Quadratic Formula

The solutions to a quadratic equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$ are given by the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Write the quadratic equation in standard form, $ax^2 + bx + c = 0$. Identify the values of a, b, and c. Write the Quadratic Formula. Then substitute in the values of a, b, and c. Simplify. Check the solutions.

Solve by using the Quadratic Formula: $2x2 + 9x - 5 = 0$.

Step 1. Write the quadratic equation in standard form. Identify the *a*, *b*, *c* values.	This equation is in standard form.	$ax^2 + bx + c = 0$ $2x^2 + 9x - 5 = 0$ $a = 2, b = 9, c = -5$
Step 2. Write the quadratic formula. Then substitute in the values of *a*, *b*, *c*.	Substitute in , $b = 9$,	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $x = \frac{-9 \pm \sqrt{9^2 - 4 \cdot \quad \cdot (\quad)}}{2 \cdot}$
Step 3. Simplify the fraction, and solve for *x*.		$x = \frac{-9 \pm \sqrt{81 - (-40)}}{4}$ $x = \frac{-9 \pm \sqrt{121}}{4}$ $x = \frac{-9 \pm 11}{4}$ $x = \frac{-9 + 11}{4}$ $\quad$ $x = \frac{-9 - 11}{4}$ $x = \frac{2}{4}$ $\quad$ $x = \frac{-20}{4}$ $x = \frac{1}{2}$ $\quad$ $x = -5$

Step 4. Check the solutions.	Put each answer in the original equation to check.	$2x^2 + 9x - 5 = 0$
	Substitute $x = \frac{1}{2}$.	$2\left(\frac{1}{2}\right)^2 + 9 \cdot \frac{1}{2} - 5 \stackrel{?}{=} 0$
		$2 \cdot \frac{1}{4} + 9 \cdot \frac{1}{2} - 5 \stackrel{?}{=} 0$
		$2 \cdot \frac{1}{4} + 9 \cdot \frac{1}{2} - 5 \stackrel{?}{=} 0$
		$\frac{1}{2} + \frac{9}{2} - 5 \stackrel{?}{=} 0$
		$\frac{10}{2} - 5 \stackrel{?}{=} 0$
		$5 - 5 \stackrel{?}{=} 0$
		$0 = 0 \checkmark$
		$2x^2 + 9x - 5 = 0$
	Substitute $x = -5$.	$2(-5)^2 + 9(-5) - 5 \stackrel{?}{=} 0$
		$2 \cdot 25 - 45 - 5 \stackrel{?}{=} 0$
		$50 - 45 - 5 \stackrel{?}{=} 0$
		$0 = 0 \checkmark$

Solve by using the Quadratic Formula: $x2 - 6x = -5$.

	$x^2 - 6x = -5$
Write the equation in standard form by adding 5 to each side.	$x^2 - 6x + 5 = 0$
This equation is now in	

sta [redacted]

Identify the values of a, b, c.

[redacted]

Write the Quadratic Formula.

[redacted]

Then substitute in the values of a, b, c.

[redacted]

Simplify.

[redacted]

[redacted]

[redacted]

Rewrite to show two solutions.

[redacted]

Simplify.

[redacted]

[redacted]

Check:

$$
\begin{array}{rcl}
x^2 - 6x + 5 = 0 & \quad & x^2 - 6x + 5 = 0 \\
5^2 - 6 \cdot 5 + 5 \stackrel{?}{=} 0 & & 1^2 - 6 \cdot 1 + 5 \stackrel{?}{=} 0 \\
25 - 30 + 5 \stackrel{?}{=} 0 & & 1 - 6 + 5 \stackrel{?}{=} 0 \\
0 = 0 \checkmark & & 0 = 0 \checkmark
\end{array}
$$

When we solved quadratic equations by using the Square Root Property, we sometimes got answers that had radicals. That can happen, too, when using the Quadratic Formula. If we get a radical as a solution, the final answer must have the radical in its simplified form.

Solve by using the Quadratic Formula: $2x2+10x+11=0$.

$$2x^2 + 10x + 11 = 0$$

This equation is in standard form.

[redacted]

Identify the values of *a*, *b*, and *c*.

[redacted]

Write the Quadratic Formula.

[redacted]

Then substitute in the values of *a*, *b*, and *c*.

[redacted]

Simplify.

[redacted]

[redacted]

Simplify the radical.

[redacted]

Factor out the common factor in the nu[redacted]

Remove the common factors.

[redacted]

Rewrite to show two solutions.

[redacted]

Check:
We leave the check for

you!

Solve by using the Quadratic Formula: $3m2+12m+7=0$.

$m=-6+153, m=-6-153$

When we substitute *a*, *b*, and *c* into the Quadratic Formula and the radicand is negative, the quadratic equation will have imaginary or complex solutions. We will see this in the next example.

Solve by using the Quadratic Formula: $3p2+2p+9=0$.

[redacted]

This equation is in standard form

[redacted]

Identify the values of a,b,c.

[redacted]

Write the Quadratic Formula.

[redacted]

Then substitute in the values of a,b,c.

[redacted]

Simplify.

[redacted]

[redacted]

Simplify the radical using complex nu[redacted]

Simplify the radical.

[redacted]

Factor the common factor in the nu[redacted]

Remove the common factors.

$$p = \frac{-1 \pm \sqrt{26}\, i}{3}$$

Rewrite in standard a + bi form.

$$p = -\frac{1}{3} \pm \frac{\sqrt{26}\, i}{3}$$

Write as two solutions.

$$p = -\frac{1}{3} + \frac{\sqrt{26}\, i}{3}, \quad p = -\frac{1}{3} - \frac{\sqrt{26}\, i}{3}$$

Solve by using the Quadratic Formula: 4a2 − 2a + 8 = 0.

a = 14 + 314i, a = 14 − 314i

Discriminant

When we solved the quadratic equations in the previous examples, sometimes we got two real solutions, one real solution, and sometimes two

complex solutions. Is there a way to predict the number and type of solutions to a quadratic equation without actually solving the equation?

Yes, the expression under the radical of the Quadratic Formula makes it easy for us to determine the number and type of solutions. This expression is called the **discriminant**.

Discriminant

In the Quadratic Formula, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$,

the quantity $b^2 - 4ac$ is called the discriminant.

Let's look at the discriminant of the equations in some of the examples and the number and type of solutions to those quadratic equations. The quadratic formula not only generates the solutions to a quadratic equation, it tells us about the nature of the solutions when we consider the discriminant, or the expression under the radical, $b^2 - 4ac$. The discriminant tells us whether the solutions are real numbers or complex numbers, and how many solutions of each type to expect. [link] relates the value of the discriminant to the solutions of a

quadratic equation.

When the discriminant is positive, the quadratic equation has **2 real solutions.**	$x = \frac{-b \pm \sqrt{+}}{2a}$
When the discriminant is zero, the quadratic equation has **1 real solution.**	$x = \frac{-b \pm \sqrt{0}}{2a}$
When the discriminant is negative, the quadratic equation has **2 complex solutions.**	$x = \frac{-b \pm \sqrt{-}}{2a}$

Value of Discriminant	Results
$b^2-4ac=0$	One rational solution (double solution)
$b^2-4ac>0$, perfect square	Two rational solutions
$b^2-4ac>0$, not a perfect square	Two irrational solutions
$b^2-4ac<0$	Two complex solutions

Determine the number of solutions to each quadratic equation.

ⓐ $3x^2+7x-9=0$ ⓑ $5n^2+n+4=0$ ⓒ $9y^2-6y+1=0$.

To determine the number of solutions of each quadratic equation, we will look at its discriminant.

ⓐ

$3x^2+7x-9=0$ The equation is in standard form, identify $a, b,$ and c. $a=3, b=7, c=-9$ Write the discriminant. b^2-4ac Substitute in the values of $a, b,$ and c. $(7)^2-4\cdot3\cdot(-9)$ Simplify. $49+108$ 157

Since the discriminant is positive, there are 2 real solutions to the equation.

ⓑ

$5n^2+n+4=0$ The equation is in standard form, identify $a, b,$ and c. $a=5, b=1, c=4$ Write the discriminant. b^2-4ac Substitute in the values of $a, b,$ and c. $(1)^2-4\cdot5\cdot4$ Simplify. $1-80$ -79

Since the discriminant is negative, there are 2 complex solutions to the equation.

ⓒ

$9y^2-6y+1=0$ The equation is in standard form, identify $a, b,$ and c. $a=9, b=-6, c=1$ Write the discriminant. b^2-4ac Substitute in the values of $a, b,$ and c. $(-6)^2-4\cdot9\cdot1$ Simplify. $36-36$ 0

Since the discriminant is 0, there is 1 real

solution to the equation.

For each Equation, compute the determinant and determine the number and type of solution.

For each Equation, compute the determinant and determine the number and type of solution.

1. $x^2+4x+4=0$
2. $8x^2+14x+3=0$
3. $3x^2-5x-2=0$
4. $3x^2-10x+15=0$

Calculate the discriminant b^2-4ac for each equation and state the expected type of solutions.

1. $x^2+4x+4=0$

   $b^2-4ac=(4)^2-4(1)(4)=0$. There will be one rational double solution.

2. $8x^2+14x+3=0$

   $b^2-4ac=(14)^2-4(8)(3)=100$. As 100 is a perfect square, there will be

two rational solutions.

3. $3x^2-5x-2=0$

   $b^2-4ac=(-5)^2-4(3)(-2)=49.$
   As 49 is a perfect square, there will be two rational solutions.

4. $3x^2-10x+15=0$

   $b^2-4ac=(-10)^2-4(3)(15)=-80.$ There will be two complex solutions.

Identify the Most Appropriate Method to Use to Solve a Quadratic Equation

Given that we have four methods to use to solve a quadratic equation, how do you decide which one to use? Factoring is often the quickest method and so we try it first. If the equation is $ax^2=k$ or $a(x-h)^2=k$ we use the Square Root Property. For any other equation, it is probably best to use the Quadratic Formula. Remember, you can solve any quadratic equation by using the Quadratic Formula, but that is not always the easiest method.

Identify the most appropriate method to use to solve each quadratic equation.

ⓐ $5z2=17$ ⓑ $4x2-12x+9=0$ ⓒ $8u2+6u=11$.

ⓐ

$5z2=17$

Since the equation is in the $ax2=k$, the most appropriate method is to use the Square Root Property.

ⓑ

$4x2-12x+9=0$

We recognize that the left side of the equation is a perfect square trinomial, and so factoring will be the most appropriate method.

ⓒ

$8u2+6u=11$ Put the equation in standard form. $8u2+6u-11=0$

While our first thought may be to try factoring, thinking about all the possibilities for trial and error method leads us to choose the Quadratic Formula as the most appropriate method.

Access these online resources for additional instruction and practice with using the Quadratic Formula.

- Using the Quadratic Formula
- Solve a Quadratic Equation Using the Quadratic Formula with Complex Solutions
- Discriminant in Quadratic Formula

Pythagorean Theorem

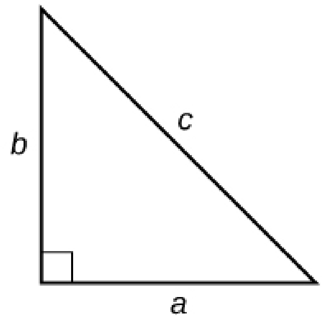
One of the most famous formulas in mathematics is the **Pythagorean Theorem**. It is based on a right triangle, and states the relationship among the lengths of the sides as $a^2 + b^2 = c^2$, where a and b refer to the legs of a right triangle adjacent to the $90°$ angle, and c refers to the hypotenuse. It has immeasurable uses in architecture, engineering, the sciences, geometry, trigonometry, and algebra, and in everyday applications.

We use the Pythagorean Theorem to solve for the length of one side of a triangle when we have the lengths of the other two. Because each of the terms is squared in the theorem, when we are solving for a side of a triangle, we have a quadratic equation. We can use the methods for solving quadratic equations

that we learned in this section to solve for the missing side.

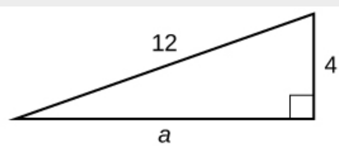
The Pythagorean Theorem is given as
$$a^2 + b^2 = c^2$$

where a and b refer to the legs of a right triangle adjacent to the $90°$ angle, and c refers to the hypotenuse, as shown in [link].



Finding the Length of the Missing Side of a Right Triangle

Find the length of the missing side of the right triangle in [link].



As we have measurements for side *b* and the

hypotenuse, the missing side is *a*.

a 2 + b 2 = c 2 a 2 + (4) 2 = (12) 2 a 2 +16 = 144 a 2 = 128 a = 128 = 8 2

Use the Pythagorean Theorem to solve the right triangle problem: Leg *a* measures 4 units, leg *b* measures 3 units. Find the length of the hypotenuse.

5 units

Key Concepts

- **Polynomial Equation:** A polynomial equation is an equation that contains a polynomial expression. The degree of the polynomial equation is the degree of the polynomial.
- **Quadratic Equation:** An equation of the form ax2+bx+c=0 is called a quadratic equation. a,b,care real numbers anda≠0
- **Zero Product Property:** If a·b=0, then either

$a = 0$ or $b = 0$ or both.

- **How to use the Zero Product Property**

  Set each factor equal to zero. Solve the linear equations. Check.

- **How to solve a quadratic equation by factoring.**

  Write the quadratic equation in standard form, $ax2 + bx + c = 0$. Factor the quadratic expression. Use the Zero Product Property. Solve the linear equations. Check. Substitute each solution separately into the original equation.

- Square Root Property
  - If $x2 = k$, then $x = korx = -k$ or $x = \pm k$

  How to solve a quadratic equation using the square root property.

  Isolate the quadratic term and make its coefficient one. Use Square Root Property. Simplify the radical. Check the solutions.

- The solutions to a quadratic equation of the form $ax2 + bx + c = 0$, $a \neq 0$ are given by the formula:
  $x = -b \pm b2 - 4ac2a$

- Using the Discriminant, $b^2 - 4ac$, to Determine the Number and Type of Solutions of a Quadratic Equation
    - For a quadratic equation of the form $ax^2 + bx + c = 0$, $a \neq 0$,
        - If $b^2 - 4ac > 0$, the equation has 2 real solutions.
        - if $b^2 - 4ac = 0$, the equation has 1 real solution.
        - if $b^2 - 4ac < 0$, the equation has 2 complex solutions.

Section Exercises

Practice Makes Perfect

Use the Zero Product Property

In the following exercises, solve.

$(3a-10)(2a-7)=0$

$a=10/3, a=7/2$

$6m(12m-5)=0$

$m=0, m=5/12$

Solve Quadratic Equations by Factoring

In the following exercises, solve.

$5a2-26a=24$

$a=-45, a=6$

$4m2=17m-15$

$m=5/4, m=3$

$7a2+14a=7a$

$a=-1, a=0$

$49m2=144$

$m=12/7, m=-12/7$

$(x+6)(x-3)=-8$

$x=2, x=-5$

$20x2-60x=-45$

$x=3/2$

$2x3+72x=24x2$

$x=0, x=6$

Solve Quadratic Equations of the Form $ax^2 = k$ Using the Square Root Property

In the following exercises, solve each equation.

$u2-300=0$

$u=\pm 103$

$4m2=36$

$m=\pm 3$

$7p2+10=26$

$p=\pm 477$

$(u-6)2=64$

$u=14, u=-2$

$(r-12)2=34$

$r=12\pm 32$

$(a-7)2+5=55$

$a=7\pm 52$

$(4x-3)2+11=-17$

$x=34\pm 72i$

$x2-6x+9=12$

$x = 3 + 23, x = 3 - 23$

$3m2 + 30m - 27 = 6$

$m = -11, m = 1$

Solve Quadratic Equations Using the Quadratic Formula

$4m2 + m - 3 = 0$

$m = -1, m = 34$

$3u2 + 7u - 2 = 0$

$u = -7 \pm 736$

$x2 + 8x - 4 = 0$

$x = -4 \pm 25$

$2x2 + 3x + 3 = 0$

$x = -34 \pm 154i$

$34b2 + 12b = 38$

$b = -2 \pm 116$

Glossary

degree of the polynomial equation
: The degree of the polynomial equation is the degree of the polynomial.

polynomial equation
: A polynomial equation is an equation that contains a polynomial expression.

quadratic equation
: Polynomial equations of degree two are called quadratic equations.

zero of the function
: A value of x where the function is 0, is called a zero of the function.

Zero Product Property
: The Zero Product Property says that if the product of two quantities is zero, then at least one of the quantities is zero.

discriminant

In the Quadratic Formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, the quantity $b^2 - 4ac$ is called the discriminant.

Other Types of Equations (1.6)
By the end of this section, you will be able to:

- Solve polynomial equations
- Solve radical equations
- Solve equations with two radicals
- Solve equations in Quadratic Form

This Module supports section **1.6 of Mat 1023**.
In case you missed something in class, or just want to review a specific topic covered in this Module, here is a list of topics covered:

1. Solve Polynomial Equations [link]
2. Solve Radical Equations [link]
3. Solve Radical Equations, two radicals [link]
4. Solve Equations with Rational Exponents [link]
5. Solve Equations in the Quadratic form [link]
6. Absolute Value Equations [link]
7. Key Concepts [link]

Solving Polynomial equations

We have already solved polynomial equations of degree one. Polynomial equations of degree one are **linear equations** are of the form $ax+b=c$. Module 8

We have now also solved polynomial equations of degree two. A polynomial equation of degree two is called a **quadratic equation**. Module 11

But what about those equations with a degree of 3 or higher? Some of them we will be able to solve by **Factoring** (remember Module 5 and Module 11).

Solve a Polynomial by Grouping

Solve a polynomial by grouping: $x^3+x^2-9x-9=0$.

This polynomial consists of 4 terms, which we can solve by grouping. Grouping procedures require factoring the first two terms and then factoring the last two terms. If the factors in the parentheses are identical, we can continue the process and solve, unless more factoring is suggested.

$$x^3+x^2-9x-9=0$$
$$x^2(x+1)-9(x+1)=0$$
$$(x^2-9)(x+1)=0$$

The grouping process ends here, as we can factor x^2-9 using the difference of squares formula.

$$(x^2-9)(x+1)=0 \quad (x-3)(x+3)(x+1)=0 \quad x=3 \quad x=-3 \quad x=-1$$

The solutions are 3, -3, and -1. Note that the highest exponent is 3 and we obtained 3 solutions.

Analysis

We looked at solving quadratic equations by factoring when the leading coefficient is 1. When the leading coefficient is not 1, we solved by grouping. Grouping requires four terms, which we obtained by splitting the linear term of quadratic equations. We can also use grouping for some polynomials of degree higher than 2, as we saw here, since there were already four terms.

Solve: $8x3=24x2-18x$.

$x=0, x=32$

$4y3 - 9y = 0$
$y = 0, 3 2, -3 2$

Solve Radical Equations

Radical equations are equations that contain variables in the radicand (the expression under a radical symbol), such as
$3x+18 = x \quad x+3 = x-3 \quad x+5 - x-3 = 2$

Radical Equation
An equation in which a variable is in the radicand of a radical expression is called a **radical equation**.

As usual, when solving these equations, what we do to one side of an equation we must do to the other side as well. Once we isolate the radical, our strategy will be to raise both sides of the equation to the power of the index. This will eliminate the

radical.

Solving radical equations containing an even index by raising both sides to the power of the index may introduce an algebraic solution that would not be a solution to the original radical equation. For Example, when we write a we mean the principal square root. So $a \geq 0$ always. When we solve radical equations by squaring both sides we may get an algebraic solution that would make a negative. This algebraic solution would not be a solution to the original radical equation; it is an **extraneous solution.**

For the equation $x+2=x$:

ⓐ Is $x=2$ a solution? ⓑ Is $x=-1$ a solution?

Solution

ⓐ Is $x=2$ a solution?

[redacted]

Let $x = 2$.

[redacted]

Simplify.

[redacted]

[redacted]

2 is a solution.

ⓑ Is $x = -1$ a solution?

[redacted]

Let $x = -1$.

[redacted]

Simplify.

[redacted]

[redacted]

	−1 is not a solution. −1 is an extraneous solution to the equation.

In the next example, we will see how to solve a radical equation. Our strategy is based on raising a radical with index *n* to the *n*th power. This will eliminate the radical.

For $a \geq 0, (an)n = a$.

Solve a radical equation with one radical.

Isolate the radical expression on one side of the equal sign. Put all remaining terms on the other side. If the radical is a square root, then square both sides of the equation. If it is a cube root, then raise both sides of the equation to the third power. In other words, for an *n*th root radical, raise both sides to the *n*th power. Doing so eliminates the radical symbol. Solve the remaining equation. Solve the new equation. Check the answer in the original equation.

How to Solve a Radical Equation

Solve: $5n - 4 - 9 = 0$.

Step 1. Isolate the radical on one side of the equation.	To isolate the radical, add 9 to both sides. Simplify.	$\sqrt{5n-4} - 9 = 0$ $\sqrt{5n-4} - 9 + 9 = 0 + 9$ $\sqrt{5n-4} = 9$
Step 2. Raise both sides of the equation to the power of the index.	Since the index of a square root is 2, we square both sides.	$\left(\sqrt{5n-4}\right)^2 = (9)^2$
Step 3. Solve the new equation.	Remember, $\left(\sqrt{a}\right)^2 = a$.	$5n - 4 = 81$ $5n = 85$ $n = 17$
Step 4. Check the answer in the original equation.		Check the answer. $\sqrt{5n-4} - 9 = 0$ $\sqrt{5(17)-4} - 9 \stackrel{?}{=} 0$ $\sqrt{85-4} - 9 \stackrel{?}{=} 0$ $\sqrt{81} - 9 \stackrel{?}{=} 0$ $9 - 9 \stackrel{?}{=} 0$ $0 = 0$ ✓ The solution is $n = 17$.

Solve: $3m+2-5=0$.

$m=233$

Sometimes there could be two solutions, but one of them may be extraneous!

Solve: $r+4-r+2=0$.

	$r+4-r+2=0$
Isolate the radical.	$r+4=r-2$
Square both sides of the equation.	$(r+4)2=(r-2)2$
Simplify and then solve the equation	$r+4=r2-4r+4$
It is a quadratic equation, so get zero on one side.	$0=r2-5r$
Factor the right side.	$0=r(r-5)$
Use the Zero Product Property.	$0=r0=r-5$

Solve the equation. Check your answer.	$r=0 \quad r=5$
	The solution is $r = 5$.

$$
\begin{array}{rlrl}
r=0, & \sqrt{r+4}-r+2=0 & r=5, & \sqrt{r+4}-r+2=0 \\
& \sqrt{0+4}-0+2\stackrel{?}{=}0 & & \sqrt{5+4}-5+2\stackrel{?}{=}0 \\
& \sqrt{4}+2\stackrel{?}{=}0 & & \sqrt{9}-3\stackrel{?}{=}0 \\
& 4\neq 0 & & 0=0\checkmark
\end{array}
$$

$r=0$ is an extraneous solution.

Solve: $\sqrt{m+9}-m+3=0$.

$m=7$

When we use a radical sign, it indicates the principal or positive root. If an equation has a radical with an even index equal to a negative number, that equation will have no solution.

Solve: $\sqrt{9k-2}+1=0$.

[redacted]

To isolate the radical, subtract 1 to both sid[redacted]

Simplify.

[redacted]

Because the square root is equal to a negative number, the equation has no solution.

Solve: $7s-3+2=0$.

no solution

If one side of an equation with a square root is a binomial, we use the Product of Binomial Squares Pattern when we square it.

Binomial Squares

$(a+b)2=a2+2ab+b2(a-b)2=a2-2ab+b2$

Don’t forget the middle term!

Solve: $p-1+1=p$.

[redacted]

To isolate the radical, subtract 1 from both sid[redacted]

Simplify.

[redacted]

Square both sides of the equation.

[redacted]

Simplify, using the Product of Binomial Sq[redacted]

right. Then solve the new equation.

It is a quadratic equation, so get zero on one side.

[redacted]

Factor the right side.

[redacted]

Use the Zero Product Property.

[redacted]

Solve each equation.

[redacted]

Check the answers.

[redacted]

The solutions are p=1,p=2.

When the index of the radical is 3, we cube both sides to remove the radical.

(a3)3=a

Solve: 5x + 13 + 8 = 4.

	$5x+13+8=4$
To isolate the radical, subtract 8 from both sides.	$5x+13=-4$
Cube both sides of the equation.	$(5x+13)3=(-4)3$
Simplify.	$5x+1=-64$
Solve the equation.	$5x=-65$
	$x=-13$
Check the answer.	

$$x=-13 \qquad \sqrt[3]{5x+1}+8=4$$
$$\sqrt[3]{5(-13)+1}+8\stackrel{?}{=}4$$
$$\sqrt[3]{-64}+8\stackrel{?}{=}4$$
$$-4+8\stackrel{?}{=}4$$
$$4=4\ \checkmark$$

The solution is $x=-13$.

Solve: $6x - 103 + 1 = -3$

$x = -9$

Solve equations with two Radicals

If the radical equation has two radicals, we start out by isolating one of them. It often works out easiest to isolate the more complicated radical first.

In the next example, when one radical is isolated, the second radical is also isolated.

Solve: $4x - 33 = 3x + 23$.

The radical terms are isolated.$4x - 33 = 3x + 23$ Since the index is 3, cube both sides of theequation.$(4x - 33)3 = (3x + 23)3$ Simplify, then solve the new equation.$4x - 3 = 3x + 2$ $x - 3 = 2$ $x = 5$ The solution is$x = 5$. Check the answer. We leave it to you to show that 5 checks!

Solve: $5x-43=2x+53$.
$x=3$

Sometimes after raising both sides of an equation to a power, we still have a variable inside a radical. When that happens, we repeat Step 1 and Step 2 of our procedure. We isolate the radical and raise both sides of the equation to the power of the index again.

Solve if 1 or more Radicals

Isolate one of the radical terms on one side of the equation. Raise both sides of the equation to the power of the index. Are there any more radicals?
If yes, repeat Step 1 and Step 2 again.
If no, solve the new equation. Check the answer in the original equation.

How to Solve a Radical Equation

Solve: $m+1=m+9$.

Step 1. Isolate one of the radical terms on one side of the equation.	The radical on the right is isolated.	$\sqrt{m}+1=\sqrt{m+9}$
Step 2. Raise both sides of the equation to the power of the index.	We square both sides. Simplify—be very careful as you multiply!	$\left(\sqrt{m}+1\right)^2=\left(\sqrt{m+9}\right)^2$
Step 3. Are there any more radicals? If yes, repeat Step 1 and Step 2 again.	There is still a radical in the equation. So we must repeat the previous steps. Isolate the radical term.	$m+2\sqrt{m}+1=m+9$
	Here, we can easily isolate the radical by dividing both sides by 2.	$2\sqrt{m}=8$
If no, solve the new equation.	Square both sides.	$\sqrt{m}=4$ $\left(\sqrt{m}\right)^2=(4)^2$ $m=16$
Step 4. Check the answer in the original equation.		$\sqrt{m}+1=\sqrt{m+9}$ $\sqrt{16}+1\stackrel{?}{=}\sqrt{16+9}$ $4+1\stackrel{?}{=}5$ $5=5\checkmark$ The solution is $m=16$.

Solve: $x+2=x+16$.

$x=9$

Be careful as you square binomials in the next example. Remember the pattern is $(a+b)^2=a^2+2ab+b^2$ or $(a-b)^2=a^2-2ab+b^2$.

Solve: $q-2+3=4q+1$.

$$\sqrt{q-2}+3=\sqrt{4q+1}$$

The radical on the right is isolated. Square both sides.

$$\left(\sqrt{q-2}+3\right)^2=\left(\sqrt{4q+1}\right)^2$$

Simplify.

$$q-2+6\sqrt{q-2}+9=4q+1$$

There is still a radical in the equation so we must repeat the

$$6\sqrt{q-2}=3q-6$$

previous steps. Isolate the radical.

Square both sides. It would not help to div

Re

sq

the q = 2.

$$(6\sqrt{q-2})^2 = (3q-6)^2$$

$$6^2(\sqrt{q-2})^2 = (3q)^2 - 2\cdot 3q\cdot 6 + 6^2$$

Simplify, then solve the new equation.

$$36(q-2) = 9q^2 - 36q + 36$$

Distribute.

$$36q - 72 = 9q^2 - 36q + 36$$

It is a quadratic equation, so get zero on one side.

$$0 = 9q^2 - 72q + 108$$

Factor the right side.

$$0 = 9(q^2 - 8q + 12)$$

$$0 = 9(q-6)(q-2)$$

Use the Zero Product Property.

$$q - 6 = 0 \qquad q - 2 = 0$$

$$q = 6 \qquad q = 2$$

The checks are left to you.

The solutions are $q = 6$ and $q = 2$.

Solve: $x+2=3x+4$

$x=0x=4$

Solve Equation with Rational Exponents

We touched on Rational exponents in Module 3. Recall that we can look at amn in two ways. Remember the Power Property tells us to multiply the exponents and so (a1n)m and (am)1n both equal amn.

Rational Exponent amn
For any positive integers *m* and *n*,
$amn=(an)mandamn=amn$

Sometimes an equation will contain rational exponents instead of a radical. We use the same techniques to solve the equation as when we have a radical. We raise each side of the equation to the power of the denominator of the rational exponent.

Since $(am)n = am\cdot n$, we have for example,
$(x12)2 = x,(x13)3 = x$

Remember, $x12 = x$ and $x13 = x3$.

Solve: $(3x - 2)14 + 3 = 5$.

	$(3x - 2)14 + 3 = 5$
To isolate the term with the rational exponent, subtract 3 from both sides.	$(3x - 2)14 = 2$
Raise each side of the equation to the fourth power.	$((3x - 2)14)4 = (2)4$
Simplify.	$3x - 2 = 16$
Solve the equation.	$3x = 18$
	$x = 6$
Check the answer.	

$$
\begin{aligned}
x = 6 \qquad (3x-2)^{\frac{1}{4}} + 3 &= 5 \\
(3 \cdot 6 - 2)^{\frac{1}{4}} + 3 &\stackrel{?}{=} 5 \\
(16)^{\frac{1}{4}} + 3 &\stackrel{?}{=} 5 \\
2 + 3 &\stackrel{?}{=} 5 \\
5 &= 5 \checkmark
\end{aligned}
$$

The solution is $x=6$.

Solve the Equation Including a Variable Raised to a Rational Exponent

Solve the equation in which a variable is raised to a rational exponent: $x^{\frac{5}{4}}=32$.

The way to remove the exponent on *x* is by raising both sides of the equation to a power that is the reciprocal of $\frac{5}{4}$, which is $\frac{4}{5}$.

$x^{\frac{5}{4}} = 32$ $\left(x^{\frac{5}{4}}\right)^{\frac{4}{5}} = (32)^{\frac{4}{5}}$ $x = (2)^4$ The fifth root of 32 is 2. $= 16$

Solving an Equation Involving Rational Exponents and Factoring

Solve 3 x 3 4 = x 1 2 .

This equation involves rational exponents as well as factoring rational exponents. Let us take this one step at a time. First, put the variable terms on one side of the equal sign and set the equation equal to zero.

3 x 3 4 −(x 1 2) = x 1 2 −(x 1 2) 3 x 3 4 − x 1 2 = 0

Now, it looks like we should factor the left side, but what do we factor out? We can always factor the term with the lowest exponent. Rewrite x 1 2 as x 2 4 . Then, factor out x 2 4 from both terms on the left.

3 x 3 4 − x 2 4 = 0 x 2 4 (3 x 1 4 −1) = 0

Where did x 1 4 come from? Remember, when we multiply two numbers with the same base, we add the exponents. Therefore, if we multiply x 2 4 back in using the distributive property, we get the expression we had before the factoring, which is what should happen. We need an exponent such that when added to 2 4 equals 3 4 . Thus, the exponent on *x* in the parentheses is 1 4 .

Let us continue. Now we have two factors and can use the zero factor theorem.

x 2 4 (3 x 1 4 −1) = 0 x 2 4 = 0 x = 0 3 x 1 4 −1 = 0 3 x 1 4 = 1 x 1 4 = 1 3 Divide

both sides by 3. (x 1 4) 4 = (1 3) 4 Raise both sides to the reciprocal of 1 4 . x = 1 81

The two solutions are 0 and 1 81 .

Solve: (9x+9)14−2=1.

x=8

Sove equations in Quadratic Form

Sometimes when we factored trinomials, the trinomial did not appear to be in the $ax^2 + bx + c$ form. So we factored by substitution allowing us to make it fit the $ax^2 + bx + c$ form. We used the standard u for the substitution.

To factor the expression $x^4 - 4x^2 - 5$, we noticed the variable part of the middle term is x^2 and its square, x^4, is the variable part of the first term. (We know (x2)2=x4.) So we let $u = x^2$ and factored.

[redacted]

[redacted]

Let u = x2 and substitute.

[redacted]

Factor the trinomial.

[redacted]

Replace *u* with x2.

[redacted]

Quadratic Form

If the exponent on the middle term is one-half of the exponent on the leading term, we have an **equation in quadratic form**, which we can solve as if it were a quadratic. We substitute a variable for the middle term to solve equations in quadratic form.

Given an equation quadratic in form, solve it.

1. Identify the exponent on the leading term and determine whether it is double the exponent

on the middle term.
2. If it is, substitute a variable, such as u, for the variable portion of the middle term.
3. Rewrite the equation so that it takes on the standard form of a quadratic.
4. Solve using one of the usual methods for solving a quadratic.
5. Replace the substitution variable with the original term.
6. Solve the remaining equation.

Similarly, sometimes an equation is not in the $ax^2 + bx + c = 0$ form but looks much like a quadratic equation. Then, we can often make a thoughtful substitution that will allow us to make it fit the $ax^2 + bx + c = 0$ form. If we can make it fit the form, we can then use all of our methods to solve quadratic equations.

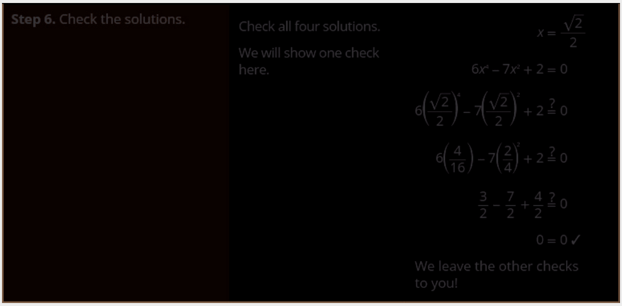
Notice that in the quadratic equation $ax^2 + bx + c = 0$, the middle term has a variable, x, and its square, x^2, is the variable part of the first term. Look for this relationship as you try to find a substitution.

The next example shows the steps for solving an equation in quadratic form.

How to Solve Equations in Quadratic Form

Solve: 6x4 − 7x2 + 2 = 0

Step 1. Identify a substitution that will put the equation in quadratic form.	Since $(x^2)^2 = x^4$, we let $u = x^2$.	$6x^4 - 7x^2 + 2 = 0$
Step 2. Rewrite the equation with the substitution to put it in quadratic form.	Rewrite to prepare for the substitution. Substitute $u = x^2$.	$6(x^2)^2 - 7x^2 + 2 = 0$ $6u^2 - 7u + 2 = 0$
Step 3. Solve the quadratic equation for u.	We can solve by factoring. Use the Zero Product Property.	$(2u - 1)(3u - 2) = 0$ $2u - 1 = 0,\ 3u - 2 = 0$ $2u = 1,\ 3u = 2$ $u = \frac{1}{2}\quad u = \frac{2}{3}$
Step 4. Substitute the original variable back into the results, using the substitution.	Replace u with x^2.	$x^2 = \frac{1}{2}\qquad x^2 = \frac{2}{3}$
Step 5. Solve for the original variable.	Solve for x, using the Square Root Property.	$x = \pm\sqrt{\frac{1}{2}}\qquad x = \pm\sqrt{\frac{2}{3}}$ $x = \pm\frac{\sqrt{2}}{2}\qquad x = \pm\frac{\sqrt{6}}{3}$ There are four solutions. $x = \frac{\sqrt{2}}{2}\qquad x = \frac{\sqrt{6}}{3}$ $x = -\frac{\sqrt{2}}{2}\qquad x = -\frac{\sqrt{6}}{3}$



Step 6. Check the solutions.	Check all four solutions. We will show one check here.	$x = \frac{\sqrt{2}}{2}$ $6x^4 - 7x^2 + 2 = 0$ $6\left(\frac{\sqrt{2}}{2}\right)^4 - 7\left(\frac{\sqrt{2}}{2}\right)^2 + 2 \stackrel{?}{=} 0$ $6\left(\frac{4}{16}\right) - 7\left(\frac{2}{4}\right) + 2 \stackrel{?}{=} 0$ $\frac{3}{2} - \frac{7}{2} + \frac{4}{2} \stackrel{?}{=} 0$ $0 = 0 \checkmark$ We leave the other checks to you!

Solve: $x^4 - 6x^2 + 8 = 0$.

$x = 2, x = -2, x = 2, x = -2$

In the next example, the binomial in the middle term, (x − 2) is squared in the first term. If we let $u = x - 2$ and substitute, our trinomial will be in $ax^2 + bx + c$ form.

Solve: $(x - 2)^2 + 7(x - 2) + 12 = 0$.

[redacted]

Prepare for the substitution.

[redacted]

Let u = x − 2 and substitute.

[redacted]

Solve by factoring.

[redacted]

[redacted]

[redacted]

Replace u with x − 2.

[redacted]

Solve for x.

[redacted]

Check:

$$\begin{array}{rr}
x=-1 & x=-2 \\
(x-2)^2+7(x-2)+12=0 & (x-2)^2+7(x-2)+12=0 \\
(-1-2)^2+7(-1-2)+12 \stackrel{?}{=} 0 & (-2-2)^2+7(-2-2)+12 \stackrel{?}{=} 0 \\
(-3)^2+7(-3)+12 \stackrel{?}{=} 0 & (-4)^2+7(-4)+12 \stackrel{?}{=} 0 \\
9-21+12 \stackrel{?}{=} 0 & 16-28+12 \stackrel{?}{=} 0 \\
0=0\ \checkmark & 0=0\ \checkmark
\end{array}$$

Solve: $(y-4)2+8(y-4)+15=0$.

$y=-1, y=1$

In the next example, we notice that $(x)2=x$. Also, remember that when we square both sides of an equation, we may introduce extraneous roots. Be sure to check your answers!

Solve: $x-3x+2=0$.

The x in the middle term, is squared in the first term $(x)2=x$. If we let $u=x$ and substitute, our trinomial will be in $ax2 + bx +$

$c = 0$ form.

$$x - 3\sqrt{x} + 2 = 0$$

Rewrite the trinomial to prepare for the su

$$\left(\sqrt{x}\right)^2 - 3\sqrt{x} + 2 = 0$$

Let u = x and substitute.

$$u^2 - 3u + 2 = 0$$

Solve by factoring.

$$(u - 2)(u - 1) = 0$$

$$u - 2 = 0, \quad u - 1 = 0$$

$$u = 2, \quad u = 1$$

Replace u with x.

$$\sqrt{x} = 2, \quad \sqrt{x} = 1$$

Solve for x, by squaring both sides.

$$x = 4, \quad x = 1$$

Check:

$$
\begin{array}{cc}
x = 4 & x = 1 \\
x - 3\sqrt{x} + 2 = 0 & x - 3\sqrt{x} + 2 = 0 \\
4 - 3\sqrt{4} + 2 \stackrel{?}{=} 0 & 1 - 3\sqrt{1} + 2 \stackrel{?}{=} 0 \\
4 - 6 + 2 \stackrel{?}{=} 0 & 1 - 3 + 2 \stackrel{?}{=} 0 \\
0 = 0 \checkmark & 0 = 0 \checkmark
\end{array}
$$

Solve: $x - 7x + 12 = 0$.

$x = 9, x = 16$

Substitutions for rational exponents can also help us solve an equation in quadratic form. Think of the properties of exponents as you begin the next example.

Solve: $x23 - 2x13 - 24 = 0$.

The x13 in the middle term is squared in the first term (x13)2=x23. If we let u=x13 and substitute, our trinomial will be in $ax2 + bx + c = 0$ form.

[redacted]

Rewrite the trinomial to prepare for the su[redacted]

Let u=x13 and substitute.

[redacted]

Solve by factoring.

[redacted]

[redacted]

[redacted]

Replace *u* with x13.

$$x^{\frac{1}{3}} = 6, \qquad x^{\frac{1}{3}} = -4$$

Solve for x by cubing both sides.

$$\left(x^{\frac{1}{3}}\right)^3 = (6)^3, \qquad \left(x^{\frac{1}{3}}\right)^3 = (-4)^3$$

$$x = 216, \qquad x = -64$$

Check:

$$\begin{array}{rr} x = 216 & x = -64 \\ x^{\frac{2}{3}} - 2x^{\frac{1}{3}} - 24 = 0 & x^{\frac{2}{3}} - 2x^{\frac{1}{3}} - 24 = 0 \\ (216)^{\frac{2}{3}} - 2(216)^{\frac{1}{3}} - 24 \stackrel{?}{=} 0 & (-64)^{\frac{2}{3}} - 2(-64)^{\frac{1}{3}} - 24 \stackrel{?}{=} 0 \\ 36 - 12 - 24 \stackrel{?}{=} 0 & 16 + 8 - 24 \stackrel{?}{=} 0 \\ 0 = 0 \checkmark & 0 = 0 \checkmark \end{array}$$

Solve: $x^{\frac{2}{3}} - 5x^{\frac{1}{3}} - 14 = 0$.

$x = -8, x = 343$

In the next example, we need to keep in mind the definition of a negative exponent as well as the properties of exponents from Module 2.

Tip

If n is an integer and $a \neq 0$, then $a - n = 1an$ or $1a - n = an$.

Solve: $3x - 2 - 7x - 1 + 2 = 0$.

The $x - 1$ in the middle term is squared in the first term $(x - 1)2 = x - 2$. If we let $u = x - 1$ and substitute, our trinomial will be in $ax2 + bx + c = 0$ form.

	[redacted]
Rewrite the trinomial to prepare for the substitution.	[redacted]
Let $u = x - 1$ and substitute.	[redacted]
Solve by	

fac [redacted]

[redacted]

[redacted]

Replace u with $x-1$.

[redacted]

Solve for x by taking the rec [redacted] $x-1=1x$.

Check:

[redacted]

Solve: $8x-2-10x-1+3=0$.

$x=43x=2$

Access this online resource for additional instruction and practice with solving quadratic equations.

- Solving Equations in Quadratic Form

The numbers 5 and -5 are both five units away from zero.

Absolute Value Equations

As we prepare to solve absolute value equations, we review our definition of **absolute value** from Module 1.

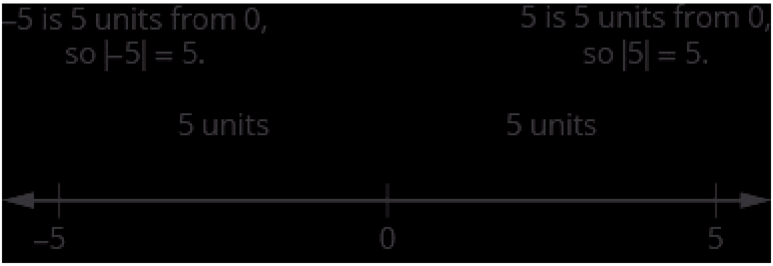
Absolute Value
The absolute value of a number is its distance from zero on the number line.
The absolute value of a number *n* is written as $|n|$ and $|n| \geq 0$ for all numbers.
Absolute values are always greater than or equal to zero.

We learned that both a number and its opposite are the same distance from zero on the number line. Since they have the same distance from zero, they have the same absolute value. For example:

-5 is 5 units away from 0, so $|-5| = 5$.

5 is 5 units away from 0, so $|5| = 5$.

[link] illustrates this idea.



For the equation $|x| = 5$, we are looking for all numbers that make this a true statement. We are looking for the numbers whose distance from zero is 5. We just saw that both 5 and -5 are five units from zero on the number line. They are the solutions to the equation.

$$\text{If } |x| = 5 \text{ then } x = -5 \text{ or } x = 5$$

The solution can be simplified to a single statement by writing $x = \pm 5$. This is read, "*x* is equal to positive or negative 5".

We can generalize this to the following property for absolute value equations.

Absolute Value Equations

For any algebraic expression, *u*, and any positive real number, *a*,

$\text{if}\ |u| = a\ \text{then}\ u = -a\ \text{or}\ u = a$

Remember that an absolute value cannot be a negative number.

Solve: ⓐ $|x| = 8$ ⓑ $|y| = -6$ ⓒ $|z| = 0$

ⓐ

$|x| = 8$ Write the equivalent equations. $x = -8$ or $x = 8$ $x = \pm 8$

ⓑ

$|y| = -6$ No solution

Since an absolute value is always positive, there are no solutions to this equation.

ⓒ

$|z| = 0$ Write the equivalent equations. $z = -0$ or $z = 0$ Since $-0 = 0$, $z = 0$

Both equations tell us that $z = 0$ and so there is only one solution.

To solve an absolute value equation, we first isolate the absolute value expression using the same procedures we used to solve linear equations. Once we isolate the absolute value expression we rewrite it as the two equivalent equations.

Solve absolute value equations.

Isolate the absolute value expression. Write the equivalent equations. Solve each equation. Check each solution.

How to Solve Absolute Value Equations

Solve $|5x-4|-3=8$.

Step 1. Isolate the absolute value expression.	Add 3 to both sides.	$\lvert 5x-4\rvert-3=8$ $\lvert 5x-4\rvert=11$
Step 2. Write the equivalent equations.		$5x-4=-11$ or $5x-4=11$
Step 3. Solve each equation.	Add 4 to each side. Divide each side by 5.	$5x=-7$ $\quad$ $5x=15$ $x=-\frac{7}{5}$ or $x=3$

Step 4. Check each solution.	Substitute 3 and $-\frac{7}{5}$ into the original equation.	$\|5x-4\|-3=8$ $x=3$ $\|5\cdot 3-4\|-3\stackrel{?}{=}8$ $\|15-4\|-3\stackrel{?}{=}8$ $\|11\|-3\stackrel{?}{=}8$ $11-3\stackrel{?}{=}8$ $8=8\checkmark$ $\|5x-4\|-3=8$ $x=-\frac{7}{5}$ $\left\|5\left(-\frac{7}{5}\right)-4\right\|-3\stackrel{?}{=}8$ $\|-7-4\|-3\stackrel{?}{=}8$ $\|-11\|-3\stackrel{?}{=}8$ $11-3\stackrel{?}{=}8$ $8=8\checkmark$

Solve: $|3x-5|-1=6$.

$x=4, x=-23$

Solve $2|x-7|+5=9$.

	$2\|x-7\|+5=9$
Isolate the absolute value expression.	$2\|x-7\|=4$
	$\|x-7\|=2$
Write the equivalent equations.	$x-7=-2$ or $x-7=2$
Solve each equation.	$x=5$ or $x=9$
Check: [missing_resource: CNX_IntAlg_Figure_02_07_003a_img.jpg]	

Solve: $3|x-4|-4=8$.

$x=8, x=0$

Remember, an absolute value is always positive!

Solve: $|23x-4|+11=3$.
$|23x-4|+11=3$Isolate the absolute value term.$|23x-4|=-8$An absolute value cannot be negative.No solution

Solve: $|56x+3|+8=6$.
No solution

Use Radicals in Applications

As you progress through your college courses, you'll encounter formulas that include radicals in many disciplines. We will modify our Problem Solving Strategy for Geometry Applications slightly to give us a plan for solving applications with formulas from any discipline.

Use a problem solving strategy for applications with formulas.

Read the problem and make sure all the words and ideas are understood. When appropriate, draw a figure and label it with the given information. **Identify** what we are looking for. **Name** what we are looking for by choosing a variable to represent it. **Translate** into an equation by writing the appropriate formula or model for the situation. Substitute in the given information. **Solve the equation** using good algebra techniques. **Check** the answer in the problem and make sure it makes sense. **Answer** the question with a complete sentence.

One application of radicals has to do with the effect of gravity on falling objects. The formula allows us to determine how long it will take a fallen object to hit the gound.

Falling Objects

On Earth, if an object is dropped from a height of *h* feet, the time in seconds it will take to reach the ground is found by using the formula

t=h4.

For example, if an object is dropped from a height of 64 feet, we can find the time it takes to reach the ground by substituting h=64 into the formula.

$t=\frac{\sqrt{h}}{4}$	
$t=\frac{\sqrt{64}}{4}$	
Take the square root of 64.	
$t=\frac{8}{4}$	
Simplify the fraction.	
$t=2$	

It would take 2 seconds for an object dropped from a height of 64 feet to reach the ground.

Marissa dropped her sunglasses from a bridge 400 feet above a river. Use the formula t=h4

to find how many seconds it took for the sunglasses to reach the river.

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	the time it takes for the sunglasses to reach the river
Step 3. Name what we are looking.	Let t = time.
Step 4. Translate into an equation by writing the ... ap... Su... n inf... $t = \frac{\sqrt{h}}{4}$, and $h = 400$ $t = \frac{\sqrt{400}}{4}$	
Step 5. Solve the equation. $t = \frac{20}{4}$ $t = 5$	
Step 6. Check the answer in the problem and make sure it makes sense.	

$5 \stackrel{?}{=} \frac{\sqrt{400}}{4}$ $5 \stackrel{?}{=} \frac{20}{4}$ $5 = 5\ \checkmark$	
Does 5 seconds seem like a reasonable length of time?	Yes.
Step 7. Answer the question.	It will take 5 seconds for the sunglasses to reach the river.

Police officers investigating car accidents measure the length of the skid marks on the pavement. Then they use square roots to determine the speed, in miles per hour, a car was going before applying the brakes.

Skid Marks and Speed of a Car

If the length of the skid marks is *d* feet, then the speed, *s*, of the car before the brakes were applied can be found by using the formula

$s = 24d$

After a car accident, the skid marks for one car measured 190 feet. Use the formula $s = 24d$ to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

Step 1. Read the problem	
Step 2. Identify what we are looking for.	the speed of a car
Step 3. Name what weare looking for,	Let $s =$ the speed.
Step 4. Translate into an equation by writing the [redacted] for [redacted] the given information.	
Step 5. Solve the equation.	
[redacted]	
[redacted]	

Round to 1 decimal place.

[redacted]

[redacted]

The speed of the car before the brakes were applied
was 67.5 miles per hour.

Access these online resources for additional instruction and practice with solving radical equations.

- Solving an Equation Involving a Single Radical
- Solving Equations with Radicals and Rational Exponents
- Solving Radical Equations
- Solve Radical Equations
- Radical Equation Application

Key Concepts

- **Binomial Squares**
  $(a+b)2=a2+2ab+b2(a-b)2=a2-2ab+b2$
- **Solve a Radical Equation**

  Isolate one of the radical terms on one side of the equation. Raise both sides of the equation to the power of the index. Are there any more radicals?
  If yes, repeat Step 1 and Step 2 again.
  If no, solve the new equation. Check the answer in the original equation.

- **Absolute Value Equations**
  For any algebraic expression, u, and any positive real number, a,
  $\text{if}|u|=\text{athenu}=-\text{aoru}=a$
  Remember that an absolute value cannot be a negative number.
- **How to Solve Absolute Value Equations**

  Isolate the absolute value expression. Write the equivalent equations. Solve each equation. Check each solution.

Practice Makes Perfect

Solve Radical Equations

In the following exercises, solve.

$5x-6=8$

$x=14$

$5x+1=-3$

---

no solution

$2x3=-2$

$x=-4$

$2m-3-5=0$

$m=14$

$2u-3+2=0$

---

no solution

$u - 3 + 3 = u$

$u = 3, u = 4$

$4x + 53 - 2 = -5$

$x = -8$

$(8x + 5)13 + 2 = -1$

$x = -4$

$(12x - 3)14 - 5 = -2$

$x = 7$

$x + 1 - x + 1 = 0$

$x = 3$

$$32x-3-20=7$$

$$x=42$$

$$28r+1-8=2$$

$$r=3$$

Solve Radical Equations with Two Radicals

In the following exercises, solve.

$$3u+7=5u+1$$

$$u=3$$

$$8+2r=3r+10$$

$$r=-2$$

$$5x-13=x+33$$

$$x=1$$

$a+2=a+4$

$a=0$

$u+1=u+4$

$u=94$

$2x+1=1+x$

$x=0x=4$

$2x-1-x-1=1$

$x=1x=5$

$x+7-x-5=2$

$x=9$

Solve Equations in Quadratic Form

$x4-7x2+12=0$

$x = \pm 3, x = \pm 2$
$2x4 - 5x2 + 3 = 0$
$x = \pm 1, x = \pm 62$
$(x - 3)2 - 5(x - 3) - 36 = 0$
$x = -1, x = 12$
$x - x - 20 = 0$
$x = 25$
$6x + x - 2 = 0$
$x = 14$
$x23 + 9x13 + 8 = 0$
$x = -1, x = -512$

$6x23-x13=12$

$x=278, x=-6427$

$6x-2+13x-1+5=0$

$x=-2, x=-35$

Solve Absolute Value Equations

$|4x-1|-3=0$

$x=1, x=-12$

$|4x+7|+2=5$

$x=-1, x=-52$

$3|x-4|+2=11$

$x=7, x=1$

$|35x-2|+5=2$

no solution

Glossary

radical equation
: An equation in which a variable is in the radicand of a radical expression is called a radical equation.

Inequalities (1.7)
By the end of this section, you will be able to:

- Graph inequalities on the number line
- Solve linear inequalities
- Solve Compound Inequalities
- Solve Absolut Value Inequalities
- Solve applications with linear inequalities

This Module supports section **1.7 of Mat 1023**.
In case you missed something in class, or just want to review a specific topic covered in this Module, here is a list of topics covered:

1. Graph Inequalities on a Number Line [link]
2. Solve Linear Inequalities [link]
3. Compound Inequalities [link]
4. Absolute Value Inequalities [link]
5. Applications with Linear Inequalities [link]
6. Key Concepts [link]

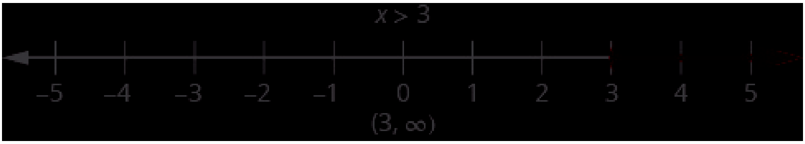
The inequality $x > 3$ is graphed on this number line and written in interval notation. The inequality $x \le 1$ is graphed on this number line and written in interval notation.

Graph Inequalities on the Number Line

What number would make the inequality $x > 3$ true? Are you thinking, "*x* could be four"? That's correct, but *x* could be 6, too, or 37, or even 3.001. Any number greater than three is a solution to the inequality $x > 3$.

Indicating the solution to an inequality can be achieved in several ways.

- We show all the solutions to the inequality $x > 3$ on the number line by shading in all the numbers to the right of three, to show that all numbers greater than three are solutions. Because the number three itself is not a solution, we put an open parenthesis at three. [link] shows the number line.
- We can also represent inequalities using **interval notation**, in which solution sets are indicated with parentheses or brackets. There is no upper end to the solution to this inequality. In interval notation, we express $x > 3$ as $(3, \infty)$. The symbol ∞ is read as "**infinity**." It is not an actual number.
- We can use set-builder notation: $\{ x | x > 3 \}$, which translates to "all real numbers *x* such that *x* is greater than 3." Notice that braces are used to indicate a set.

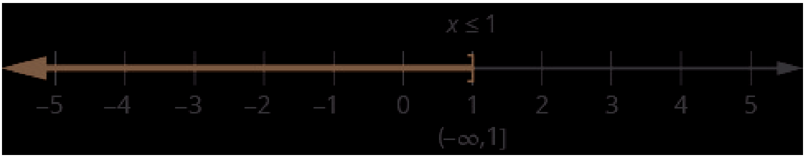


To write in Interval Notation

- The smallest number from the interval is written first.
- The largest number in the interval is written second, following a comma.
- Parentheses, (or), are used to signify that an endpoint value is not included, called exclusive.
- Brackets, [or], are used to indicate that an endpoint value is included, called inclusive.

The inequality $x \leq 1$ means all numbers less than or equal to one. Here we need to show that one is a solution, too. We do that by putting a bracket at $x = 1$. We then shade in all the numbers to the left of one, to show that all numbers less than one are solutions. See [link].

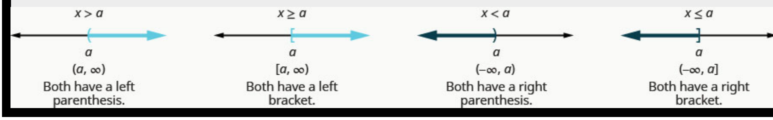
There is no lower end to those numbers. We write $x \leq 1$ in interval notation as $(-\infty, 1]$. The symbol $-\infty$ is read as "negative infinity." [link] shows both the number line and interval notation.



The notation for inequalities on a number line and

in interval notation use the same symbols to express the endpoints of intervals.

Inequalities, Number Lines, and Interval Notation



The main concept to remember is that **parentheses** represent *solutions greater or less than* the number, and **brackets** represent *solutions that are greater than or equal to or less than or equal* to the number. Use parentheses to represent infinity or negative infinity, since positive and negative infinity are not numbers in the usual sense of the word and, therefore, cannot be "equaled."

A few examples of an **interval**, or a set of numbers in which a solution falls, are $[-2,6)$, or all numbers between -2 and 6, including -2, but not including 6; $(-1,0)$, all real numbers between, but not including -1 and 0; and $(-\infty,1]$, all real numbers less than and including 1. [link] outlines the possibilities.

Inequality	Interval Notation	Graph on Number Line	Description
$x > a$	(a, ∞)		x is greater than a
$x < a$	$(-\infty, a)$		x is less than a
$x \geq a$	$[a, \infty)$		x is greater than or equal to a
$x \leq a$	$(-\infty, a]$		x is less than or equal to a
$a < x < b$	(a, b)		x is strictly between a and b
$a \leq x < b$	$[a, b)$		x is between a and b, to include a
$a < x \leq b$	$(a, b]$		x is between a and b, to include b
$a \leq x \leq b$	$[a, b]$		x is between a and b, to include a and b

First we will look at interval notation with inequalities. We will return to the set-builder format in a little bit.

Using Interval Notation to Express All Real Numbers Greater Than or Equal to *a*

Use interval notation to indicate all real numbers greater than or equal to -2.

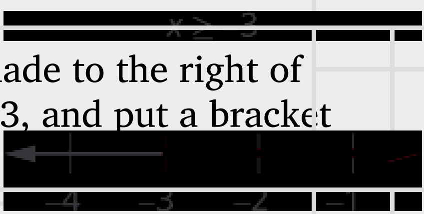
Use a bracket on the left of -2 and parentheses after infinity: $[-2, \infty)$. The bracket indicates that -2 is included in the set with all real numbers greater than -2 to infinity.

Graph and write in Interval Notation

Graph each inequality on the number line and write in interval notation.

ⓐ $x \geq -3$ ⓑ $x < 2.5$ ⓒ $x \leq -35$

ⓐ



Shade to the right of -3, and put a bracket at

Write in interval

no [redacted]

ⓑ

[redacted]

Shade to the left of 2.5 and put a parenthesis at [redacted]

Write in interval notation.

[redacted]

ⓒ

[redacted]

Shade to the left of −35, and put a bracket

at 

Write in interval notation.

$\left(-\infty, -\frac{3}{5}\right]$

What numbers are greater than two but less than five? Are you thinking say, 2.5,3,323,4,4,99? We can represent all the numbers between two and five with the inequality $2<x<5$. We can show $2<x<5$ on the number line by shading all the numbers between two and five. Again, we use the parentheses to show the numbers two and five are not included. See [link].

Graph each inequality on the number line and write in interval notation.

ⓐ $-3<x<4$ ⓑ $-6\le x<-1$ ⓒ $0\le x\le 2.5$

ⓐ

[redacted]

Shade between -3 and 4.

Put a parentheses at -3 and 4.

Write in interval notation.

[redacted]

ⓑ

Shade between -6 and

−1.
Put a bracket at −6, and a parenthesis at −1. Write in interval no[redacted].

©

[redacted]

Shade between 0 an[redacted]
Put a bracket at 0 and at 2.5. Write in interval no[redacted].

Use interval notation to indicate all real numbers between and including -3 and 5.

$[-3,5]$

This format changes slightly in **set-builder notation**. For example, $\{x|10 \le x < 30\}$ describes the behavior of x in set-builder notation. The braces $\{\}$ are read as "the set of," and the vertical bar | is read as "such that," so we would read $\{x|10 \le x < 30\}$ as "the set of *x*-values such that 10 is less than or equal to x, and x is less than 30."

[link] compares inequality notation, set-builder notation, and interval notation.

Set Indicated	Set-Builder Notation	Interval Notation
All real numbers	$\{x|a < x < b\}$	(a,b)

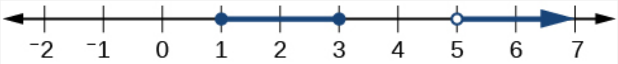
between *a* and *b*, but not including *a* or *b*		
All real numbers greater than *a*, but not including *a*	$\{x \mid x > a\}$	(a, ∞)
All real numbers less than *b*, but not including *b*	$\{x \mid x < b\}$	$(-\infty, b)$
All real numbers greater than *a*, including *a*	$\{x \mid x \geq a\}$	$[a, \infty)$
All real numbers less than *b*, including *b*	$\{x \mid x \leq b\}$	$(-\infty, b]$
All real numbers between *a* and *b*, including *a*	$\{x \mid a \leq x < b\}$	$[a, b)$
All real numbers between *a* and *b*, including *b*	$\{x \mid a < x \leq b\}$	$(a, b]$
All real numbers between *a* and *b*, including *a* and *b*	$\{x \mid a \leq x \leq b\}$	$[a, b]$
All real numbers less than *a* or greater than *b*	$\{x \mid x < a \text{ or } x > b\}$	$(-\infty, a) \cup (b, \infty)$
All real numbers	$\{x \mid x \text{ is a real number}\}$	$(-\infty, \infty)$

So what about when you have two sets of intervals that you need to graph? To combine two intervals using inequality notation or set-builder notation, we use the word “or," and we use the union symbol, $\cup$, to combine two unconnected intervals. For example, the union of the sets $\{2,3,5\}$ and $\{4,6\}$ is the set $\{2,3,4,5,6\}$. It is the set of all elements that belong to one *or* the other (or both) of the original two sets. For sets with a finite number of elements like these, the elements do not have to be listed in ascending order of numerical value. If the original two sets have some elements in common, those elements should be listed only once in the union set. For sets of real numbers on intervals, another example of a union is

$\{ x| \mid x \mid \geq 3 \} = (-\infty, -3] \cup [3, \infty)$

Describing Sets on the Real-Number Line

Describe the intervals of values shown in [link] using inequality notation, set-builder notation, and interval notation.



To describe the values, x, included in the intervals shown, we would say, “ x is a real number greater than or equal to 1 and less

than or equal to 3, or a real number greater than 5."

Inequality	$1 \leq x \leq 3$ or $x > 5$
Set-builder notation	$\{x \mid 1 \leq x \leq 3 \text{ or } x > 5\}$
Interval notation	$[1,3] \cup (5, \infty)$

Remember that, when writing or reading interval notation, using a square bracket means the boundary is included in the set. Using a parenthesis means the boundary is not included in the set.

Solve Linear Inequalities

A linear inequality is much like a linear equation—but the equal sign is replaced with an inequality sign. A **linear inequality** is an inequality in one variable that can be written in one of the forms, $ax + b < c, ax + b \leq c, ax + b > c$, or $ax + b \geq c$.

Linear Inequality

A linear inequality is an inequality in one variable that can be written in one of the following forms where *a*, *b*, and *c* are real numbers and $a \neq 0$:

$ax + b < c, ax + b \leq c, ax + b > c, ax + b \geq c.$

When we solved linear equations, we were able to use the properties of equality to add, subtract, multiply, or divide both sides and still keep the equality. Similar properties hold true for inequalities.

We can add or subtract the same quantity from both sides of an inequality and still keep the inequality. For example:

$$\begin{array}{rcl} -4 &<& 2 \\ -4-5 &<& 2-5 \\ -9 &<& -3 \text{ True} \end{array} \qquad \begin{array}{rcl} -4 &<& 2 \\ -4+7 &<& 2+7 \\ 3 &<& 9 \text{ True} \end{array}$$

Notice that the inequality sign stayed the same.

This leads us to the Addition and Subtraction Properties of Inequality.

Addition and Subtraction Property of Inequality

For any numbers *a*, *b*, and *c*, if $a<b$,then
$a+c<b+c a-c<b-c$ $a+c>b+c a-c>b-c$
We can add or subtract the same quantity from both sides of an inequality and still keep the inequality.

What happens to an inequality when we divide or multiply both sides by a constant?

Let's first multiply and divide both sides by a positive number.

$10<15$	$10<15$
$10(5)<15(5)$	$\frac{10}{5}<\frac{15}{5}$
$50<75$ True	$2<3$ True

The inequality signs stayed the same.

Does the inequality stay the same when we divide or multiply by a negative number?

$10<15$	$10<15$
$10(-5)\ ?\ 15(-5)$	$\frac{10}{-5}\ ?\ \frac{15}{-5}$
$-50\ ?\ -75$	$-2\ ?\ -3$
$-50>-75$	$-2>-3$

Notice that when we filled in the inequality signs, the inequality signs reversed their direction.

When we divide or multiply an inequality by a positive number, the inequality sign stays the same. When we *divide or multiply an inequality by a negative number,* the **inequality sign reverses.**

Multiplication and Division Property of Inequality
For any numbers *a*, *b*, and *c*,
multiply or divide by a positive
$ifa < bandc > 0, thenac < bcandac < bc.$
$ifa > bandc > 0, thenac > bcandac > bc.$ multiply or divide by a
$negativeifa < bandc < 0, thenac > bcandac > bc.$
$ifa > bandc < 0, thenac < bcandac < bc.$

When we divide or multiply an inequality by *a*:

- positive number, the inequality stays the same.
- negative number, the inequality reverses.

Sometimes when solving an inequality, as in the next example, the variable ends upon the right. We can rewrite the inequality in reverse to get the variable to the left.
$x > a$has the same meaning as$a < x$

Think about it as “If Xander is taller than Andy, then Andy is shorter than Xander.”

Solve each inequality. Graph the solution on the number line, and write the solution in interval notation.

ⓐ $x - 38 \leq 34$ ⓑ $9y < 54$ ⓒ $-15 < 35z$

ⓐ

[redacted]

Add 38 to both sides of the inequality.

[redacted]

Simplify.

[redacted]

Graph the solution on the number line.

[redacted]

Write the solution in interval notation.

[redacted]

ⓑ

[redacted]

Divide both sides of the inequality by 9; sin[redacted] 9 is positive, the inequality stays the same.

Simplify.

[redacted]

Graph the solution on the number line.

[redacted]

Write the solution in interval notation.

[redacted]

ⓒ

[redacted]

Multiply both sides of the inequality by 53.

Si[redacted]
the inequality stays the same.

Simplify.

[redacted]

Rewrite with the variable on the left.

[redacted]

Graph the solution on the number line.

[redacted]

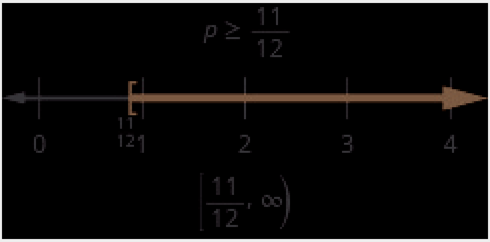
Write the solution in interval notation.

[redacted]

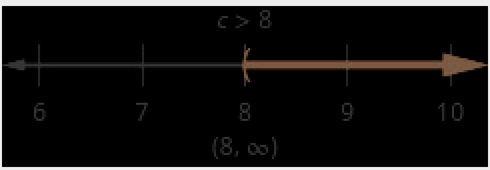
Solve each inequality, graph the solution on the number line, and write the solution in interval notation:

ⓐ $p-34\geq 16$ ⓑ $9c>72$ ⓒ $24\leq 38m$

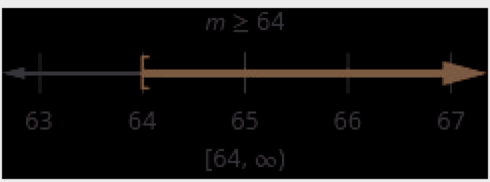
ⓐ



ⓑ



ⓒ



Be careful when you multiply or divide by a negative number—remember to reverse the inequality sign.

Solve each inequality, graph the solution on the number line, and write the solution in interval notation.

ⓐ $-13m \geq 65$ ⓑ $n-2 \geq 8$

ⓐ

[redacted]

Divide both sides of the inequality by -13.
Si[redacted]
negative, the inequality reverses.
Simplify.

[redacted]

Graph the solution on the number line.

[redacted]

Write the solution in interval notation.

[redacted]

ⓑ

[redacted]

Multiply both sides of the inequality by -2. Si [redacted] the inequality reverses.

Simplify.

[redacted]

Graph the solution on the number line.

[redacted]

Write the solution in interval notation.

[redacted]

Most inequalities will take more than one step to solve. We follow the same steps we used in the

general strategy for solving linear equations, but make sure to pay close attention when we multiply or divide to isolate the variable.

Solve the inequality $6y \le 11y + 17$, graph the solution on the number line, and write the solution in interval notation.

[redacted]

Subtract $11y$ from both sides to collect the [redacted] left.

Simplify.

[redacted]

Divide both sides of the inequality by -5, an[redacted] inequality.

Simplify.

$y > \frac{17}{-5}$

Graph the solution on the number line.

Write the solution in interval notation.

$\left[-\frac{17}{5}, \infty\right)$

Solve the inequality, graph the solution on the number line, and write the solution in interval notation: $3q \geq 7q - 23$.

$q \leq \frac{23}{4}$

$\left(-\infty, \frac{23}{4}\right]$

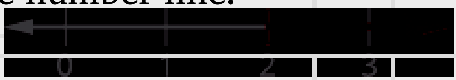
When solving inequalities, it is usually easiest to

collect the variables on the side where the coefficient of the variable is largest. This eliminates negative coefficients and so we don't have to multiply or divide by a negative—which means we don't have to remember to reverse the inequality sign.

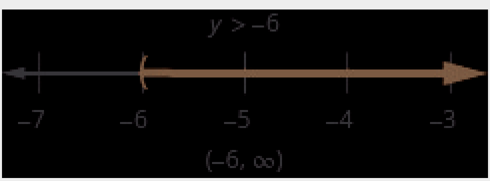
Solve the inequality $8p+3(p-12)>7p-28$, graph the solution on the number line, and write the solution in interval notation.

	$8p+3(p-12)>7p-28$
Simplify each side as much as possible.	
Distribute.	$8p+3p-36>7p-28$
Combine like terms.	$11p-36>7p-28$
Subtract 7p from both sides to collect the variables on the left, since $11>7$.	$11p-36-7p>7p-28-7p$
Simplify.	$4p-36>-28$
Add 36 to both sides to	$4p-36+36>$

collect the constants on the right.	$-28+36$
Simplify.	$4p>8$
Divide both sides of the inequality by 4; the inequality stays the same.	$\frac{4p}{4}>\frac{8}{4}$
Simplify.	$p>2$
Graph the solution on the number line.	
Write the solution in interval notation.	$(2,\infty)$



Solve the inequality $9y+2(y+6)>5y-24$, graph the solution on the number line, and write the solution in interval notation.



Unusual Solution Sets

Just like some equations are identities and some are contradictions, inequalities may be identities or contradictions, too. We recognize these forms when we are left with only constants as we solve the inequality. If the result is a true statement, we have an identity. If the result is a false statement, we have a contradiction.

Solve the inequality $8x-2(5-x)<4(x+9)+6x$, graph the solution on the number line, and write the solution in interval notation.

Simplify each side as much as possible.	$8x-2(5-x)<4(x+9)+6x$
Distribute.	$8x-10+2x<4x+36+6x$
Combine like terms.	$10x-10<10x+36$

Subtract $10x$ from both sides to collect the variables on the left.	$10x - 10 - 10x < 10x + 36 - 10x$
Simplify.	$-10 < 36$
The x's are gone, and we have a true statement.	The inequality is an identity. The solution is all real numbers.
Graph the solution on the nu	
Write the solution in interval notation.	$(-\infty, \infty)$

Unusual Solution Sets

Solve the inequality $4b - 3(3 - b) > 5(b - 6) + 2b$, graph the solution on the number line, and write the solution in interval notation.

Fractions and Inequalities
We can clear fractions in inequalities much as we did in equations. Again, be careful with the signs when multiplying or dividing by a negative.

Solve the inequality $13a-18a>524a+34$, graph the solution on the number line, and write the solution in interval notation.

$$\frac{1}{3}a-\frac{1}{8}a>\frac{5}{24}a+\frac{3}{4}$$

Multiply both sides by the LCD, 24,

to [redacted]

Simplify.

[redacted]

Combine like terms.

[redacted]

Subtract 5a from both sides to collect the variables on the left.

[redacted]

Simplify.

[redacted]

The statement is false. The inequality is a contradiction.

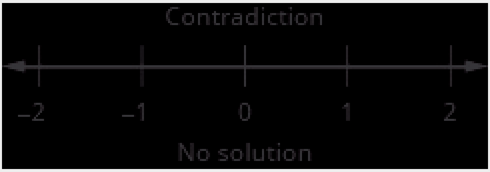
There is no solution.

Graph the solution on the number line.

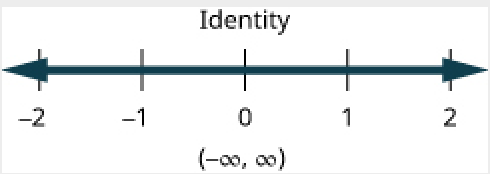
[redacted]

Write the solution in interval notation. There is no solution.

Solve the inequality $14x - 112x > 16x + 78$, graph the solution on the number line, and write the solution in interval notation.



Solve the inequality $25z - 13z < 115z - 35$, graph the solution on the number line, and write the solution in interval notation.



Compound Inequalities

Now that we know how to solve linear inequalities, the next step is to look at compound inequalities. A

compound inequality is made up of two inequalities connected by the word “and” or the word “or.” For example, the following are compound inequalities.

$x + 3 > -4 \text{and} 4x - 5 \leq 3$

$2(y + 1) < 0 \text{or} y - 5 \geq -2$

Compound Inequality

A **compound inequality** is made up of two inequalities connected by the word “and” or the word “or.”

For now, we are going to only focus on the "and" form of compound inequalities. There are two ways to solve compound inequalities: separating them into two separate inequalities or leaving the compound inequality intact and performing operations on all three parts at the same time. We will illustrate both methods.

Method 1 - Seperate inequalities

To solve a compound inequality means to find all values of the variable that make the compound inequality a true statement. We solve compound inequalities using the same techniques we used to solve linear inequalities. We solve each inequality separately and then consider the two solutions.

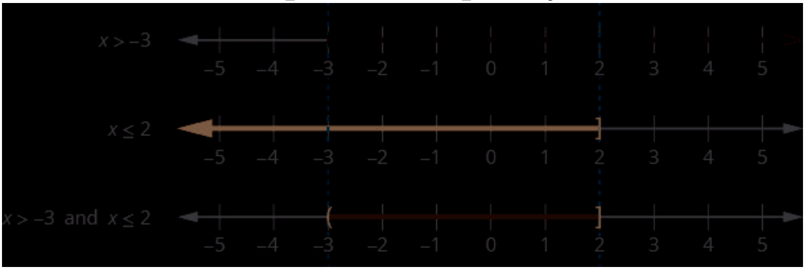
To solve a compound inequality with the word "and," we look for all numbers that make *both* inequalities true.

Let's start with the compound inequalities with "and." Our solution will be the numbers that are solutions to *both* inequalities known as the intersection of the two inequalities. Consider the intersection of two streets—the part where the streets overlap—belongs to both streets.

To find the solution of the compound inequality, we look at the graphs of each inequality and then find the numbers that belong to both graphs—where the graphs overlap.

For the compound inequality $x > -3$ and $x \leq 2$, we graph each inequality. We then look for where the

graphs “overlap”. The numbers that are shaded on both graphs, will be shaded on the graph of the solution of the compound inequality. See [link].



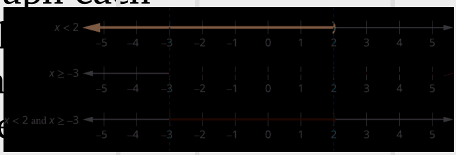
We can see that the numbers between -3 and 2 are shaded on both of the first two graphs. They will then be shaded on the solution graph.

The number -3 is not shaded on the first graph and so since it is not shaded on both graphs, it is not included on the solution graph.

The number two is shaded on both the first and second graphs. Therefore, it is be shaded on the solution graph.

This is how we will show our solution in the next examples.

Solve $6x-3<9$ and $2x+9\geq 3$. Graph the solution and write the solution in interval notation.

Step 1. Solve each inequality.	$6x-3<9$	and	$2x+9\geq 3$
	$6x-3<9$		$2x+9\geq 3$
	$6x<12$		$2x\geq -6$
	$x<2$	and	$x\geq -3$
Step 2. Graph each so Th the numbers that make both inequalities true. The final graph will show all the numbers that make both inequalities true—the numbers shaded on *both* of the first two graphs.			
Step 3. Write the	$[-3,2)$		

solution in interval notation.

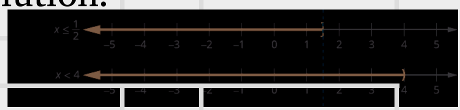
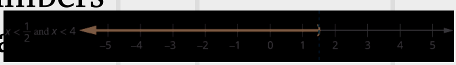
All the numbers that make both inequalities true are the solution to the compound inequality.

Solve a compound inequality with "and."

Solve each inequality. Graph each solution. Then graph the numbers that make *both* inequalities true.
This graph shows the solution to the compound inequality. Write the solution in interval notation.

Solve $3(2x+5) \leq 18$ and $2(x-7) < -6$. Graph the solution and write the solution in interval

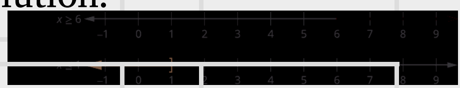
notation.

Solve each inequality.	$3(2x+5) \le 18$	and	$2(x-7) < -6$
	$6x+15 \le 18$		$2x-14 < -6$
	$6x \le 3$		$2x < 8$
	$x \le 12$	and	$x < 4$
Graph each solution.			
Graph the numbers that make both inequalities true.			
Write the solution in interval notation.	$(-\infty, 12]$		

Solve $13x - 4 \geq -2$ and $-2(x - 3) \geq 4$. Graph the solution and write the solution in interval notation.

	$13x - 4 \geq -2$	and	$-2(x - 3) \geq 4$
Solve each inequality.	$13x - 4 \geq -2$		$-2x + 6 \geq 4$
	$13x \geq 2$		$-2x \geq -2$
	$x \geq 6$	and	$x \leq 1$

Graph each solution.



Graph the numbers that make both inequalities true.

There are no numbers that make both inequalities

true.

This is a contradiction so there is no solution.

Method 2 - Operate on all 3 parts
Sometimes we have a compound inequality that can be written more concisely. For example, $a < x$ and $x < b$ can be written simply as $a < x < b$ and then we call it a double inequality. The two forms are equivalent.

Double Inequality
A double inequality is a compound inequality such as $a < x < b$. It is equivalent to $a < x$ and $x < b$.
Other forms: $a < x < b$ is equivalent to $a < x$ and $x < b$
$a \leq x \leq b$ is equivalent to $a \leq x$ and $x \leq b$ $a > x > b$ is equivalent to $a > x$ and $x > b$ $a \geq x \geq b$ is equivalent to $a \geq x$ and $x \geq b$

To solve a double inequality we perform the same

operation on all three “parts” of the double inequality with the goal of isolating the variable in the center.

Solve $-4 \leq 3x-7 < 8$. Graph the solution and write the solution in interval notation.

Add 7 to all three parts.	
Simplify.	
Divide each part by three.	
Simplify.	
Graph the solution.	

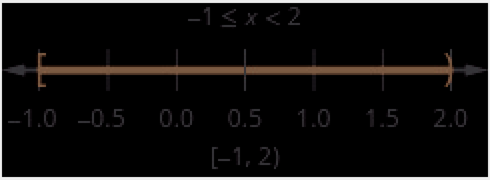
Write the solution in interval notation.

[redacted]

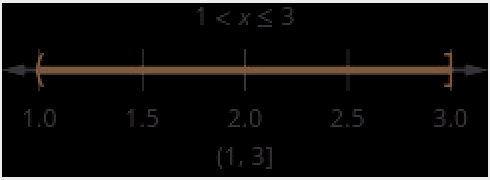
When written as a double inequality, $1 \le x < 5$, it is easy to see that the solutions are the numbers caught between one and five, including one, but not five. We can then graph the solution immediately as we did above.

Another way to graph the solution of $1 \le x < 5$ is to graph both the solution of $x \ge 1$ and the solution of $x < 5$. We would then find the numbers that make both inequalities true as we did in previous examples.

Solve the compound inequality. Graph the solution and write the solution in interval notation: $-5 \le 4x - 1 < 7$.



Solve the compound inequality. Graph the solution and write the solution in interval notation: $-3 < 2x - 5 \leq 1$.



The next example demonstrates solving the inequality using both methods described above.

Solving a Compound Inequality

Solve the compound inequality: $3 \le 2x + 2 < 6$.

The first method is to write two separate inequalities: $3 \le 2x + 2$ and $2x + 2 < 6$. We solve them independently.

$3 \le 2x + 2$ and $2x + 2 < 6$ $1 \le 2x$ $2x < 4$ $\frac{1}{2} \le x$ $x < 2$

Then, we can rewrite the solution as a compound inequality, the same way the problem began.

$\frac{1}{2} \le x < 2$

In interval notation, the solution is written as $\left[\frac{1}{2}, 2\right)$.

The second method is to leave the compound inequality intact, and perform solving procedures on the three parts at the same time.

$3 \le 2x + 2 < 6$ $1 \le 2x < 4$ Isolate the variable term, and subtract 2 from all three parts. $\frac{1}{2} \le x < 2$ Divide through all three parts by 2.

We get the same solution: $\left[\frac{1}{2}, 2\right)$.

Access this online resource for additional instruction and practice with solving compound inequalities.

- Compound inequalities

Absolute Value Inequalities

Recall from the previous module, the work we did with Absolute Value Equations.

Absolute Value Review

The absolute value of a number is its distance from zero on the number line.

The absolute value of a number *n* is written as $|n|$ and $|n| \geq 0$ for all numbers.

Absolute values are always greater than or equal to zero.

Example

$\text{If} |x| = 5 \text{then} x = -5 \text{or} x = 5$

The solution can be simplified to a single statement by writing $x = \pm 5$. This is read, "*x* is equal to positive or negative 5".

How To Solve Equations with Absolute Value

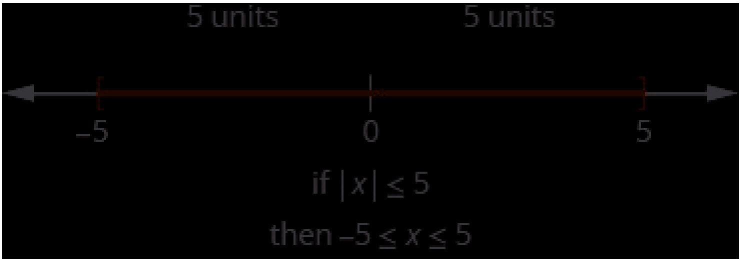
Isolate the absolute value expression. Write the

equivalent equations. Solve each equation. Check each solution.

Now that we have reviewed absolute value and solving equations with them, what about inequalities?

Again we will look at our definition of absolute value. The absolute value of a number is its distance from zero on the number line. For the equation $|x| = 5$, we saw that both 5 and -5 are five units from zero on the number line. They are the solutions to the equation.

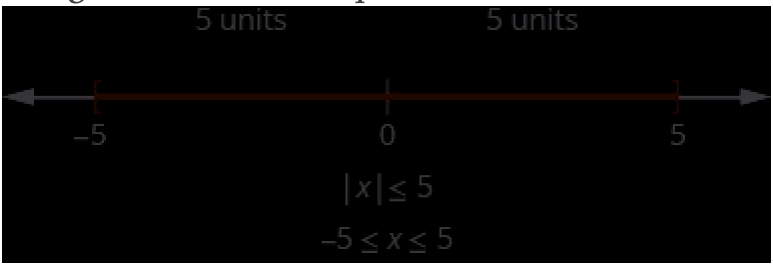
What about the inequality $|x| \leq 5$? Where are the numbers whose distance is less than or equal to 5? We know -5 and 5 are both five units from zero. All the numbers between -5 and 5 are less than five units from zero. See [link].



In a more general way, we can see that if $|x| \leq k$,

then $-k \leq x \leq k$.

What about for those inequalities using "greater than?" Now we want to look at the inequality $|x| \geq 5$. Where are the numbers whose distance from zero is greater than or equal to five?



In a more general way, we can see that if $|x| \geq k$, then $x \leq -k$ or $x \leq k$.

Absolute Value Inequality

For an algebraic expression *X*, and $k > 0$, an **absolute value inequality** is an inequality of the form

$|\ X\ | < k$ is equivalent to $-k < X < k$ $|\ X\ | > k$ is equivalent to $X < -k$ or $X > k$

These statements also apply to $|\ X\ | \leq k$ and $|\ X\ | \geq k$.

Solve Absolute Value Inequalities

Isolate the absolute value expression. Write the equivalent compound inequality.
$|X| < k$ is equivalent to $-k < X < k$ $|X| > k$ is equivalent to $X < -k$ or $X > k$ Solve the compound inequality. Graph the solution Write the solution using interval notation.

Solve $|x| < 7$. Graph the solution and write the solution in interval notation.

[redacted]

Write the equivalent inequality.

[redacted]

Graph the solution.

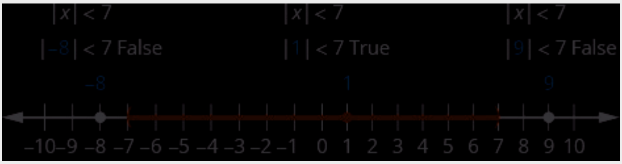
[redacted]

Write the solution using interval notation.

[redacted]

Check:

To verify, check a value in each section of the number line showing the solution. Choose numbers such as -8, 1, and 9.



Solve $|x| > 4$. Graph the solution and write the solution in interval notation.

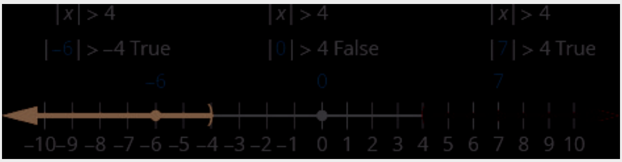
	$|x| > 4$
Write the equivalent inequality.	$x < -4 \text{ or } x > 4$
Graph the solution.	

Write the solution	$(-\infty, -4) \cup (4, \infty)$

using interval notation.

Check:

To verify, check a value in each section of the number line showing the solution. Choose numbers such as -6, 0, and 7.



Solve $|2x-3|\geq 5$. Graph the solution and write the solution in interval notation.

Step 1. Isolate the absolute value expression. It is isolated.

$$|2x-3|\geq 5$$

Step 2. Write the equivalent compound inequality.

$2x-3\leq-5 \text{ or } 2x-3\geq5$

Step 3. Solve the compound inequality.

$2x\leq-2 \text{ or } 2x\geq8$

$x\leq-1 \text{ or } x\geq4$

Step 4. Graph the solution.

Step 5. Write the solution using interval notation.

$(-\infty,-1]\cup[4,\infty)$

Check:

The check is left to you.

Solve $|5x-6|\leq4$. Graph the solution and write the solution in interval notation.

Step 1. Isolate the absolute value expression.

$|5x-6|\leq4$

It is isolated.	
Step 2. Write the equivalent compound inequality.	$-4 \le 5x-6 \le 4$
Step 3. Solve the compound inequality.	$2 \le 5x \le 10$ $25 \le x \le 2$
Step 4. Graph the solution.	

Step 5. Write the solution using interval notation.	[25,2]
Check:	

The check is left to you.

Using a Graphical Approach to Solve Absolute Value Inequalities

Given the equation $y = -\ 1\ 2\,|\,4x-5\,|+3$, determine the *x*-values for which the *y*-values are negative.

We are trying to determine where $y<0$, which is when $-\ 1\ 2\,|\,4x-5\,|+3<0$. We begin by isolating the absolute value.

$-\frac{1}{2}|4x-5|<-3$ Multiply both sides by –2, and reverse the inequality. $|4x-5|>6$

Next, we solve for the equality $|4x-5|=6$.
$4x-5=6$ $4x-5=-6$ $4x=11$ or $4x=-1$ $x=\frac{11}{4}$ $x=-\frac{1}{4}$

Now, we can examine the graph to observe where the *y*-values are negative. We observe where the branches are below the *x*-axis. Notice that it is not important exactly what the graph looks like, as long as we know that it crosses the horizontal axis at $x=-\frac{1}{4}$ and $x=\frac{11}{4}$, and that the graph opens downward. See [link].

[missing_resource: CNX_CAT_Figure_02_07_006.jpg]

Access these online resources for additional instruction and practice with linear inequalities and absolute value inequalities.

- Interval notation
- How to solve linear inequalities
- How to solve an inequality
- Absolute value equations
- Compound inequalities
- Absolute value inequalities

Solve Applications with Linear Inequalities

Many real-life situations require us to solve inequalities. The method we will use to solve applications with linear inequalities is very much like the one we used when we solved applications with equations.

We will read the problem and make sure all the words are understood. Next, we will identify what we are looking for and assign a variable to represent it. We will restate the problem in one sentence to make it easy to translate into an inequality. Then, we will solve the inequality.

Sometimes an application requires the solution to be a whole number, but the algebraic solution to the inequality is not a whole number. In that case, we must round the algebraic solution to a whole number. The context of the application will determine whether we round up or down.

Dawn won a mini-grant of $4,000 to buy tablet computers for her classroom. The tablets

she would like to buy cost \$254.12 each, including tax and delivery. What is the maximum number of tablets Dawn can buy?

Step 1. Readthe problem. Step 2. Identifywhat you are looking for.the maximum number of tablets Dawn can buy Step 3. Namewhat you are looking for. Choose a variable to represent thatquantity.Letn = the number of tablets. Step 4. Translate. Write a sentence that gives theinformation to find it. Translate into an inequality.\$254.12 times the number of tablets isno more than \$4,000.$254.12n \leq 4000$ Step 5. Solvethe inequality.Butnmust be a whole number oftablets, so round to 15.$n \leq 15.74$ $n \leq 15$ Step 6. Checkthe answer in the problemand make sure it makes sense.Rounding down the price to \$250, 15tablets would cost \$3,750, while 16tablets would be \$4,000. So amaximum of 15 tablets at \$254.12seems reasonable. Step 7. Answerthe question with a complete sentence.Dawn can buy a maximum of 15 tablets.

Taleisha’s phone plan costs her \$28.80 a

month plus \$0.20 per text message. How many text messages can she send/receive and keep her monthly phone bill no more than \$50?

Step 1. Readthe problem. Step 2. Identifywhat you are looking for.the number of text messages Taleisha can make Step 3. Namewhat you are looking for. Choose a variable to represent thatquantity.Lett = the number of text messages. Step 4. TranslateWrite a sentence thatgives the information to find it.Translate into an inequality.\$28.80 plus \$0.20 times the number oftext messages is less than or equal to \$50. $28.80 + 0.20t \le 50$ Step 5. Solvethe inequality. $0.2t \le 21.2$ $t \le 106$ text messages Step 6. Checkthe answer in the problemand make sure it makes sense.Yes, $28.80 + 0.20(106) = 50$. Step 7. Writea sentence that answers the question.Taleisha can send/receive no more than106 text messages to keep her bill nomore than \$50.

Sergio and Lizeth have a very tight vacation budget. They plan to rent a car from a company that charges \$75 a week plus \$0.25 a

mile. How many miles can they travel during the week and still keep within their $200 budget?

Sergio and Lizeth can travel no more than 500 miles.

Profit is the money that remains when the costs have been subtracted from the revenue. In the next example, we will find the number of jobs a small businesswoman needs to do every month in order to make a certain amount of profit.

Felicity has a calligraphy business. She charges $2.50 per wedding invitation. Her monthly expenses are $650. How many invitations must she write to earn a profit of at least $2,800 per month?

Step 1. Readthe problem. Step 2. Identifywhat you are looking for.the number of invitations Felicity needs to write Step 3. Namewhat you are looking for.Let$j =$ the number of invitations. Choose a variable to represent it.

Step 4. Translate.Write a sentence thatgives the information to find it. Translate into an inequality.$2.50 times the number of invitationsminus $650 is at least $2,800.$2.50j - 650 \geq 2,800$ Step 5. Solvethe inequality. $2.5j \geq 3,450$ $j \geq 1,380$invitations Step 6. Checkthe answer in the problemand make sure it makes sense. If Felicity wrote 1400 invitations, herprofit would be$2.50(1400) - 650$,or$2,850. This is more than $2800. Step 7. Writea sentence that answers the question.Felicity must write at least 1,380 invitations.

Elliot has a landscape maintenance business. His monthly expenses are $1,100. If he charges $60 per job, how many jobs must he do to earn a profit of at least $4,000 a month?

Elliot must work at least 85 jobs.

There are many situations in which several quantities contribute to the total expense. We must make sure to account for all the individual expenses

when we solve problems like this.

Malik is planning a six-day summer vacation trip. He has \$840 in savings, and he earns \$45 per hour for tutoring. The trip will cost him \$525 for airfare, \$780 for food and sightseeing, and \$95 per night for the hotel. How many hours must he tutor to have enough money to pay for the trip?
Step 1. Readthe problem. Step 2. Identifywhat you are looking for.the number of hours Malik must tutor Step 3. Namewhat you are looking for.Choose a variable to represent thatquantity. Let$h=$the number of hours. Step 4. Translate.Write a sentence thatgives the information to find it. The expenses must be less than or equal tothe income. The cost of airfare plus thecost of food and sightseeing and the hotelbill must be less than the savings plus theamount earned tutoring. Translate into an inequality. $525+780+95(6)\leq 840+45h$ Step 5. Solvethe inequality. $1{,}875\leq 840+45h$ $1{,}035\leq 45h$ $23\leq h$ $h\geq 23$ Step 6. Checkthe answer in the problemand make sure it makes sense.We substitute 23 into the inequality. $1{,}875\leq 840+45h$ $1{,}875\leq 840+45(23)$ $1{,}875\leq 1875$

Step 7. Writea sentence that answers the question.Malik must tutor at least 23 hours.

Brenda’s best friend is having a destination wedding and the event will last three days. Brenda has \$500 in savings and can earn \$15 an hour babysitting. She expects to pay \$350 airfare, \$375 for food and entertainment and \$60 a night for her share of a hotel room. How many hours must she babysit to have enough money to pay for the trip?

Brenda must babysit at least 27 hours.

Key Concepts

- **Inequalities, Number Lines, and Interval Notation**
  $x > a x \geq a x < a x \leq a$

- **Linear Inequality**
    - A **linear inequality** is an inequality in one variable that can be written in one of the following forms where *a*, *b*, and *c* are real numbers and $a \neq 0$: $ax+b<c, ax+b\leq c, ax+b>c, ax+b\geq c.$
- **Addition and Subtraction Property of Inequality**
    - For any numbers *a*, *b*, and *c*, if $a<b$, then $a+c<b+c \quad a-c<b-c \quad a+c>b+c \quad a-c>b-c$
    - We can add or subtract the same quantity from both sides of an inequality and still keep the inequality.
- **Multiplication and Division Property of Inequality**
    - For any numbers *a*, *b*, and *c*, multiply or divide by a positive if $a<b$ and $c>0$, then $ac<bc$ and $\frac{a}{c}<\frac{b}{c}$. if $a>b$ and $c>0$, then $ac>bc$ and $\frac{a}{c}>\frac{b}{c}$. multiply or divide by a negative if $a<b$ and $c<0$, then $ac>bc$ and $\frac{a}{c}>\frac{b}{c}$. if $a>b$ and $c<0$, then $ac<bc$ and $\frac{a}{c}<\frac{b}{c}$.

- **Phrases that indicate inequalities**

$>$	$\geq$	$<$	$\leq$
is greater than is more than is larger than exceeds	is greater than or equal to is at least is no less than is the minimum	is less than is smaller than has fewer than is lower than	is less than or equal to is at most is no more than is the maximum

- **Absolute Value**
  The absolute value of a number is its distance from 0 on the number line.
  The absolute value of a number n is written as $|n|$ and $|n| \geq 0$ for all numbers.
  Absolute values are always greater than or equal to zero.

Practice Makes Perfect

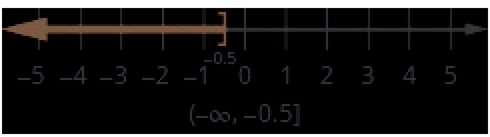
Graph Inequalities on the Number Line

In the following exercises, graph each inequality on the number line and write in interval notation.

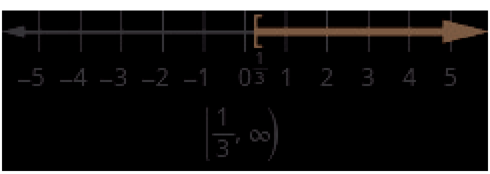
ⓐ $x > 3$
ⓑ $x \leq -0.5$
ⓒ $x \geq 13$

ⓐ

ⓑ

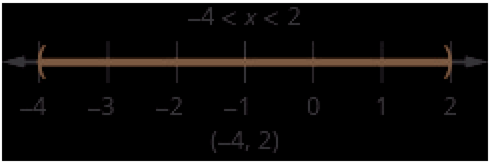


ⓒ



ⓐ $-4 < x < 2$
ⓑ $-5 < x \leq -2$
ⓒ $-3.75 \leq x \leq 0$

ⓐ



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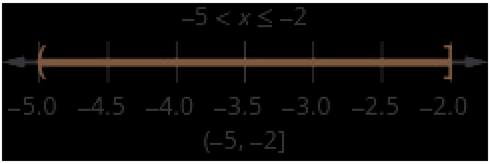

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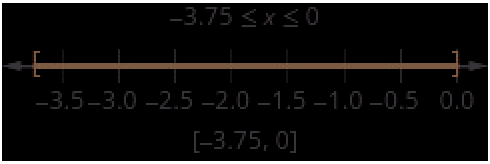


For the following exercises, write the set in interval notation or in set-builder notation.

$\{ x | -1 < x < 3 \}$

$(-1,3)$

$(-\infty,6)$

$\{ x | x < 6 \}$

$\{ x | x < 4 \}$

$(-\infty,4)$

$[-3,5)$

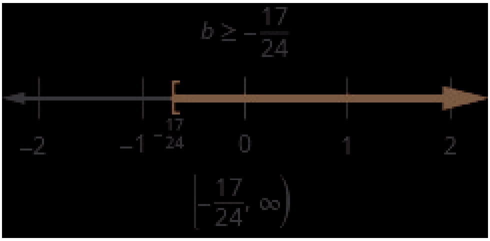
$\{ x | -3 \le x < 5 \}$

Solve Linear Inequalities

In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

ⓐ $b + 78 \geq 16$
ⓑ $6y < 48$
ⓒ $40 < 58k$

ⓐ



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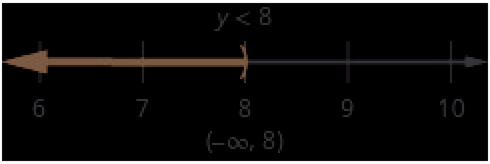
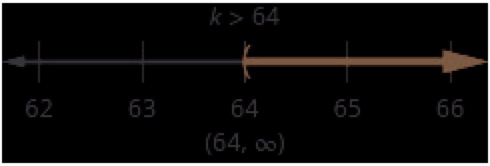
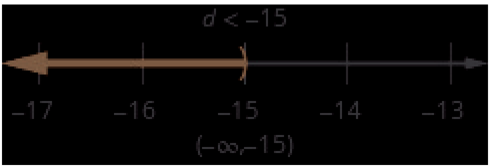


ⓒ

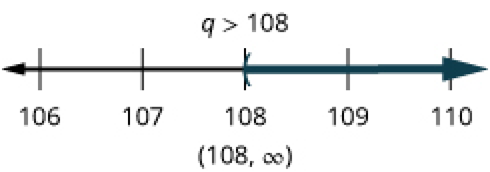


ⓐ $-7d > 105$
ⓑ $-18 > q - 6$

ⓐ

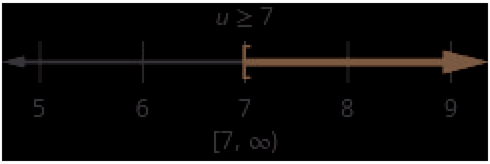


ⓑ

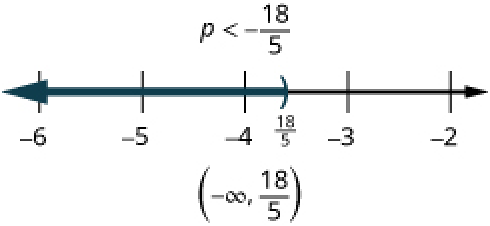


In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

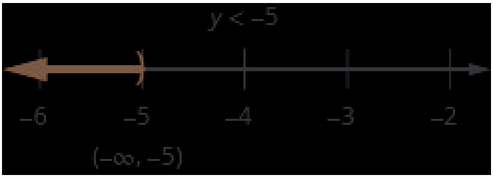
$5u \le 8u - 21$



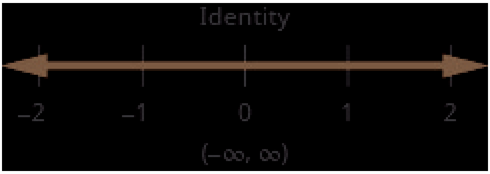
$9p > 14p - 18$



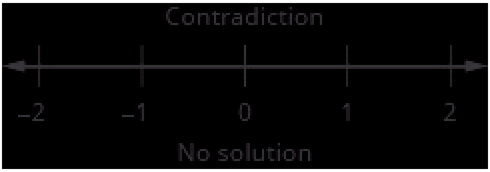
$9y + 5(y + 3) < 4y - 35$



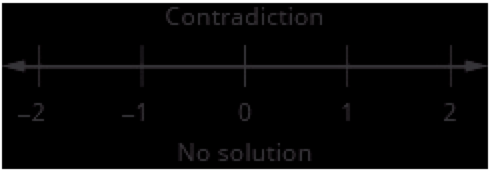
$$6n - 12(3 - n) \leq 9(n - 4) + 9n$$



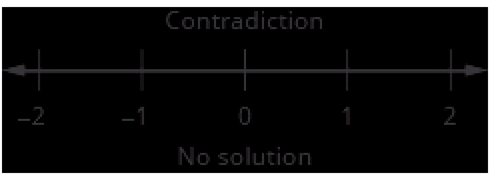
$$9u + 5(2u - 5) \geq 12(u - 1) + 7u$$



$$45h - 23(h - 9) \geq 115(2h + 90)$$

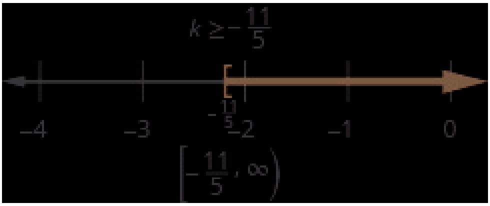


$$12v + 3(4v - 1) \leq 19(v - 2) + 5v$$

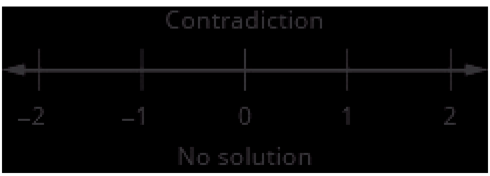


In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

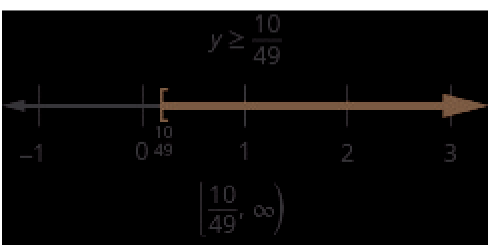
$$35k \geq -77$$



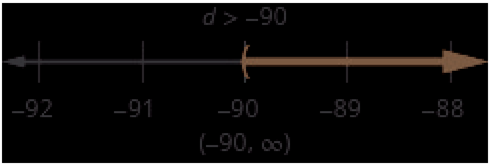
$$18q - 4(10 - 3q) < 5(6q - 8)$$



$$-218y \leq -1528$$



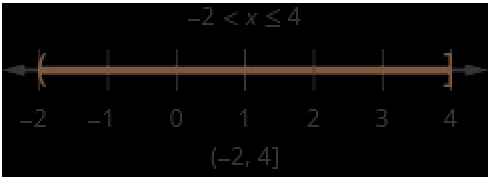
$d+29>-61$



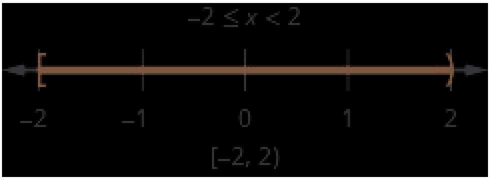
Solve Compound Inequalities with "and"

In the following exercises, solve each inequality, graph the solution, and write the solution in interval notation.

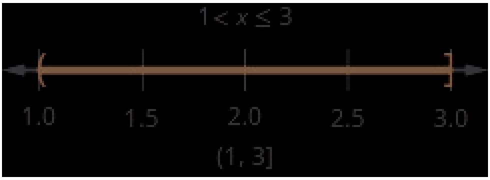
$x \le 4$ and $x>-2$



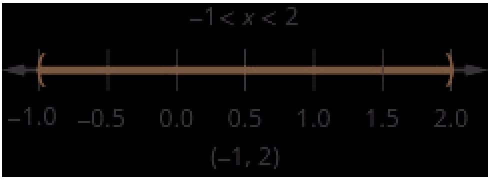
$4x-1<7$ and $2x+8 \ge 4$



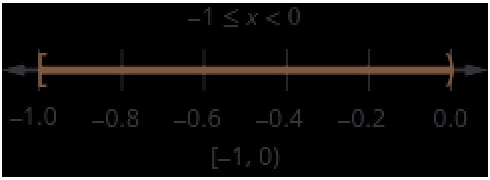
$-3 < 2x - 5 \leq 1$



$-1 < 3x + 2 < 8$



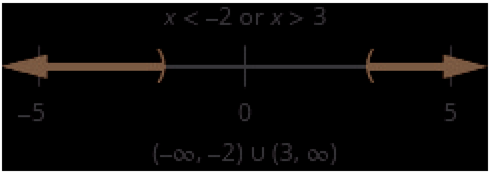
$-6 \leq 4x - 2 < -2$



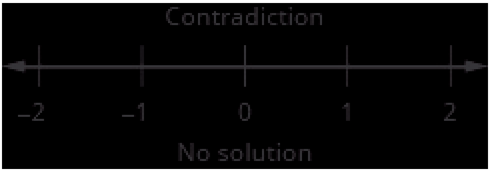
Solve Absolute Value Inequalities

In the following exercises, solve each inequality. Graph the solution and write the solution in interval notation.

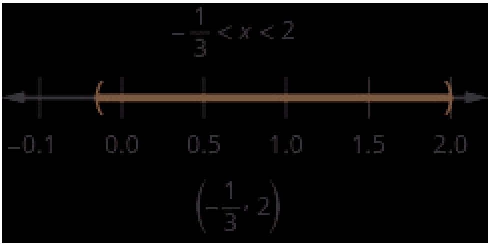
$|2x-1|>5$



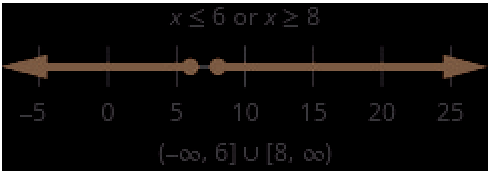
$|3x-7|+3<1$



$|6x - 5| < 7$



$|x - 7| \geq 1$



$|x - 2| + 4 \geq 10$

$(-\infty, -4] \cup [8, +\infty)$

Solve Applications with Linear Inequalities

In the following exercises, solve.

The elevator in Yehire’s apartment building has a sign that says the maximum weight is 2100 pounds. If the average weight of one person is 150 pounds, how many people can safely ride the elevator?

A maximum of 14 people can safely ride in the elevator.

Kimuyen needs to earn $4,150 per month in order to pay all her expenses. Her job pays her $3,475 per month plus 4% of her total sales. What is the minimum Kimuyen’s total sales must be in order for her to pay all her expenses?

$16,875

Nataly is considering two job offers. The first job would pay her $83,000 per year. The second would pay her $66,500 plus 15% of her total sales. What would her total sales need to be for her salary on the second offer be higher than the first?

$110,000

Kiyoshi's phone plan costs $17.50 per month plus $0.15 per text message. What is the maximum number of text messages Kiyoshi can use so the phone bill is no more than $56.60?

---

260 messages

Kellen wants to rent a banquet room in a restaurant for her cousin's baby shower. The restaurant charges $350 for the banquet room plus $32.50 per person for lunch. How many people can Kellen have at the shower if she wants the maximum cost to be $1,500?

---

35 people

Noe installs and configures software on home computers. He charges $125 per job. His monthly expenses are $1,600. How many jobs must he work in order to make a profit of at least $2,400?

---

32 jobs

Melissa makes necklaces and sells them online. She charges $88 per necklace. Her monthly

expenses are $3,745. How many necklaces must she sell if she wants to make a profit of at least $1,650?

---

62 necklaces

Cesar is planning a four-day trip to visit his friend at a college in another state. It will cost him $198 for airfare, $56 for local transportation, and $45 per day for food. He has $189 in savings and can earn $35 for each lawn he mows. How many lawns must he mow to have enough money to pay for the trip?

---

seven lawns

Eun-Kyung works as a tutor and earns $60 per hour. She has $792 in savings. She is planning an anniversary party for her parents. She would like to invite 40 guests. The party will cost her $1,520 for food and drinks and $150 for the photographer. She will also have a favor for each of the guests, and each favor will cost $7.50. How many hours must she tutor to have enough money for the party?

---

20 hours

Everyday Math

Maximum weight on a boat In 2004, a water taxi sank in Baltimore harbor and five people drowned. The water taxi had a maximum capacity of 3,500 pounds (25 people with average weight 140 pounds). The average weight of the 25 people on the water taxi when it sank was 168 pounds per person. What should the maximum number of people of this weight have been?

---

20 people

Shower budget Penny is planning a baby shower for her daughter-in-law. The restaurant charges $950 for up to 25 guests, plus $31.95 for each additional guest. How many guests can attend if Penny wants the total cost to be no more than $1,500?

---

42 guests

Glossary

compound inequality

A compound inequality is made up of two inequalities connected by the word “and” or the word “or.”

interval
: an interval describes a set of numbers within which a solution falls

interval notation
: a mathematical statement that describes a solution set and uses parentheses or brackets to indicate where an interval begins and ends

linear inequality
: similar to a linear equation except that the solutions will include sets of numbers

Relations and Functions (2.1)
By the end of this section, you will be able to:

- Find the domain and range of a relation
- Determine if a relation is a function
- Find the value of a function
- Use the Vertical Line Test

This Module supports section **2.1 of Mat 1023**.
In case you missed something in class, or just want to review a specific topic covered in this Module, here is a list of topics covered:

1. Domain and Range [link]
2. Identify if a Relation is a Function [link]
3. Find the Value of a Function [link]
4. Vertical Line Test [link]
5. Read Information from Graphs [link]
6. Key Concepts [link]

Find the Domain and Range of a Relation

As we go about our daily lives, we have many data items or quantities that are paired to our names.

Our social security number, student ID number, email address, phone number and our birthday are matched to our name. There is a relationship between our name and each of those items.

When your professor gets her class roster, the names of all the students in the class are listed in one column and then the student ID number is likely to be in the next column. If we think of the correspondence as a set of ordered pairs, where the first element is a student name and the second element is that student's ID number, we call this a **relation**.
(Student name, Student ID #)

The set of all the names of the students in the class is called the **domain** of the relation and the set of all student ID numbers paired with these students is the range of the relation.

There are many similar situations where one variable is paired or matched with another. The set of ordered pairs that records this matching is a relation.

Relation
A **relation** is any set of ordered pairs,(x,y). All the *x*-values in the ordered pairs together make up the **domain**. All the *y*-values in the ordered pairs

together make up the **range**.

$\{(1,2),(2,4),(3,6),(4,8),(5,10)\}$

The domain is $\{1,2,3,4,5\}$. The range is $\{2,4,6,8,10\}$. Note that each value in the domain is also known as an **input** value, or **independent variable**, and is often labeled with the lowercase letter x. Each value in the range is also known as an **output** value, or **dependent variable**, and is often labeled lowercase letter y.

For the relation $\{(1,1),(2,4),(3,9),(4,16),(5,25)\}$:

ⓐ Find the domain of the relation.

ⓑ Find the range of the relation.

$\{(1,1),(2,4),(3,9),(4,16),(5,25)\}$

ⓐ The domain is the set of all x-values of the relation. $\{1,2,3,4,5\}$

ⓑ The range is the set of all y-values of the

relation. {1,4,9,16,25}

For the relation {(1,1),(2,8),(3,27),(4,64),(5,125)}:

ⓐ Find the domain of the relation.

ⓑ Find the range of the relation.

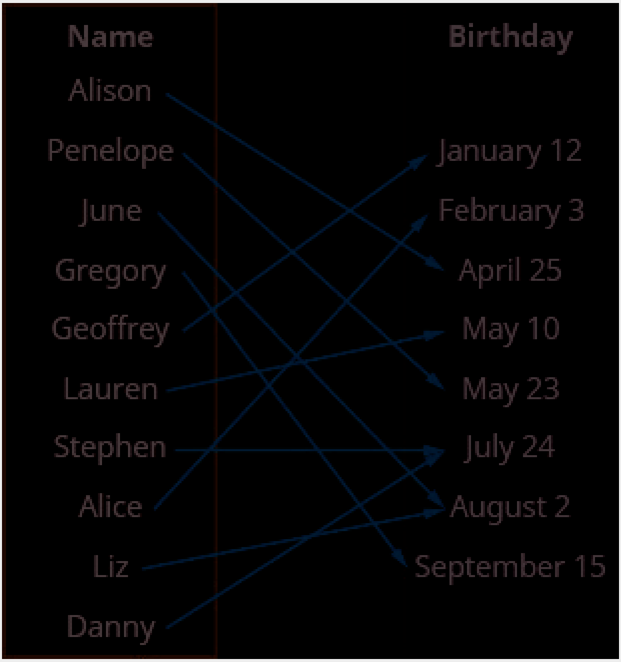
ⓐ {1,2,3,4,5}
ⓑ {1,8,27,64,125}

Mapping

A **mapping** is sometimes used to show a relation. The arrows show the pairing of the elements of the domain with the elements of the range.

Use the **mapping** of the relation shown to ⓐ list the ordered pairs of the relation, ⓑ find the domain of the relation, and ⓒ find the

range of the relation.



ⓐ The arrow shows the matching of the person to their birthday. We create ordered pairs with the person's name as the x-value and their birthday as the y-value.

{(Alison, April 25), (Penelope, May 23), (June, August 2), (Gregory, September 15), (Geoffrey, January 12), (Lauren, May 10), (Stephen, July 24), (Alice, February 3), (Liz, August 2), (Danny, July 24)}

ⓑ The domain is the set of all x-values of the

relation.

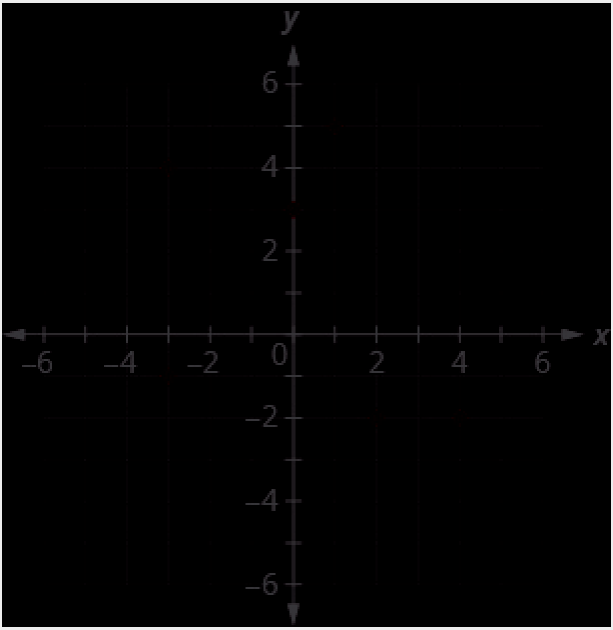
{Alison, Penelope, June, Gregory, Geoffrey, Lauren, Stephen, Alice, Liz, Danny}

ⓒ The range is the set of all y-values of the relation.

{January 12, February 3, April 25, May 10, May 23, July 24, August 2, September 15}

A graph is yet another way that a relation can be represented. The set of ordered pairs of all the points plotted is the relation. The set of all x-coordinates is the domain of the relation and the set of all y-coordinates is the range. Generally we write the numbers in ascending order for both the domain and range.

Use the graph of the relation to ⓐ list the ordered pairs of the relation ⓑ find the domain of the relation ⓒ find the range of the relation.



ⓐ The ordered pairs of the relation are: $\{(1,5),(-3,-1),(4,-2),(0,3),(2,-2),(-3,4)\}$.

ⓑ The domain is the set of all x-values of the relation: $\{-3,0,1,2,4\}$.

Notice that while -3 repeats, it is only listed once.

ⓒ The range is the set of all y-values of the relation: $\{-2,-1,3,4,5\}$.

Notice that while -2 repeats, it is only listed once.

(a) This relationship is a function because each input is associated with a single output. Note that input q and r both give output n. (b) This relationship is also a function. In this case, each input is associated with a single output. (c) This relationship is not a function because input q is associated with two different outputs.

Determine if a Relation is a Function

A special type of relation, called a **function**, occurs extensively in mathematics. A function is a relation that assigns to each element in its domain exactly one element in the range. For each ordered pair in the relation, each *x*-value is matched with only one *y*-value. In other words, no *x*-values are repeated.

Function
A **function** is a relation in which each possible input value leads to exactly one output value. We say “the output is a function of the input.”
The **input** values make up the **domain**, and the **output** values make up the **range**.

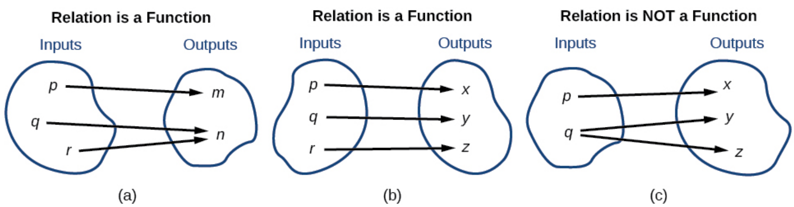
The birthday example helps us understand this definition. Every person has a birthday but no one has two birthdays. It is okay for two people to share

a birthday. It is okay that Danny and Stephen share July 24th as their birthday and that June and Liz share August 2nd. Since each person has exactly one birthday, the relation is a function.

The relation shown by the graph in [link] includes the ordered pairs $(-3,-1)$ and $(-3,4)$. Is that okay in a function? No, as this is like one person having two different birthdays.

Using the example set above,
$\{(1,2),(2,4),(3,6),(4,8),(5,10)\}$

this relation is a function because each element in the domain, $\{1,2,3,4,5\}$, is paired with exactly one element in the range, $\{2,4,6,8,10\}$. [link] compares relations that are functions and not functions.



Use the set of ordered pairs to (i) determine whether the relation is a function (ii) find the domain of the relation (iii) find the range of the relation.

ⓐ $\{(-3,27),(-2,8),(-1,1),(0,0),(1,1),(2,8),(3,27)\}$

ⓑ $\{(9,-3),(4,-2),(1,-1),(0,0),(1,1),(4,2),(9,3)\}$

ⓐ $\{(-3,27),(-2,8),(-1,1),(0,0),(1,1),(2,8),(3,27)\}$

(i) Each x-value is matched with only one y-value. So this relation is a function.

(ii) The domain is the set of all x-values in the relation.
The domain is: $\{-3,-2,-1,0,1,2,3\}$.

(iii) The range is the set of all y-values in the relation. Notice we do not list range values twice.
The range is: $\{27,8,1,0\}$.

ⓑ $\{(9,-3),(4,-2),(1,-1),(0,0),(1,1),(4,2),(9,3)\}$

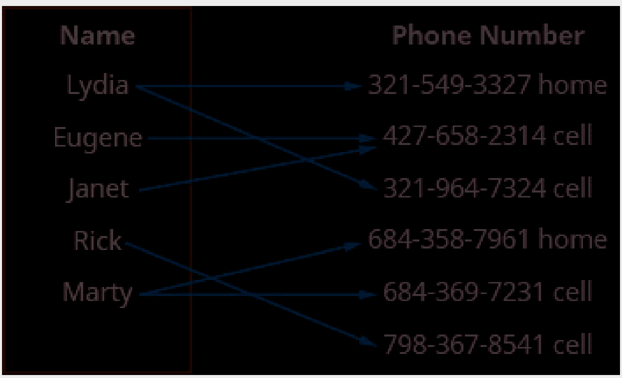
(i) The x-value 9 is matched with two y-values, both 3 and -3. So this relation is not a function.

(ii) The domain is the set of all x-values in the relation. Notice we do not list domain values twice.

The domain is: $\{0,1,2,4,9\}$.

(iii) The range is the set of all y-values in the relation.
The range is: $\{-3,-2,-1,0,1,2,3\}$.

Use the mapping to ⓐ determine whether the relation is a function ⓑ find the domain of the relation ⓒ find the range of the relation.



ⓐ Both Lydia and Marty have two phone numbers. So each x-value is not matched with only one y-value. So this relation is not a function.

ⓑ The domain is the set of all x-values in the

relation. The domain is: {Lydia, Eugene, Janet, Rick, Marty}

ⓒ The range is the set of all *y*-values in the relation. The range is:

{321-549-3327, 427-658-2314, 321-964-7324, 684-358-7961, 684-369-7231, 798-367-8541}

In algebra, more often than not, functions will be represented by an equation. It is easiest to see if the equation is a function when it is solved for *y*. If each value of *x* results in only one value of *y*, then the equation defines a function.

Determine whether each equation is a function.

ⓐ $2x+y=7$ ⓑ $y=x2+1$ ⓒ$x+y2=3$

ⓐ $2x+y=7$

For each value of *x*, we multiply it by -2 and then add 7 to get the *y*-value

[redacted]

For example, if $x=3$:

[redacted]

[redacted]

We have that when $x=3$, then $y=1$. It would work similarly for any value of *x*. Since each value of *x*, corresponds to only one value of *y* the equation defines a function.

ⓑ $y=x2+1$

For each value of *x*, we square it and then add 1 to get the *y*-value.

[redacted]

For example, if $x=2$:

[redacted]

[redacted]

We have that when $x = 2$, then $y = 5$. It would work similarly for any value of x. Since each value of x, corresponds to only one value of y the equation defines a function.

ⓒ

[redacted]

Isolate the y term.

[redacted]

Let's substitute $x = 2$.

[redacted]

[redacted]

This give us two values $y = 1$ $y = -1$ for y.

We have shown that when $x = 2$, then $y = 1$ and $y = -1$. It would work similarly for any value of x. Since each value of x does not

corresponds to only one value of *y* the equation does not define a function.

Determine whether each equation is a function.

ⓐ $x+y2=4$ ⓑ $y=x2-7$ ⓒ $y=5x-4$

ⓐ no ⓑ yes ⓒ yes

Find the Value of a Function

It is very convenient to name a function and most often we name it *f*, *g*, *h*, *F*, *G*, or *H*. In any function, for each *x*-value from the domain we get a corresponding *y*-value in the range. For the function *f*, we write this range value *y* as f(x). This is called function notation and is read *f* of *x* or the value of *f* at *x*. In this case the parentheses does not indicate multiplication.

Function Notation

For the function $y = f(x)$

f is the name of the function x is the domain value

$f(x)$ is the range value y corresponding to the value x

We read $f(x)$ as *f* of *x* or the value of *f* at *x*.

Example

h is f of a We name the function f; height is a function of age. $h = f(a)$ We use parentheses to indicate the function input. $f(a)$ We name the function f; the expression is read as “f of a.”

Remember, we can use any letter to name the function; the notation $h(a)$ shows us that h depends on a. The value a must be put into the function h to get a result. The parentheses indicate that age is input into the function; they do not indicate multiplication.

We call *x* the independent variable as it can be any value in the domain. We call *y* the dependent variable as its value depends on *x*.

Independent and Dependent Variables

For the function $y = f(x)$,

x is the independent variable as it can be any value in the domain y the dependent variable as its value depends on x

Much as when you first encountered the variable *x*, function notation may be rather unsettling. It seems strange because it is new. You will feel more comfortable with the notation as you use it.

Let’s look at the equation $y = 4x - 5$. To find the value of *y* when $x = 2$, we know to substitute $x = 2$ into the equation and then simplify.

[redacted]

Let $x = 2$.

[redacted]

[redacted]

The value of the function at $x = 2$ is 3.

We do the same thing using function notation, the equation $y = 4x - 5$ can be written as $f(x) = 4x - 5$. To find the value when $x = 2$, we write:

[redacted]

Let $x = 2.$

[redacted]

[redacted]

The value of the function at $x = 2$ is 3.

This process of finding the value of f(x) for a given value of *x* is called *evaluating the function.*

For the function $f(x) = 2x2 + 3x - 1$, evaluate the function.

ⓐ $f(3)$ ⓑ $f(-2)$ ⓒ $f(a)$

ⓐ

[redacted]

To evaluate f(3), substitute 3 for x.

[redacted]

Simplify.

[redacted]

[redacted]

[redacted]

ⓑ

[redacted]

[redacted]

Simplify.

[redacted]

[redacted]

[redacted]

ⓒ

[redacted]

To evaluate f(a), substitute a for x.

[redacted]

Simplify.

[redacted]

For the function $f(x) = 3x2 - 2x + 1$, evaluate the function.

ⓐ $f(3)$ ⓑ $f(-1)$ ⓒ $f(t)$

ⓐ $f(3)=22$ ⓑ $f(-1)=6$ ⓒ $f(t)=3t2-2t-1$

In the last example, we found f(x) for a constant value of *x*. In the next example, we are asked to find g(x) with values of *x* that are variables. We still follow the same procedure and substitute the variables in for the *x*.

For the function $g(x)=3x-5$, evaluate the function.

ⓐ $g(h2)$ ⓑ $g(x+2)$ ⓒ $g(x)+g(2)$

ⓐ

[redacted]

To evaluate g(h2),

su[redacted]
h2 for x.

[redacted]

ⓑ

[redacted]

To evaluate g(x + 2), substitute x + 2 for x.

[redacted]

Simplify.

[redacted]

[redacted]

ⓒ

$$g(x) = 3x - 5$$

To evaluate $g(x) + g(2)$, first find $g(2)$.

$$g(2) = 3 \cdot 2 - 5$$

$$g(2) = 1$$

$$g(x) + g(2) = \underbrace{3x - 5}_{g(x)} + \underbrace{1}_{g(2)}$$

Simplify.

$$g(x) + g(2) = 3x - 5 + 1$$

$$g(x) + g(2) = 3x - 4$$

Notice the difference between part ⓑ and ⓒ. We get $g(x+2) = 3x+1$ and $g(x) + g(2) = 3x - 4$. So we see that $g(x+2) \neq g(x) + g(2)$.

For the function $g(x) = 4x - 7$, evaluate the function.

ⓐ $g(m2)$ ⓑ $g(x-3)$ ⓒ $g(x) - g(3)$

ⓐ $4m2-7$ ⓑ $4x-19$
ⓒ $x-12$

Some functions are defined by mathematical rules or procedures expressed in equation form. If it is possible to express the function output with a formula involving the input quantity, then we can define a function in algebraic form. For example, the equation $2n+6p=12$ expresses a functional relationship between n and p. We can rewrite it to decide if p is a function of n.

Does the equation $x^2+y^2=1$ represent a function with x as input and y as output? If so, express the relationship as a function $y=f(x)$.

First we subtract x^2 from both sides.

$y^2=1-x^2$

We now try to solve for y in this equation.

$y=\pm\sqrt{1-x^2}=+\sqrt{1-x^2}$ and $-\sqrt{1-x^2}$

We get two outputs corresponding to the same

input, so this relationship cannot be represented as a single function $y=f(x)$. If we graph both functions on a graphing calculator, we will get the upper and lower semicircles.

If $x-8y\ 3\ =0$, express y as a function of x.

$y=f(x)=\ x\ 3\ 2$

Many everyday situations can be modeled using functions.

The number of unread emails in Sylvia’s account is 75. This number grows by 10 unread emails a day. The function $N(t)=75+10t$ represents the relation between the number of emails, *N*, and the time, *t*, measured in days.

ⓐ Determine the independent and dependent

variable.

ⓑ Find N(5). Explain what this result means.

ⓐ The number of unread emails is a function of the number of days. The number of unread emails, *N*, depends on the number of days, *t*. Therefore, the variable *N*, is the dependent variable and the variable t is the independent variable.

ⓑ Find N(5). Explain what this result means.

[redacted]

Substitute in t = 5.

[redacted]

Simplify.

[redacted]

[redacted]

Since 5 is the number of days, N(5), is the number of unread emails after 5 days. After 5 days, there are 125 unread emails in the account.

Access this online resource for additional instruction and practice with relations and functions.

- Introduction to Functions
- Find Domain and Range

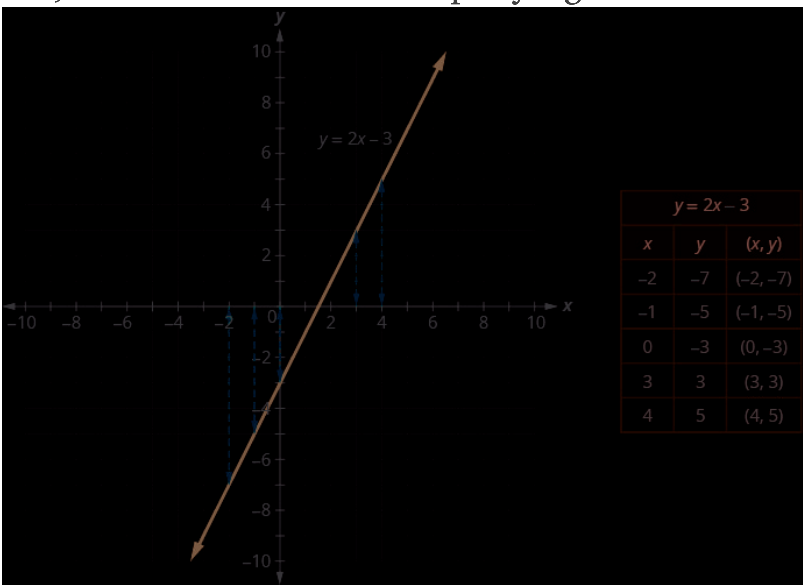
Vertical Line Test

In the last section we learned how to determine if a relation is a function. The relations we looked at were expressed as a set of ordered pairs, a mapping or an equation. We will now look at how to tell if a graph is that of a function.

An ordered pair (x,y) is a solution of a linear equation, if the equation is a true statement when the *x*- and *y*-values of the ordered pair are substituted into the equation.

The graph of a linear equation is a straight line where every point on the line is a solution of the equation and every solution of this equation is a point on this line.

In [link], we can see that, in graph of the equation $y = 2x - 3$, for every *x*-value there is only one *y*-value, as shown in the accompanying table.



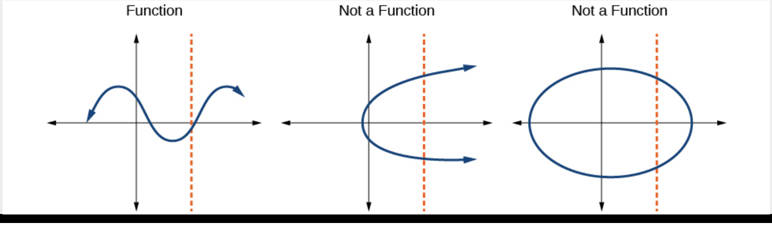
A relation is a function if every element of the domain has exactly one value in the range. So the relation defined by the equation $y = 2x - 3$ is a function. If we look at the graph, each vertical dashed line only intersects the line at one point. This makes sense as in a function, for every *x*-value there is only one *y*-value. If the vertical line hit the graph twice, the *x*-value would be mapped to two *y*-values, and so the graph would not represent a function. This leads us to the vertical line test. A set

of points in a rectangular coordinate system is the graph of a function if every vertical line intersects the graph in at most one point. If any vertical line intersects the graph in more than one point, the graph does not represent a function.

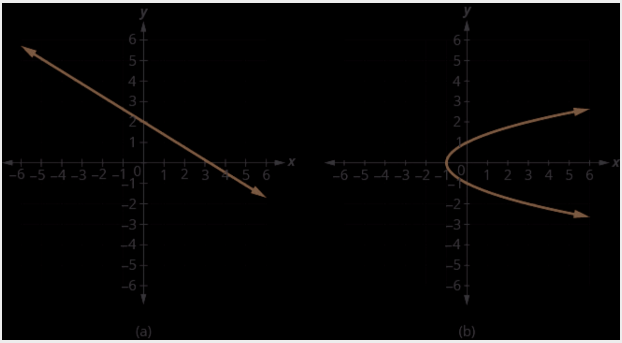
Vertical Line Test

A set of points in a rectangular coordinate system is the graph of a function if every vertical line intersects the graph in at most one point.

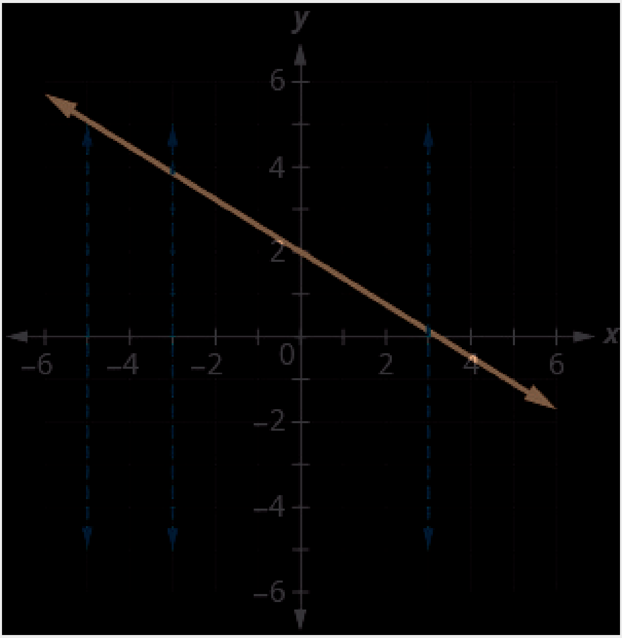
If any vertical line intersects the graph in more than one point, the graph does not represent a function.



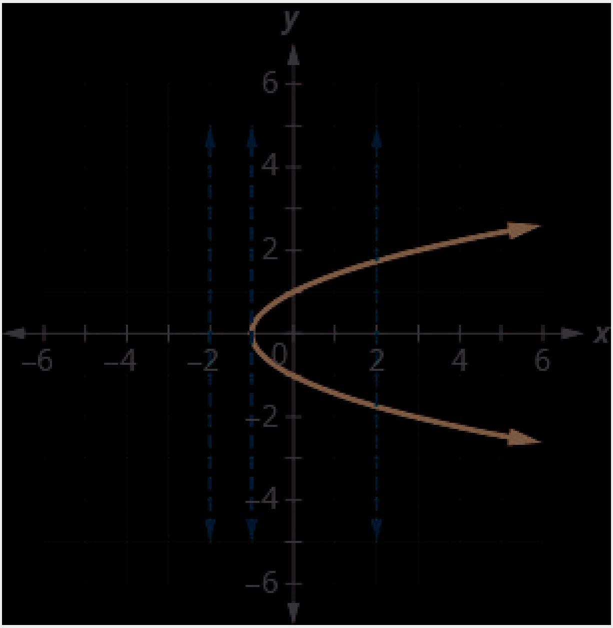
Determine whether each graph is the graph of a function.



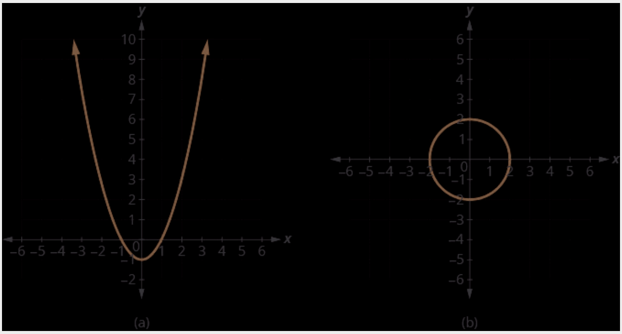
ⓐ Since any vertical line intersects the graph in at most one point, the graph is the graph of a function.



ⓑ One of the vertical lines shown on the graph, intersects it in two points. This graph does not represent a function.



Determine whether each graph is the graph of a function.



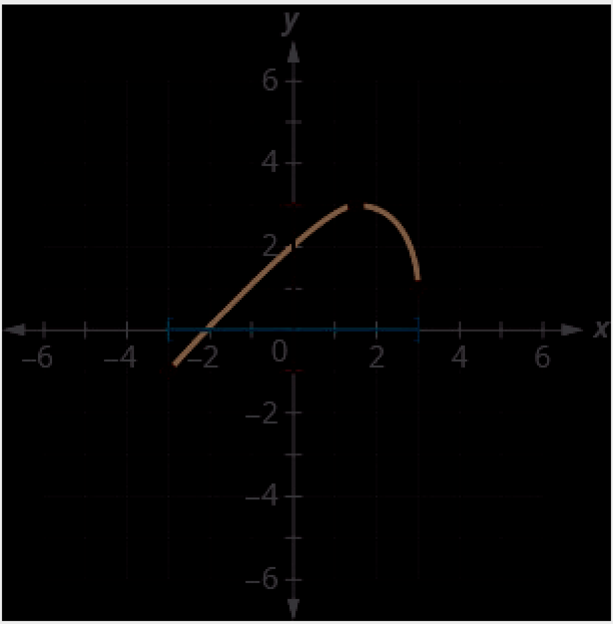
ⓐ yes ⓑ no

Read Information from Graphs

We will start by reading the domain and range of a function from its graph. Remember the domain is the set of all the x-values in the ordered pairs in the function. To find the domain we look at the graph and find all the values of x that have a corresponding value on the graph. Follow the value x up or down vertically. If you hit the graph of the function then x is in the domain. Remember the range is the set of all the y-values in the ordered pairs in the function. To find the range we look at the graph and find all the values of y that have a corresponding value on the graph. Follow the value y left or right horizontally. If you hit the graph of

the function then y is in the range.

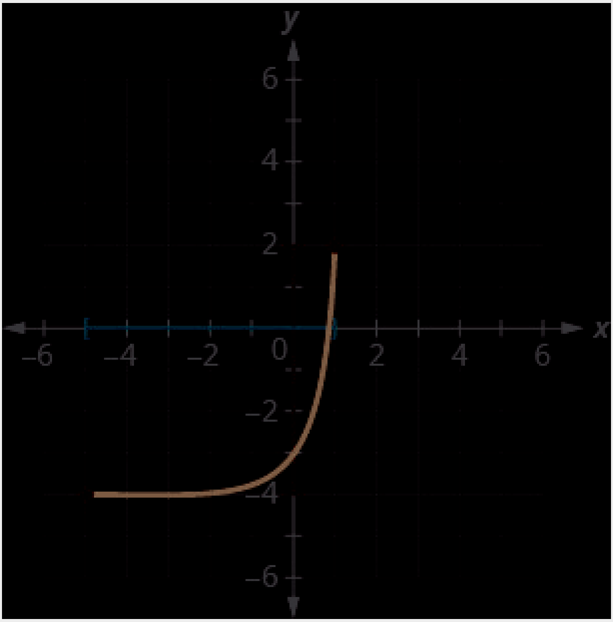
Use the graph of the function to find its domain and range. Write the domain and range in interval notation.



To find the domain we look at the graph and find all the values of x that correspond to a point on the graph. The domain is highlighted in red on the graph. The domain is $[-3,3]$.

To find the range we look at the graph and find all the values of y that correspond to a point on the graph. The range is highlighted in blue on the graph. The range is $[-1,3]$.

Use the graph of the function to find its domain and range. Write the domain and range in interval notation.



The domain is $[-5,1]$. The range is $[-4,2]$.

Key Concepts

- **Function Notation:** For the function $y = f(x)$

    - f is the name of the function
    - x is the domain value
    - $f(x)$ is the range value y corresponding to the value x
      We read $f(x)$ as f of x or the value of f at x.

- **Independent and Dependent Variables:** For the function $y = f(x)$,

    - x is the independent variable as it can be any value in the domain
    - y is the dependent variable as its value depends on x

- **Vertical Line Test**

    - A set of points in a rectangular coordinate system is the graph of a function if every vertical line intersects the graph in at most one point.
    - If any vertical line intersects the graph in more than one point, the graph does not represent a function.
    - **Graph of a Function**

- The graph of a function is the graph of all its ordered pairs, (x,y) or using function notation, $(x,f(x))$ where $y=f(x)$.
  fname of functionxx-coordinate of the ordered pairf(x)y-coordinate of the ordered pair

Practice Makes Perfect

Find the Domain and Range of a Relation

In the following exercises, for each relation ⓐ find the domain of the relation ⓑ find the range of the relation.

$\{(1,4),(2,8),(3,12),(4,16),(5,20)\}$

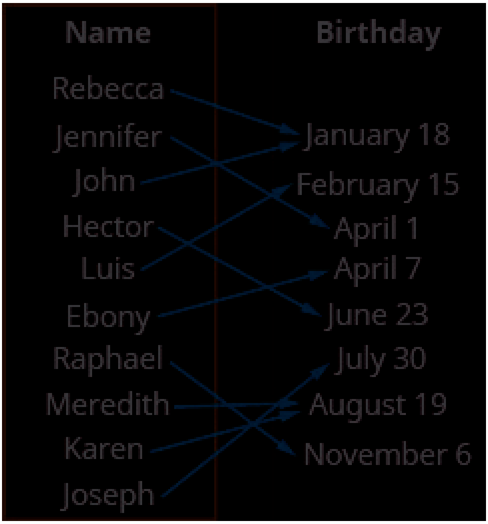
ⓐ $\{1, 2, 3, 4, 5\}$ ⓑ $\{4, 8, 12, 16, 20\}$

$\{(1,7),(5,3),(7,9),(-2,-3),(-2,8)\}$

ⓐ $\{1, 5, 7, -2\}$ ⓑ $\{7, 3, 9, -3, 8\}$

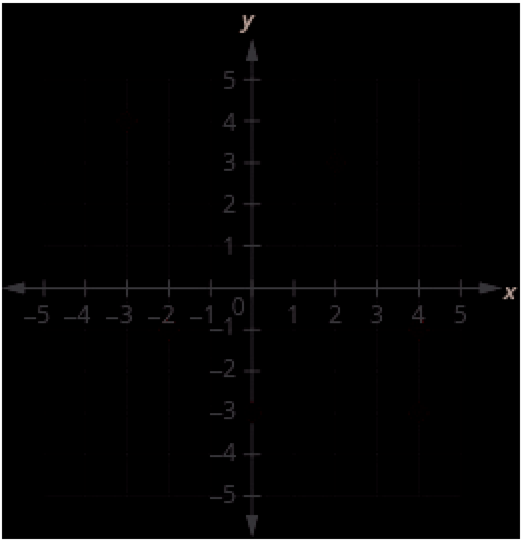
$\{(11,3),(-2,-7),(4,-8),(4,17),(-6,9)\}$

In the following exercises, use the mapping of the relation to ⓐ list the ordered pairs of the relation, ⓑ find the domain of the relation, and ⓒ find the range of the relation.



ⓐ (Rebecca, January 18), (Jennifer, April 1), (John, January 18), (Hector, June 23), (Luis, February 15), (Ebony, April 7), (Raphael, November 6), (Meredith, August 19), (Karen, August 19), (Joseph, July 30)
ⓑ {Rebecca, Jennifer, John, Hector, Luis, Ebony, Raphael, Meredith, Karen, Joseph}
ⓒ {January 18, April 1, June 23, February 15, April 7, November 6, August 19, July 30}

In the following exercises, use the graph of the relation to ⓐ list the ordered pairs of the relation ⓑ find the domain of the relation ⓒ find the range of the relation.



ⓐ $(2,\ 3),\ (4,\ -3),\ (-2,\ -1),\ (-3,\ 4),\ (4,\ -1),$ $(0,\ -3)$ ⓑ $\{-3,\ -2,\ 0,\ 2,\ 4\}$
ⓒ $\{-3,\ -1,\ 3,\ 4\}$

Determine if a Relation is a Function

In the following exercises, use the set of ordered pairs to ⓐ determine whether the relation is a function, ⓑ find the domain of the relation, and ⓒ

find the range of the relation.

$\{(-3,9),(-2,4),(-1,1),$
$(0,0),(1,1),(2,4),(3,9)\}$

ⓐ yes ⓑ $\{-3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3\}$ ⓒ $\{9,\ 4,\ 1,\ 0\}$

$\{(-3,27),(-2,8),(-1,1),$
$(0,0),(1,1),(2,8),(3,27)\}$

ⓐ yes ⓑ $\{-3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3\}$ ⓒ $0,\ 1,\ 8,\ 27\}$

In the following exercises, determine whether each equation is a function.

ⓐ $2x+y=-3$
ⓑ $y=x2$
ⓒ$x+y2=-5$

ⓐ yes ⓑ yes ⓒ no

ⓐ $y-3x3=2$

ⓑ $x+y^2=3$
ⓒ $3x-2y=6$

ⓐ yes ⓑ no ⓒ yes

Find the Value of a Function

In the following exercises, evaluate the function: ⓐ $f(2)$ ⓑ $f(-1)$ ⓒ $f(a)$.

$f(x)=5x-3$

ⓐ $f(2)=7$ ⓑ $f(-1)=-8$ ⓒ $f(a)=5a-3$

$f(x)=x^2-x+3$

ⓐ $f(2)=5$ ⓑ $f(-1)=5$
ⓒ $f(a)=a^2-a+3$

$f(x)=2x^2-x+3$

ⓐ $f(2)=9$ ⓑ $f(-1)=6$
ⓒ $f(a)=2a^2-a+3$

In the following exercises, evaluate the function.

$F(x) = 2x2 - 3x + 1;$
$F(-1)$

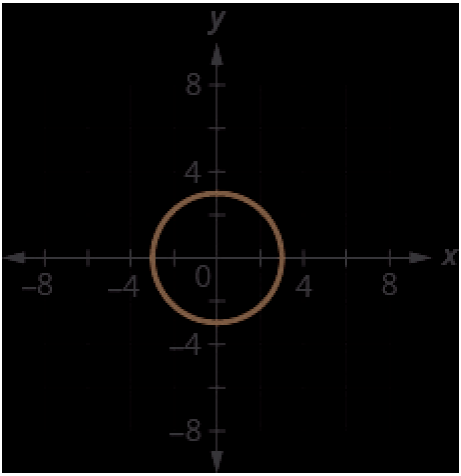
6

$f(x) = x + 2x - 1;\ f(2)$

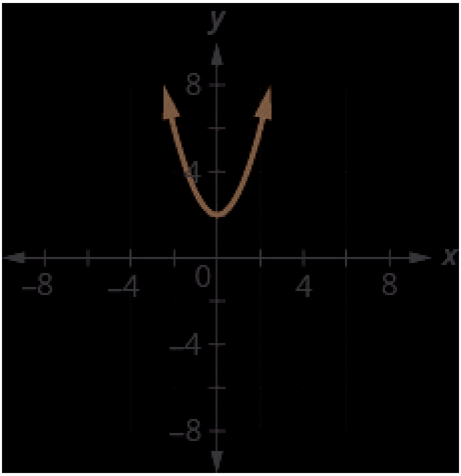
4

Use the Vertical Line Test

ⓐ

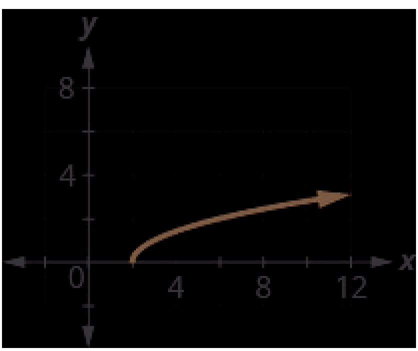


ⓑ

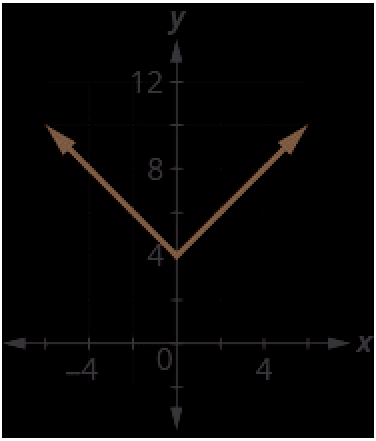


ⓐ no ⓑ yes

Read Information from a Graph of a Function
In the following exercises, use the graph of the function to find its domain and range. Write the domain and range in interval notation.



D: $[2, \infty)$, R: $[0, \infty)$



D: $(-\infty, \infty)$, R: $[4, \infty)$

Glossary

dependent variable
: an output variable

domain of a relation
: The domain of a relation is all the x-values in the ordered pairs of the relation.

independent variable
: an input variable

function
: A function is a relation that assigns to each element in its domain exactly one element in

the range.

mapping
A mapping is sometimes used to show a relation. The arrows show the pairing of the elements of the domain with the elements of the range.

range of a relation
The range of a relation is all the *y*-values in the ordered pairs of the relation.

relation
A relation is any set of ordered pairs,(x,y). All the *x*-values in the ordered pairs together make up the domain. All the *y*-values in the ordered pairs together make up the range.

vertical line test
a method of testing whether a graph represents a function by determining whether a vertical line intersects the graph no more than once

More on Functions (2.2)
This module will look at even/odd functions, and introduce difference quotients as they identify slope.

This Module supports section **2.2 of Mat 1023**.
In case you missed something in class, or just want to review a specific topic covered in this Module, here is a list of topics covered:

1. Determining Even and Odd Functions [link]
2. Difference Quotient and Slope [link]
3. Key Concepts [link]

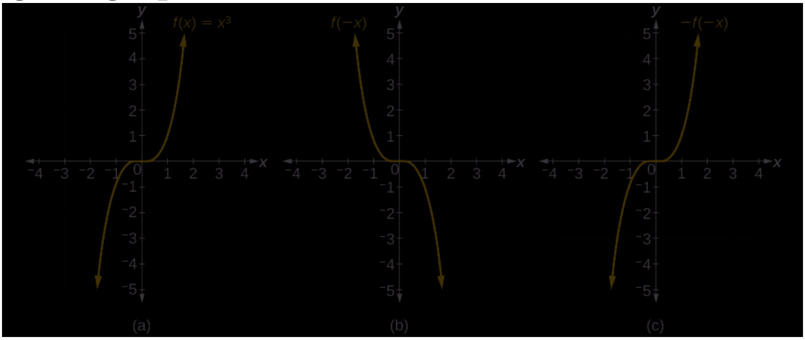
(a) The cubic toolkit function (b) Horizontal reflection of the cubic toolkit function (c) Horizontal and vertical reflections reproduce the original cubic function.

Determining Even and Odd Functions

Some functions exhibit symmetry so that reflections result in the original graph. For example, horizontally reflecting the toolkit functions $f(x) = x$ 2 or $f(x) = |x|$ will result in the original graph. We say that these types of graphs are symmetric about the *y*-axis. A function whose graph is symmetric

about the *y*-axis is called an **even function.**

If the graphs of $f(x)=x^3$ or $f(x)=\frac{1}{x}$ were reflected over *both* axes, the result would be the original graph, as shown in [link].



We say that these graphs are symmetric about the origin. A function with a graph that is symmetric about the origin is called an **odd function**.

Note: A function can be neither even nor odd if it does not exhibit either symmetry. For example, $f(x)=2^x$ is neither even nor odd. Also, the only function that is both even and odd is the constant function $f(x)=0$.

Even and Odd Functions

A function is called an **even function** if for every input x

$$f(x)=f(-x)$$

The graph of an even function is symmetric about the y-axis.

A function is called an **odd function** if for every input x

$f(x) = -f(-x)$

The graph of an odd function is symmetric about the origin.

Given the formula for a function, determine if the function is even, odd, or neither.

1. Determine whether the function satisfies $f(x) = f(-x)$. If it does, it is even.
2. Determine whether the function satisfies $f(x) = -f(-x)$. If it does, it is odd.
3. If the function does not satisfy either rule, it is neither even nor odd.

Determining whether a Function Is Even, Odd, or Neither

Is the function $f(x) = x^3 + 2x$ even, odd, or neither?

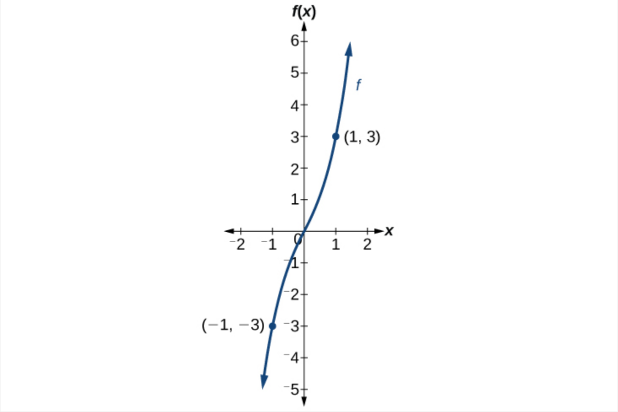
Without looking at a graph, we can determine whether the function is even or odd by finding formulas for the reflections and determining if they return us to the original function. Let's

begin with the rule for even functions.
$f(-x)=(-x)^3+2(-x)=-x^3-2x$

This does not return us to the original function, so this function is not even. We can now test the rule for odd functions.
$-f(-x)=-(-x^3-2x)=x^3+2x$

Because $-f(-x)=f(x)$, this is an odd function.

Analysis
Consider the graph of f in [link]. Notice that the graph is symmetric about the origin. For every point (x,y) on the graph, the corresponding point $(-x,-y)$ is also on the graph. For example, (1, 3) is on the graph of f, and the corresponding point $(-1,-3)$ is also on the graph.



Is the function $f(s)= s^4 +3s^2 +7$ even, odd, or neither?
even

$f(x)=3x^4$
even

$h(x)= \frac{1}{x} +3x$
odd

Difference Quotient

While we will look more closely at slope in the next section,

What is Slope?
The slope of a line, *m*, represents the veritcal change in *y* over the horizontal change in *x*. Given two points, (x_1, y_1) and (x_2, y_2), the following formula determines the slope of a line containing these points:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Slope also indicates the direction in which a line slants as well as its steepness. Slope is sometimes described as **rise over run.**

We can use the slope of a line to a function at a point (a,f(a)) to estimate the rate of change, or the rate at which one variable changes in relation to another variable. We can obtain the slope by choosing a value of x near a and drawing a line through the points (a,f(a)) and (x,f(x)), as shown in [link]. The slope of this line is given by an equation in the form of a difference quotient:

$$m = \frac{f(x) - f(a)}{x - a}.$$

We can also calculate the slope of a line to a function at a value *a* by using this equation and replacing x with $a+h$, where h is a value close to 0.

We can then calculate the slope of the line through the points $(a,f(a))$ and $(a+h,f(a+h))$. In this case, we find the line has a slope given by the following difference quotient with increment h:
$m=f(a+h)-f(a)a+h-a=f(a+h)-f(a)h.$

Difference Quotient
Let f be a function defined on an interval I containing a. If $x\neq a$ is in I, then
$Q=f(x)-f(a)x-a$
is a **difference quotient**.
Also, if $h\neq 0$ is chosen so that $a+h$ is in I, then
$Q=f(a+h)-f(a)h$
is a difference quotient with increment h.

Given the function $g(x)= x 2 +2x$, simplify $g(x)-g(a) x-a , x\neq a$.

$g(x)-g(a) x-a =x+a+2, x\neq a$

Evaluating Functions at Specific Values

Evaluate $f(x) = x^2 + 3x - 4$ at

1. 2
2. a
3. $a+h$
4. $\frac{f(a+h) - f(a)}{h}$

Replace the x in the function with each specified value.

1. Because the input value is a number, 2, we can use simple algebra to simplify.
   $f(2) = 2^2 + 3(2) - 4 = 4 + 6 - 4 = 6$
2. In this case, the input value is a letter so we cannot simplify the answer any further.
   $f(a) = a^2 + 3a - 4$
3. With an input value of $a+h$, we must use the distributive property.
   $f(a+h) = (a+h)^2 + 3(a+h) - 4 = a^2 + 2ah + h^2 + 3a + 3h - 4$
4. In this case, we apply the input values to the function more than once, and then perform algebraic operations on the result. We already found that
   $f(a+h) = a^2 + 2ah + h^2 + 3a + 3h - 4$

   and we know that
   $f(a) = a^2 + 3a - 4$

   Now we combine the results and simplify.

$$\frac{f(a+h)-f(a)}{h}=\frac{(a^2+2ah+h^2+3a+3h-4)-(a^2+3a-4)}{h}=\frac{2ah+h^2+3h}{h}=\frac{h(2a+h+3)}{h}\text{ Factor out h.}=2a+h+3\text{Simplify.}$$

Key Concepts

- Even functions are symmetric about the y- axis, whereas odd functions are symmetric about the origin.
- Even functions satisfy the condition $f(x)=f(-x)$.
- Odd functions satisfy the condition $f(x)=-f(-x)$.
- A function can be odd, even, or neither. See [link].
- **Difference quotient** $Q=\frac{f(x)-f(a)}{x-a}$
- **Difference quotient with increment** h $Q=\frac{f(a+h)-f(a)}{a+h-a}=\frac{f(a+h)-f(a)}{h}$

Practice Makes Perfect

Even / Odd

For the following exercises, determine whether the function is odd, even, or neither.

$h(x) = 1\ x\ + 3x$

---

odd

$g(x) = 2\ x\ 4$

---

even

Slope Basics (2.3)
By the end of this section, you will be able to:

- Find the slope of a line
- Graph a line given a point and the slope
- Graph a line using its slope and intercept
- Find an equation of the line given the slope and y-intercept
- Find an equation of the line given the slope and a point
- Find an equation of the line given two points

This Module supports section **2.3 of Mat 1023**.
In case you missed something in class, or just want to review a specific topic covered in this Module, here is a list of topics covered:

1. Defining Slope [link]
2. Slope and Equations [link]
3. Graphing and Slope [link]
4. Vertical and Horizontal Lines [link]
5. Key Concepts [link]

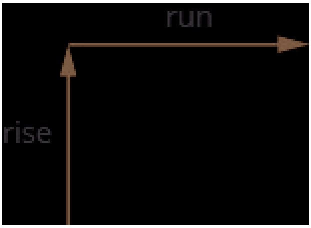
Defining Slope

When you graph linear equations, you may notice that some lines tilt up as they go from left to right and some lines tilt down. Some lines are very steep and some lines are flatter.

In mathematics, the measure of the steepness of a line is called the *slope* of the line.

The concept of slope has many applications in the real world. In construction the pitch of a roof, the slant of the plumbing pipes, and the steepness of the stairs are all applications of slope. and as you ski or jog down a hill, you definitely experience slope.

We can assign a numerical value to the slope of a line by finding the ratio of the rise and run. The *rise* is the amount the vertical distance changes while the *run* measures the horizontal change, as shown in this illustration. Slope is a **rate of change.** See [link].



Slope of a Line

The slope of a line is m=riserun.

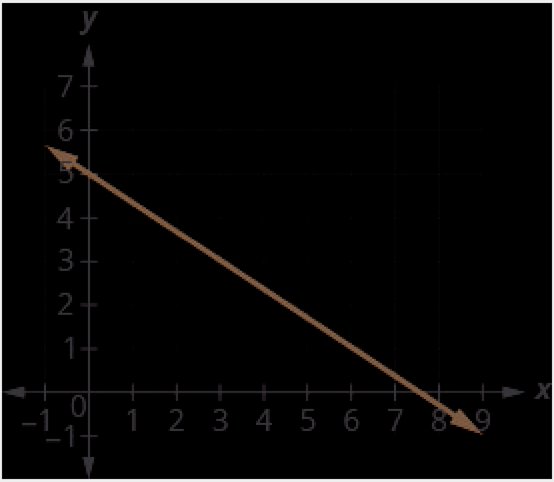
The rise measures the vertical change and the run

measures the horizontal change.

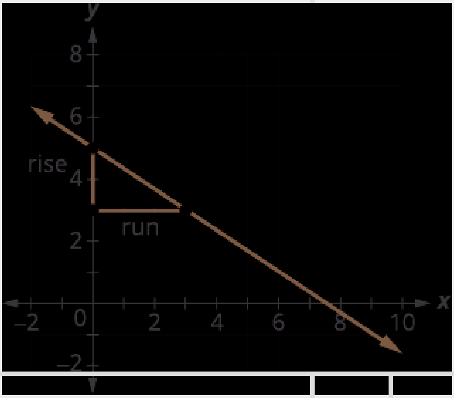
Locate two points on the line whose coordinates are integers. Starting with one point, sketch a right triangle, going from the first point to the second point. Count the rise and the run on the legs of the triangle. Take the ratio of rise to run to find the slope: $m = \text{riserun}$.

To find the slope of a line, we locate two points on the line whose coordinates are integers. Then we sketch a right triangle where the two points are vertices and one side is horizontal and one side is vertical.

Find the slope of the line shown.

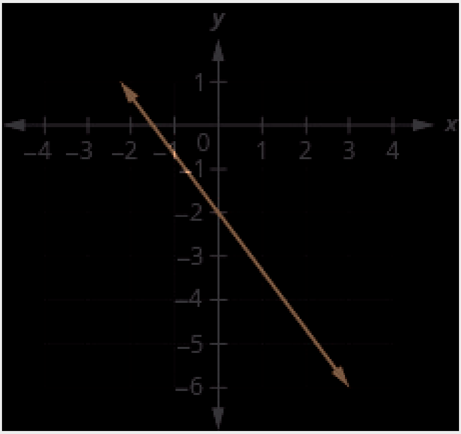


Locate two points on the graph whose coordinates are integers.	(0,5) and (3,3)
Starting at (0,5), sketch a right triangle to (3,3) as shown in this graph.	



Count the rise— since it goes down, it is negative.	The rise is -2.
Count the run.	The run is 3.
Use the slope formula.	$m = \frac{\text{rise}}{\text{run}}$
Substitute the values of the rise and run.	$m = \frac{-2}{3}$
Simplify.	$m = -\frac{2}{3}$
	The slope of the line is $-\frac{2}{3}$.
	So y decreases by 2 units as x increases by 3 units.

Find the slope of the line shown.



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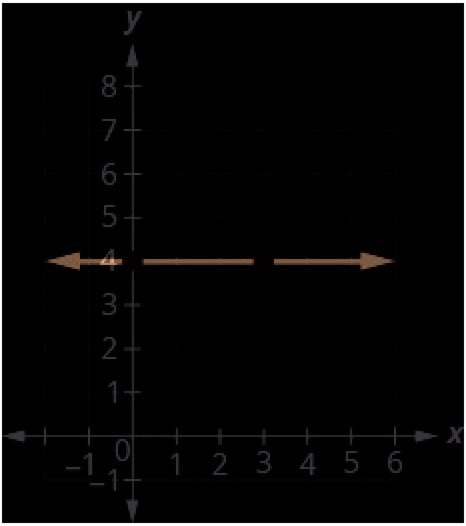
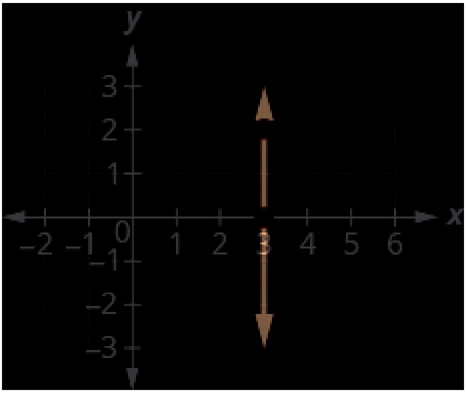
How do we find the slope of horizontal and vertical lines? To find the slope of the horizontal line, $y = 4$, we could graph the line, find two points on it, and count the rise and the run. Let’s see what happens when we do this, as shown in the graph below.



What is the rise?The rise is 0. What is the run?The run is 3. What is the slope?$m=$riserun $m=03$ $m=0$ The slope of the horizontal line$y=4$is 0.

Let's also consider a vertical line, the line $x=3$, as shown in the graph.



What is the rise?The rise is 2. What is the run?The run is 0. What is the slope?$m=$riserun $m=20$

The slope is undefined since division by zero is undefined. So we say that the slope of the vertical line $x=3$ is undefined.

All horizontal lines have slope 0. When the *y*-coordinates are the same, the rise is 0.

The slope of any vertical line is undefined. When the *x*-coordinates of a line are all the same, the run is 0.

Slope of a Horizontal and Vertical Line

The slope of a horizontal line, $y=b$, is 0.

The slope of a vertical line, $x=a$, is undefined.

Find the slope of each line: ⓐ $x=8$ ⓑ $y=-5$.

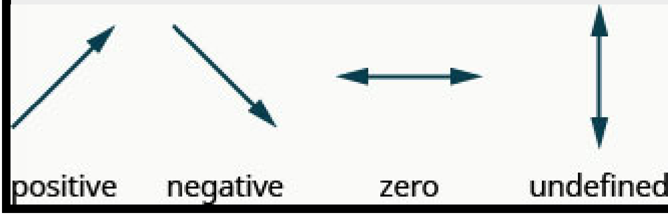
ⓐ $x=8$

This is a vertical line. Its slope is undefined.

ⓑ $y=-5$

This is a horizontal line. It has slope 0.

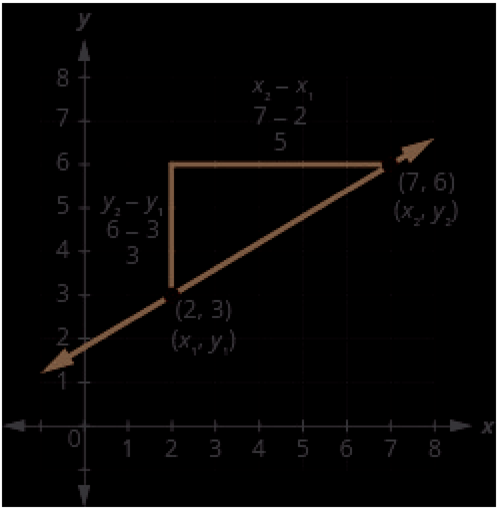
Quick Guide to the Slopes of Lines



Sometimes we'll need to find the slope of a line between two points when we don't have a graph to count out the rise and the run. We could plot the points on grid paper, then count out the rise and the run, but as we'll see, there is a way to find the slope without graphing. Before we get to it, we need to introduce some algebraic notation.

We will use (x1,y1) to identify the first point and (x2,y2) to identify the second point.

Let's see how the rise and run relate to the coordinates of the two points by taking another look at the slope of the line between the points (2,3) and (7,6), as shown in this graph.



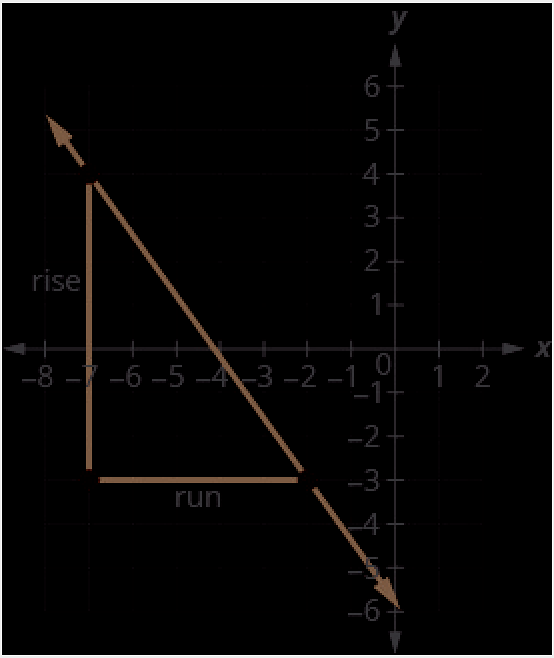
Since we have two points, we will use subscript notation.(2,x1,3y1)(7,6x2,y2) m=riserun On the graph, we counted the rise of 3 and the run of 5.m=35 Notice that the rise of 3 can be found by subtracting the y-coordinates, 6 and 3, and the run of 5 can be found by subtracting thex-coordinates 7 and 2. We rewrite the rise and run by putting in the coordinates.m=6−37−2 But 6 isy2,they-coordinate of the second point and 3 isy1,they-coordinate of the first point. So we can rewrite the slope using subscript notation.m=y2−y17−2 Also 7 is thex-coordinate of the second point and 2 is thex-coordinate of the first point. So again we rewrite the slope using subscript notation.m=y2−y1x2−x1

Slope of a line between two points

The slope of the line between two points $(x1,y1)$ and $(x2,y2)$ is:
$m=y2-y1x2-x1.$

Use the slope formula to find the slope of the line through the points $(-2,-3)$ and $(-7,4)$.

We'll call$(-2,-3)$point#1and$(-7,4)$point#2. $(x1,y1-2,-3)(x2,y2-7,4)$ Use the slope formula.$m=y2-y1x2-x1$ Substitute the values. yof the second point minusyof the first point xof the second point minusxof the first pointm$=4-(-3)-7-(-2)$ Simplify.
$m=7-5$ $m=-75$
Let's verify this slope on the graph shown.



$m = \text{riserun}$ $m = 7 - 5$ $m = -75$

Finding the Slope of a Line Given Two Points

Find the slope of a line that passes through the points $(2, -1)$ and $(-5, 3)$.

We substitute the *y*-values and the *x*-values into the formula.

$m = 3-(-1) \; -5-2 = 4 \; -7 = - \; 4 \; 7$

The slope is $- 4 7$.

Analysis

It does not matter which point is called (x_1, y_1) or (x_2, y_2). As long as we are consistent with the order of the *y* terms and the order of the *x* terms in the numerator and denominator, the calculation will yield the same result.

Use the slope formula to find the slope of the line through the pair of points: $(-2,6)$ and $(-3,-4)$.

10

Slope and Equations

Now that we've defined slope, and looked at the Standard form of a line($Ax+By=C$), there are two other forms of a linear equation in addition to the Standard form:

Point Slope Form	Slope Intercept Form
$y - y_1 = m(x - x_1)$	$y = mx + b$
where $m = \text{slope}$ (x_1, y_1) is a point on the line.	where $m = \text{slope}$ and $b = \text{y-intercept}$.

What if all you are given is 2 points of a line? We know from the previous section that you could use them to determine slope, but the this process will also allow us to find out what the linear equation would look like.

The Slope of a Line

The slope of a line, *m*, represents the veritcal change in *y* over the horizontal change in *x*. Given two points, (x_1, y_1) and (x_2, y_2), the following formula determines the slope of a line containing these points:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Slope also indicates the direction in which a line slants as well as its steepness. Slope is sometimes described as rise over run.

Suppose we have a line that has slope *m* and that contains some specific point (x1,y1) and some other point, which we will just call (x,y). We can write the slope of this line and then change it to a different form.

$m=y-y1x-x1$ Multiply both sides of the equation by$x-x1$.$m(x-x1)=(y-y1x-x1)(x-x1)$ Simplify.$m(x-x1)=y-y1$ Rewrite the equation with the$y$terms on the left.$y-y1=m(x-x1)$

Point Slope Form
Given one point and the slope, the point-slope formula will lead to the equation of a line:
$y- y 1 =m(x- x 1)$

What happens if you only have two points? Can we find the slope with just two points? Yes, using the steps we looked at in the previous section. Once we have the slope, we can use it and one of the given points to find the equation. The following example gives two points, so you need to solve for slope first and then use one of the points.

Find the slope using the given points. $m=y2-y1x2-x1$ Choose one point. Substitute the values into the point-slope form: $y-y1=m(x-x1)$. Write the equation in slope-intercept form.

Finding the Equation of a Line Passing Through Two Given Points

Find the equation of the line passing through the points $(3,4)$ and $(0,-3)$. Write the final equation in slope-intercept form.

First, we calculate the slope using the slope formula and two points.

$$m = \frac{-3-4}{0-3} = \frac{-7}{-3} = \frac{7}{3}$$

Next, we use the point-slope formula with the slope of $\frac{7}{3}$, and either point. Let's pick the point $(3,4)$ for (x_1, y_1).

$$y-4 = \frac{7}{3}(x-3)$$
$$y-4 = \frac{7}{3}x-7 \quad \text{Distribute the } \frac{7}{3}.$$
$$y = \frac{7}{3}x-3$$

The equation is written as $y = \frac{7}{3}x-3$. (this is known as Slope-intercept form).

Analysis

To prove that either point can be used, let us use the second point $(0,-3)$ and see if we get the same equation.

$$y-(-3) = \frac{7}{3}(x-0)$$
$$y+3 = \frac{7}{3}x$$
$$y = \frac{7}{3}x-3$$

We see that the same line will be obtained using either point. This makes sense because we used both points to calculate the slope.

Find an equation of a line that contains the points $(-3,5)$ and $(-3,4)$. Write the equation in slope-intercept form.

Again, the first step will be to find the slope.

Find the slope of the line through$(-3,5)$and$(-3,4)$.$m=y2-y1x2-x1$ $m=4-5-3-(-3)$ $m=-10$ The slope is undefined.

This tells us it is a vertical line. Both of our points have an *x*-coordinate of -2. So our equation of the line is $x=-2$. Since there is no *y*, we cannot write it in slope-intercept form.

You may want to sketch a graph using the two given points. Does your graph agree with our conclusion that this is a vertical line?

How to Find the Equation of a Line Given Two Points

Find an equation of a line that contains the points $(-3,-1)$ and $(2,-2)$ Write the equation in slope-intercept form.

Step 1. Find the slope using the given points.	Find the slope of the line through (−3, −1) and (2, −2).	$m = \frac{y_2 - y_1}{x_2 - x_1}$ $m = \frac{-2-(-1)}{2-(-3)}$ $m = \frac{-1}{5}$
Step 2. Choose one point.	Choose either point.	$\left(\overset{x_1}{2}, \overset{y_1}{-2}\right)$
Step 3. Substitute the values into the point–slope form, $y - y_1 = m(x - x_1)$.	Simplify.	$y - y_1 = \quad (x - x_1)$ $y - (-2) = \quad (x - 2)$ $y + 2 = -\frac{1}{5}x + \frac{2}{5}$
Step 4. Write the equation in slope–intercept form.		$y = -\frac{1}{5}x - \frac{8}{5}$

In the examples above you will note the form of the equation we get after using point-slope form is: $y = mx + b$. where $m = \text{slope}$ and $b = \text{y-intercept}$. Practice identifying them below.

Identifying the Slope and *y*-intercept of a Line Given an Equation

Identify the slope and *y*-intercept, given the equation $y = -\frac{3}{4}x - 4$.

As the line is in $y = mx + b$ form, the given line has a slope of $m = -\frac{3}{4}$. The *y*-intercept is $b = -4$.

Analysis

The *y*-intercept is the point at which the line crosses the *y*-axis. On the *y*-axis, $x = 0$. We can always identify the *y*-intercept when the line is in slope-intercept form, as it will always equal *b*. Or, just substitute $x = 0$ and solve for *y*.

We can easily determine the slope and intercept of a line if the equation is written in slope-intercept form, $y = mx + b$. Now we will do the reverse—we will start with the slope and *y*-intercept and use them to find the equation of the line.

Find the equation given a slope and a point

- Identify the slope.
- Identify the point.
- Substitute the values into the point-slope form, $y - y_1 = m(x - x_1)$.

- Write the equation in slope-intercept form.

Finding the Equation of a Line Given the Slope and One Point

Write the equation of the line with slope $m = -3$ and passing through the point $(4, 8)$. Then solve the equation for y.

Using the point-slope formula, substitute -3 for *m* and the point $(4, 8)$ for (x_1, y_1).

$$y - y_1 = m(x - x_1)$$
$$y - 8 = -3(x - 4)$$
$$y - 8 = -3x + 12$$
$$y = -3x + 20$$

Analysis

Note that any point on the line can be used to find the equation. If done correctly, the same final equation will be obtained.

Find the equation of a line with slope $m = -25$ and containing the point $(10, -5)$.

$y = -25x - 1$

Find the equation of the line shown in the graph.
[missing_resource: CNX_IntAlg_Figure_03_03_002_img_new.jpg]

We need to find the slope and *y*-intercept of the line from the graph so we can substitute the needed values into the slope-intercept form, $y = mx + b$.

To find the slope, we choose two points on the graph.

The *y*-intercept is $(0, -4)$ and the graph passes through $(3, -2)$.

Find the slope, by counting the rise and run. $m = \frac{\text{rise}}{\text{run}}$		
$m = \frac{2}{3}$		
Find the *y*-intercept.		

[redacted]

Substitute the values into y = mx + b.

[redacted]

[redacted]

Find an equation of a horizontal line that contains the point $(-2, -6)$. Write the equation in slope-intercept form.

Every horizontal line has slope 0. We can substitute the slope and points into the point-slope form, $y - y1 = m(x - x1)$.

Identify the slope.

[redacted]

Identify the point.

[redacted]

Substitute the values into $y - y1 = m(x - x1)$.

[redacted]

[redacted]

Simplify.

[redacted]

[redacted]

Write in slope-intercept form.	It is in *y*-form, but could be written $y = 0x - 6$.

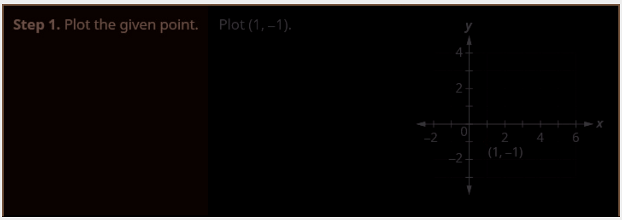
Did we end up with the form of a horizontal line, $y = a$?

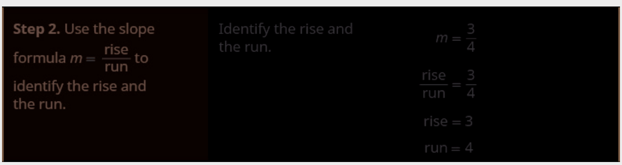
Graphing and Slope

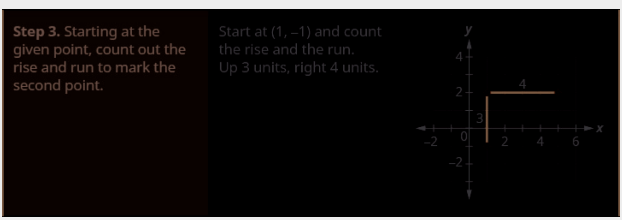
We can also graph a line when we know one point and the slope of the line. We will start by plotting the point and then use the definition of slope to draw the graph of the line.

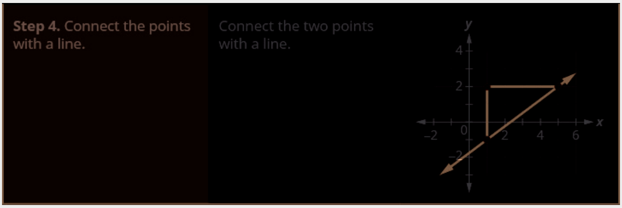
How to graph a Line Given a Point and the Slope

Graph the line passing through the point (1, −1) whose slope is m=34.



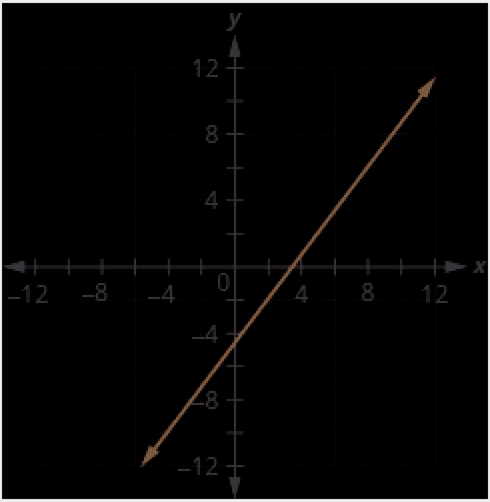




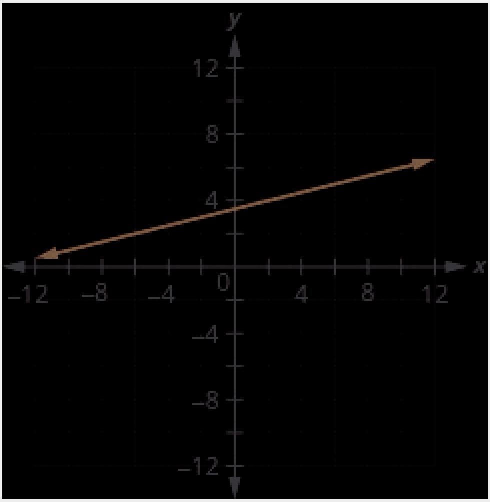


You can check your work by finding a third point. Since the slope is $m=34$, it can also be written as $m=-3-4$ (negative divided by negative is positive!). Go back to $(1,-1)$ and count out the rise, -3, and the run, -4.

Graph the line passing through the point (2, -2) with the slope$m=43$.



Graph the line passing through the point $(-2,3)$ with the slope $m=14$.



Graph a line given a point and the slope.

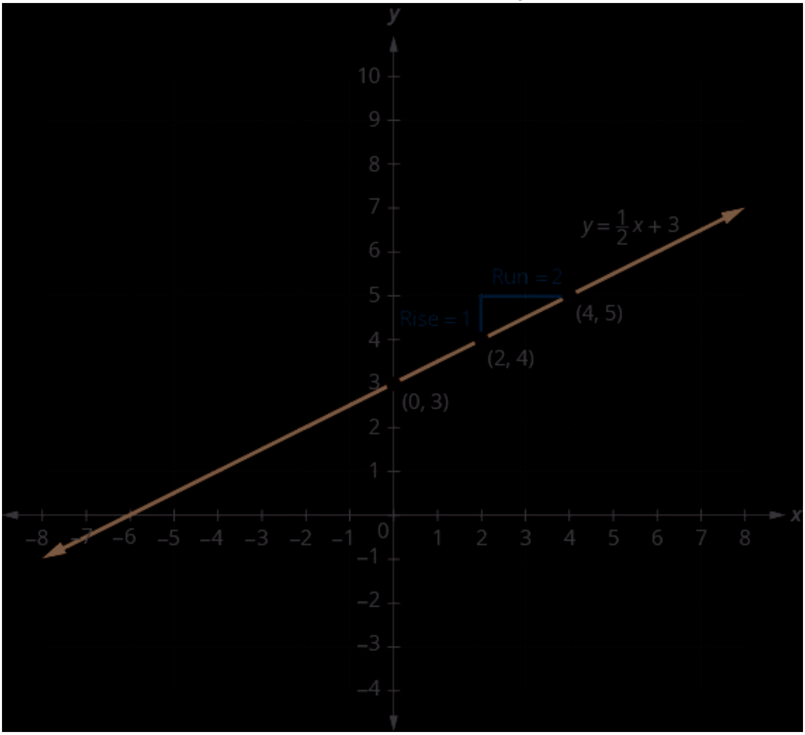
Plot the given point. Use the slope formula m=riserun to identify the rise and the run. Starting at the given point, count out the rise and run to mark the second point. Connect the points with a line.

Graphing with Slope and Intercept

We have graphed linear equations by plotting points, using intercepts, recognizing horizontal and vertical lines, and using one point and the slope of the line. Once we see how an equation in slope–intercept form and its graph are related, we'll have

one more method we can use to graph lines.

See [link]. Let’s look at the graph of the equation $y = 12x + 3$ and find its slope and *y*-intercept.



The red lines in the graph show us the rise is 1 and the run is 2. Substituting into the slope formula: $m = \text{riserun}$ $m = 12$

The *y*-intercept is (0,3).

Look at the equation of this line.

$y = \frac{1}{2}x +$

Look at the slope and *y*-intercept.

slope $m = \frac{1}{2}$ and *y*-intercept (0,).

When a linear equation is solved for *y*, the coefficient of the *x* term is the slope and the constant term is the *y*-coordinate of the *y*-intercept. We say that the equation y = 12x + 3 is in slope–intercept form. Sometimes the slope–intercept form is called the "*y*-form."

$m = \frac{1}{2}$; *y*-intercept is (0,)

$$y = \frac{1}{2}x +$$

$$y = mx +$$

Slope Intercept Form of an Equation of a Line

The slope–intercept form of an equation of a line with slope *m* and *y*-intercept, (0,b) is y = mx + b.

Let's practice finding the values of the slope and *y*-intercept from the equation of a line.

Identify the slope and *y*-intercept of the line from the equation:

ⓐ $y = -47x - 2$ ⓑ $x + 3y = 9$

ⓐ We compare our equation to the slope–intercept form of the equation.

Write the slope–intercept form of the eq[redacted]

Write the equation of the line.

[redacted]

Identify the slope.

[redacted]

Identify the *y*-intercept.

[redacted]

ⓑ When an equation of a line is not given in

slope–intercept form, our first step will be to solve the equation for *y*.

Solve for *y*.	$x + 3y = 9$
Subtract *x* from each side.	[redacted]
Divide both sides by 3.	[redacted]
Simplify.	[redacted]
Write the slope–intercept form of the equation of the line.	[redacted]
Write the equation of the line.	[redacted]
Identify the slope.	[redacted]
Identify the *y*-intercept.	[redacted]

Identify the slope and *y*-intercept from the equation of the line.

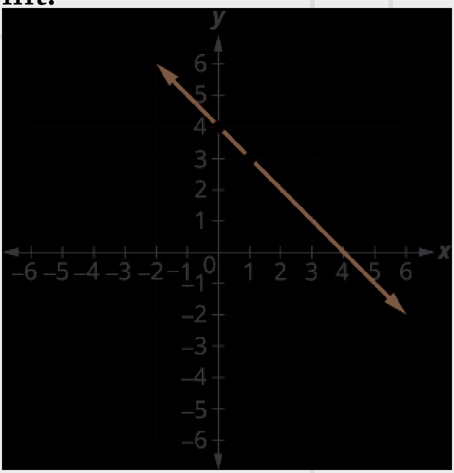
ⓐ $y = -43x + 1$ ⓑ $3x + 2y = 12$

ⓐ $m = -43; (0,1)$
ⓑ $m = -32; (0,6)$

We have graphed a line using the slope and a point. Now that we know how to find the slope and *y*-intercept of a line from its equation, we can use the *y*-intercept as the point, and then count out the slope from there.

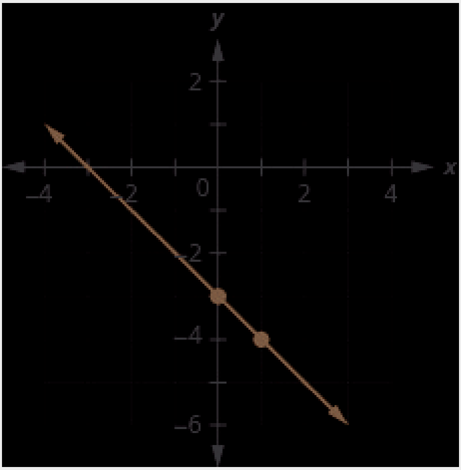
Graph the line of the equation $y = -x + 4$ using its slope and *y*-intercept.

The equation is in slope–intercept form.	$y=mx+b$ $y=-x+4$
Identify the slope and *y* intercept.	$m=-1$ *y* intercept is $(0,4)$
Plot the *y* intercept.	See the graph.
Identify the rise over the run.	$m=\frac{-1}{1}$
Count out the rise and run to mark the second point.	rise -1, run 1



Draw the line as shown in the graph.

Graph the line of the equation $y=-x-3$ using its slope and *y*-intercept.



Now that we have graphed lines by using the slope and *y*-intercept, let’s summarize all the methods we have used to graph lines.

Methods to Graph Lines			
Point Plotting	**Slope–Intercept**	**Intercepts**	**Recognize Vertical and Horizontal Lines**
x | y	$y = mx + b$	x | y 0 | | 0	
Find three points. Plot the points, make sure they line up, then draw the line.	Find the slope and *y*-intercept. Start at the *y*-intercept, then count the slope to get a second point.	Find the intercepts and a third point. Plot the points, make sure they line up, then draw the line.	The equation has only one variable. $x = a$ vertical $y = b$ horizontal

Vertical and Horizontal Lines

Some linear equations have only one variable. They may have just *x* and no *y*, or just *y* without an *x*. This changes how we make a table of values to get the points to plot.

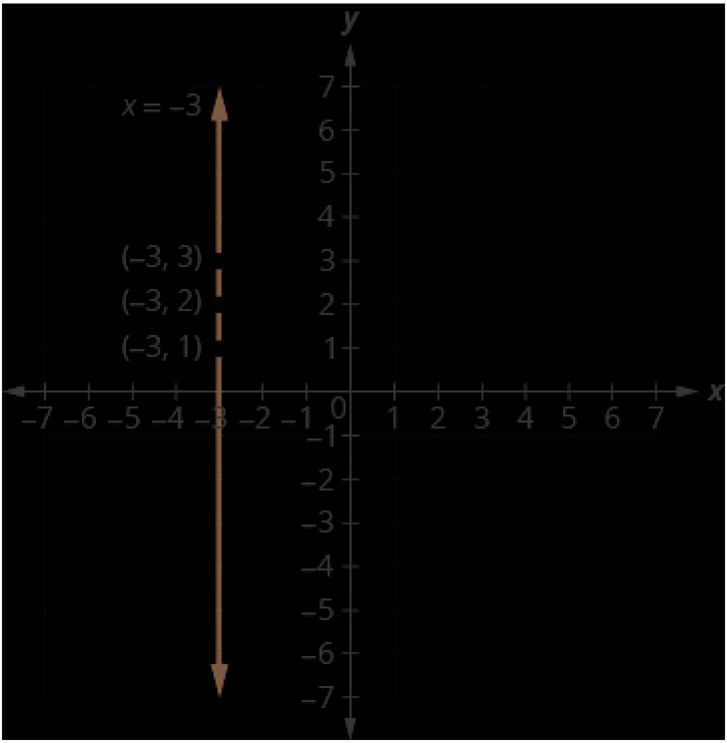
Let's consider the equation $x=-3$. This equation has only one variable, *x*. The equation says that *x* is *always* equal to -3, so its value does not depend on *y*. No matter what is the value of *y*, the value of *x* is always -3.

So to make a table of values, write -3 in for all the *x*-values. Then choose any values for *y*. Since *x* does not depend on *y*, you can choose any numbers you like. But to fit the points on our coordinate graph, we'll use 1, 2, and 3 for the *y*-coordinates. See [link].

$x=-3$		
x	y	(x,y)
-3	1	$(-3,1)$
-3	2	$(-3,2)$

-3	3			$(-3,3)$

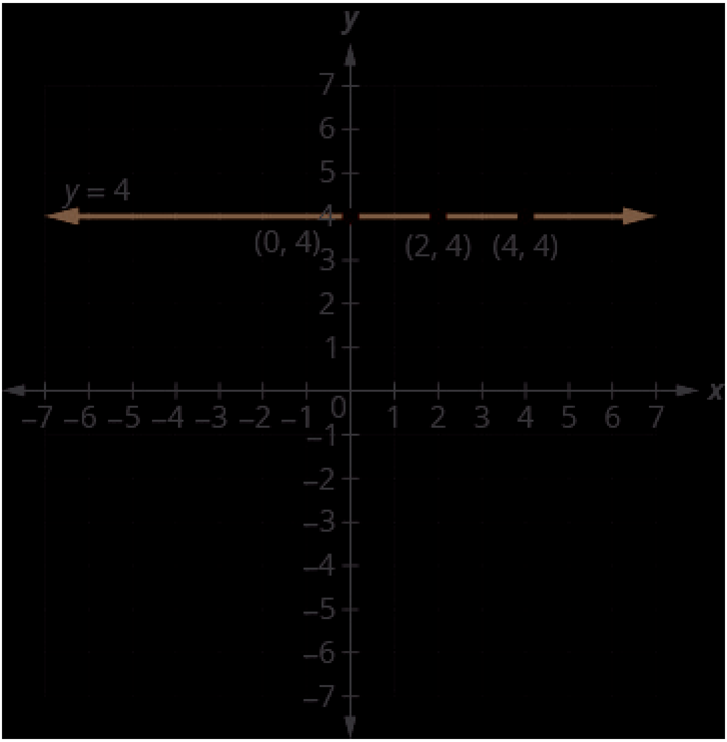
Plot the points from the table and connect them with a straight line. Notice that we have graphed a **vertical line.**



What if the equation has *y* but no *x*? Let's graph the equation $y = 4$. This time the *y*-value is a constant, so in this equation, *y* does not depend on *x*. Fill in 4 for all the *y*'s in [link] and then choose any values for *x*. We'll use 0, 2, and 4 for the *x*-coordinates.

y = 4		
x	***y***	**(x,y)**
0	4	(0,4)
2	4	(2,4)
4	4	(4,4)

In this figure, we have graphed a **horizontal line** passing through the *y*-axis at 4.



Vertical and Horizontal Lines

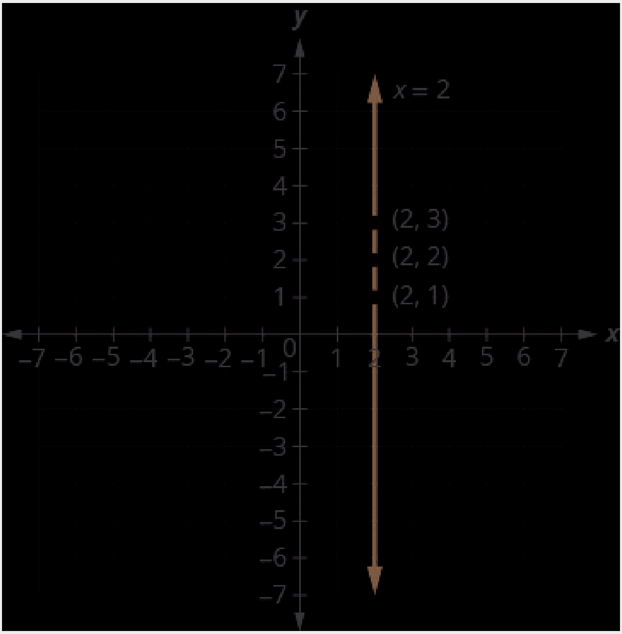
A **vertical line** is the graph of an equation of the form $x = a$.

The line passes through the *x*-axis at (a,0).
A **horizontal line** is the graph of an equation of the form $y=b$.
The line passes through the *y*-axis at (0,b).

Graph: ⓐ $x=2$ ⓑ $y=-1$.

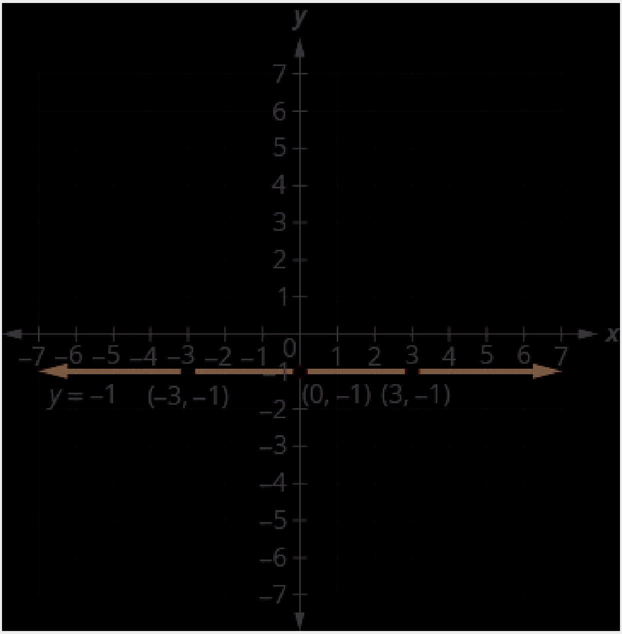
ⓐ The equation has only one variable, *x*, and *x* is always equal to 2. We create a table where *x* is always 2 and then put in any values for *y*. The graph is a vertical line passing through the *x*-axis at 2.

$x=2$		
x	***y***	**(x,y)**
2	1	(2,1)
2	2	(2,2)
2	3	(2,3)



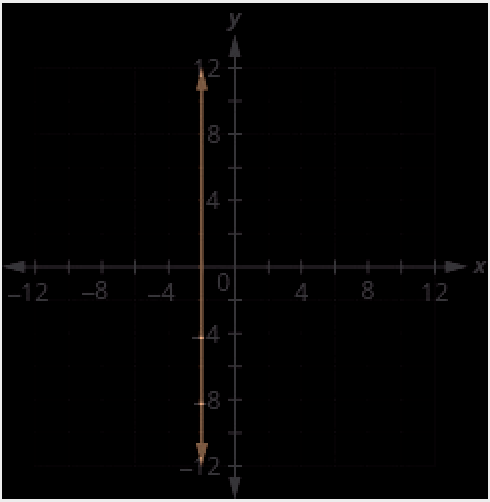
ⓑ Similarly, the equation $y = -1$ has only one variable, y. The value of y is constant. All the ordered pairs in the next table have the same y-coordinate. The graph is a horizontal line passing through the y-axis at -1.

$y = -1$		
x	y	(x, y)
0	-1	$(0, -1)$
3	-1	$(3, -1)$
-3	-1	$(-3, -1)$



Graph the equations: ⓐ $x = -2$ ⓑ $y = 3$.

ⓐ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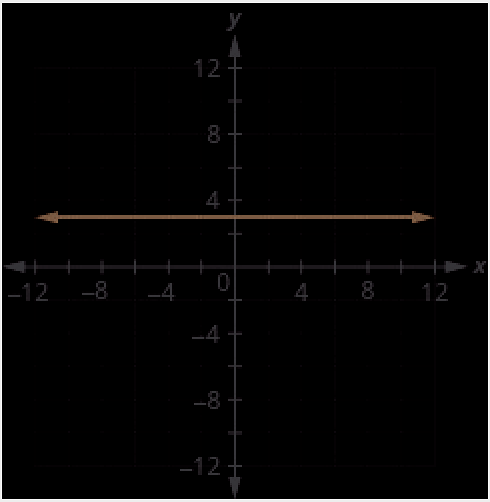
−4
4
8
12

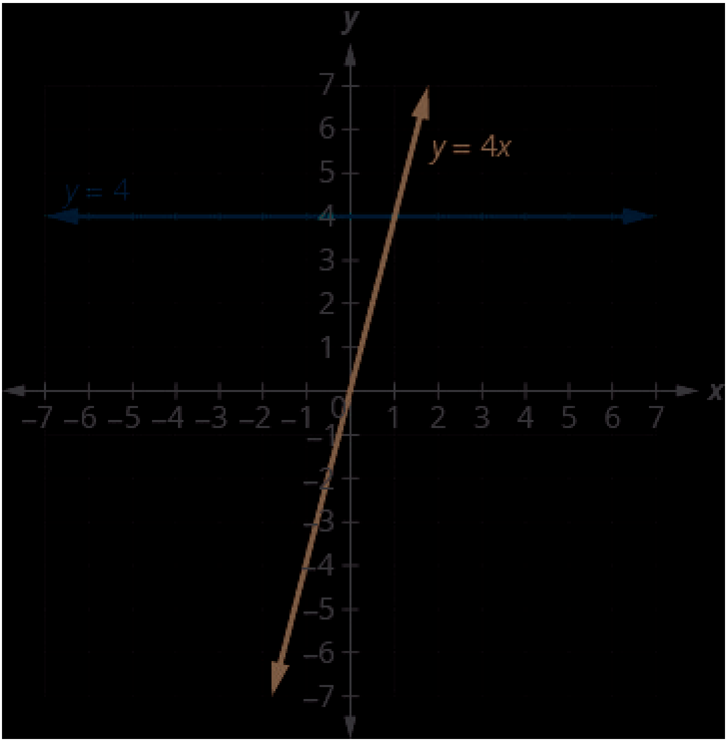
ⓑ


y
x
12
8
4
0
−4
−8
−12
−12
−8
−4
4
8
12

What is the difference between the equations $y = 4x$ and $y = 4$?

The equation $y = 4x$ has both x and y. The value of y depends on the value of x, so the y -coordinate changes according to the value of x. The equation $y = 4$ has only one variable. The value of y is constant, it does not depend on the value of x, so the y-coordinate is always 4.

$y = 4x$				$y = 4$		
x	y	(x, y)		x	y	(x, y)
0	0	(0, 0)		0	4	(0, 4)
1	4	(1, 4)		1	4	(1, 4)
2	8	(2, 8)		2	4	(2, 4)

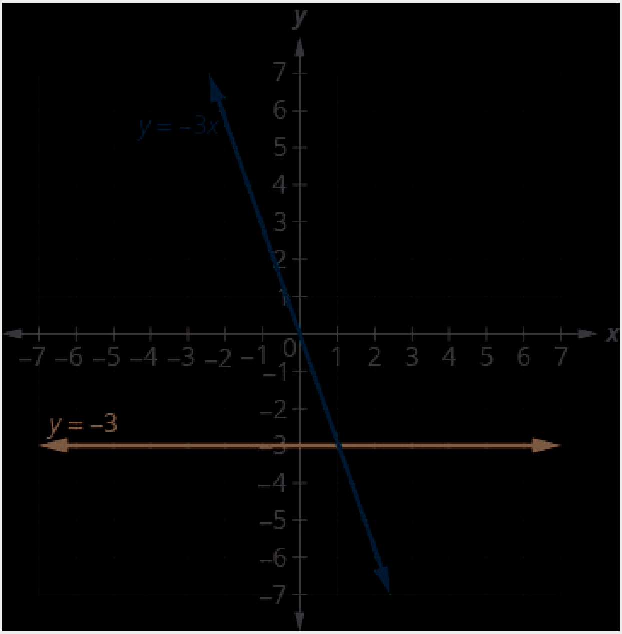


Notice, in the graph, the equation $y = 4x$ gives a slanted line, while $y = 4$ gives a horizontal line.

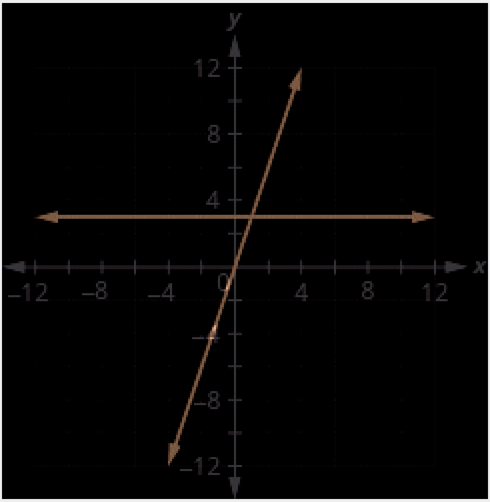
Graph $y = -3x$ and $y = -3$ in the same rectangular coordinate system.
We notice that the first equation has the variable *x*, while the second does not. We make a table of points for each equation and then graph the lines. The two graphs are

shown.

$y = -3x$			$y = -3$		
x	y	(x, y)	x	y	(x, y)
0	0	(0, 0)	0	–3	(0, –3)
1	–3	(1, –3)	1	–3	(1, –3)
2	–6	(2, –6)	2	–3	(2, –3)



Graph the equations in the same rectangular coordinate system: $y=3$ and $y=3x$.



Key Concepts

- **Slope of a Line**
  - The slope of a line is $m=\frac{\text{rise}}{\text{run}}$.
  - The rise measures the vertical change and the run measures the horizontal change.

- **How to find the slope of a line from its graph using** $m = \text{riserun}$.

  Locate two points on the line whose coordinates are integers. Starting with one point, sketch a right triangle, going from the first point to the second point. Count the rise and the run on the legs of the triangle. Take the ratio of rise to run to find the slope:
  $m = \text{riserun}$.

- **Slope of a line between two points.**

  - The slope of the line between two points $(x1,y1)$ and $(x2,y2)$ is:
    $m = y2 - y1x2 - x1$.

- **How to graph a line given a point and the slope.**

  Plot the given point. Use the slope formula $m = \text{riserun}$ to identify the rise and the run. Starting at the given point, count out the rise and run to mark the second point. Connect the points with a line.

- **Slope Intercept Form of an Equation of a**

Line

- The slope–intercept form of an equation of a line with slope *m* and *y*-intercept, (0,b) is $y=mx+b$

Methods to Graph Lines			
Point Plotting	Slope–Intercept	Intercepts	Recognize Vertical and Horizontal Lines
x | y	$y = mx + b$	x | y 0 | | 0	
Find three points. Plot the points, make sure they line up, then draw the line.	Find the slope and *y*-intercept. Start at the *y*-intercept, then count the slope to get a second point.	Find the intercepts and a third point. Plot the points, make sure they line up, then draw the line.	The equation has only one variable. $x = a$ vertical $y = b$ horizontal

We have seen that we can use either the slope-intercept form or the point-slope form to find an equation of a line. Which form we use will depend on the information we are given.

To Write an Equation of a Line

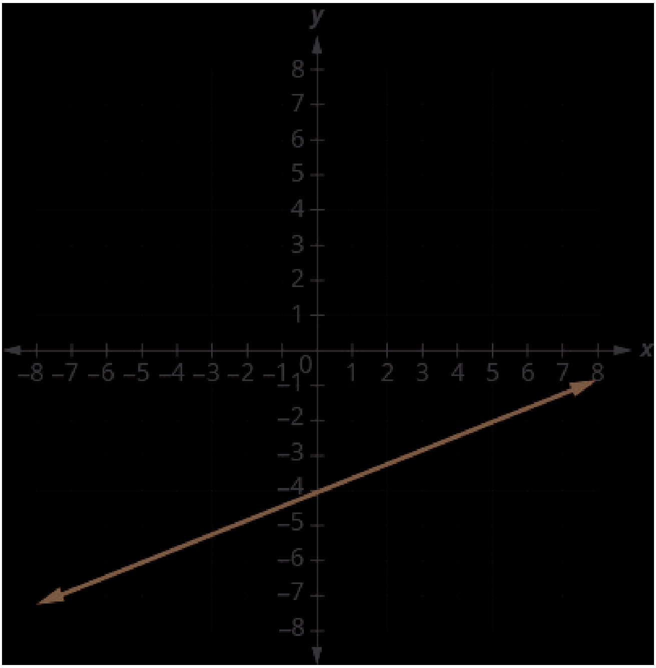
If given:	**Use:**	**Form:**
Slope and *y*-intercept	slope-intercept	$y=mx+b$
Slope and a point	point-slope	$y-y1=m(x-x1)$
Two points	point-slope	$y-y1=m(x$

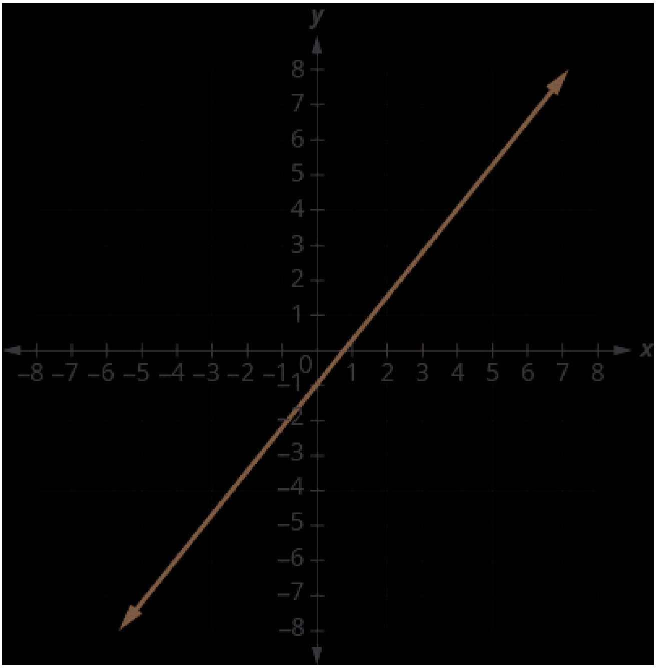
$-x1)$

Practice Makes Perfect

Find the Slope of a Line

In the following exercises, find the slope of each line shown.




y
x
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2
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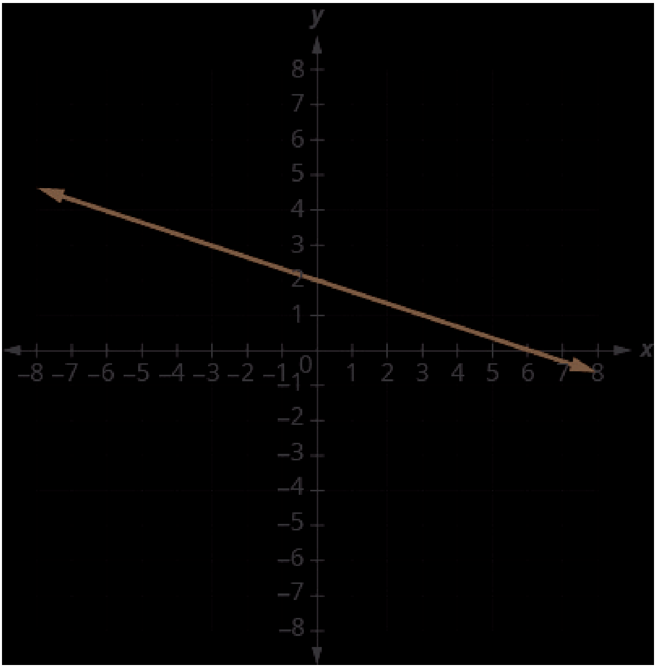
–8
–8 –7 –6 –5 –4 –3 –2 –1
1 2 3 4 5 6 7 8



-13

In the following exercises, find the slope of each line.

$y = 3$

0

$x = -5$

---

undefined

$x=4$

In the following exercises, use the slope formula to find the slope of the line between each pair of points.

$(2,5),(4,0)$

-52

$(-3,3),(4,-5)$

-87

$(-1,-2),(2,5)$

73

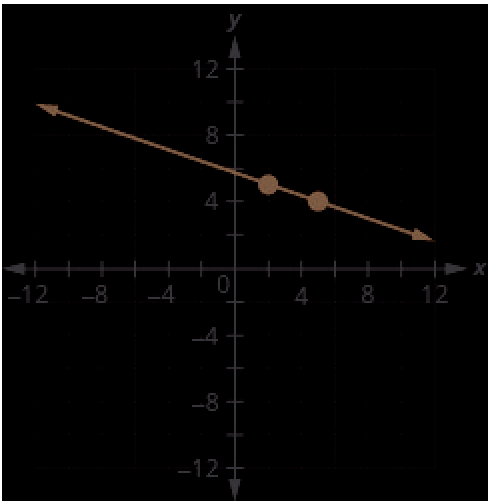
$(4,-5),(1,-2)$

-1

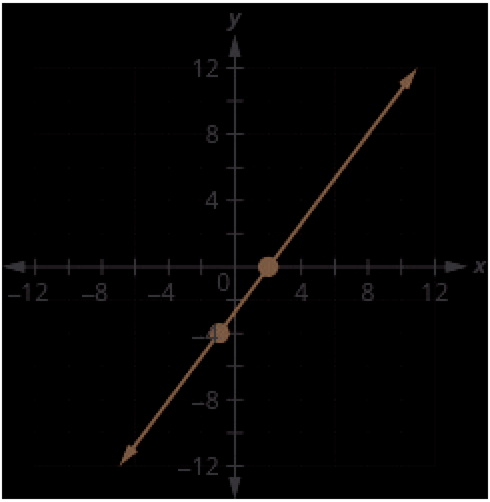
Graph a Line Given a Point and the Slope

In the following exercises, graph each line with the given point and slope.

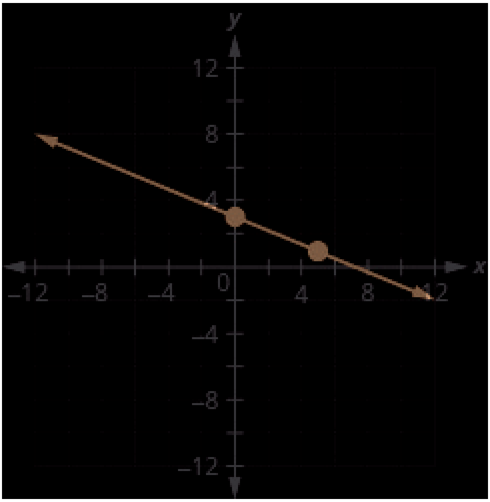
$(2,5); m=-13$



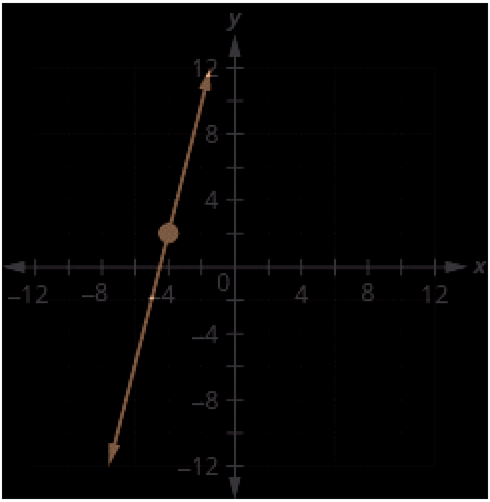
$(-1,-4);\ m=43$



y-intercept 3; $\mathrm{m} = -25$



$(-4,2)$; $m=4$



Graph a Line Using Its Slope and Intercept

In the following exercises, identify the slope and *y*-intercept of each line.

$y=-7x+3$

$m=-7;(0,3)$

$3x+y=5$

$m=-3;(0,5)$

$6x+4y=12$

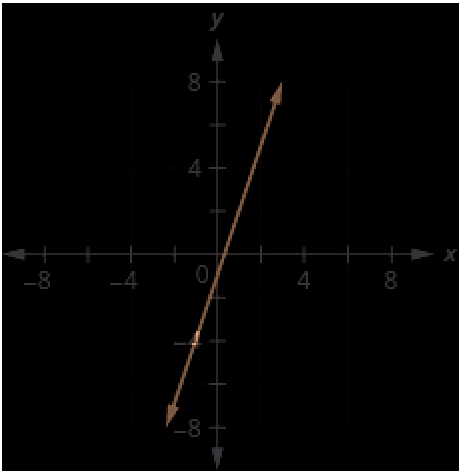
$m=-32;(0,3)$

$5x-2y=6$

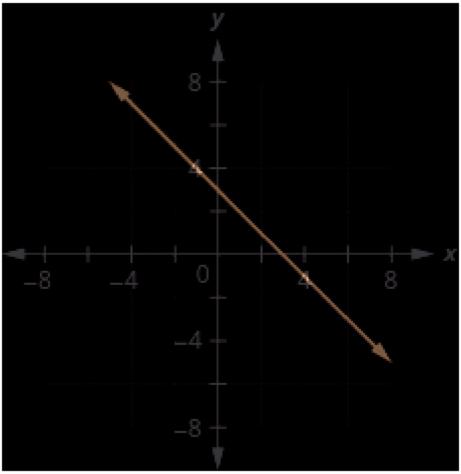
$m=52;(0,-3)$

In the following exercises, graph the line of each equation using its slope and *y*-intercept.

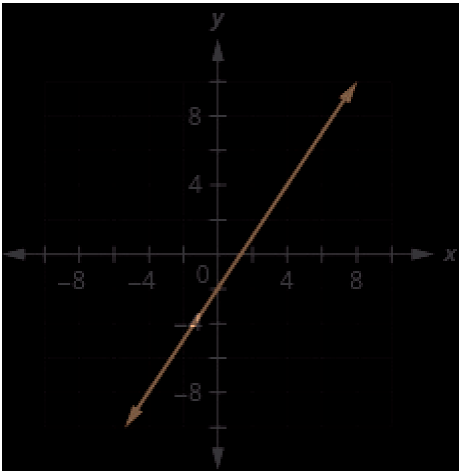
$y=3x-1$



$y = -x + 3$



$3x - 2y = 4$



In the following exercises, find the equation of a line with given slope and *y*-intercept. Write the equation in slope-intercept form.

slope 3 and
y-intercept $(0,5)$

$y=3x+5$

slope -3 and
y-intercept $(0,-1)$

$y=-3x-1$

slope 15 and

y-intercept $(0, -5)$

$y = 15x - 5$

In the following exercises, find the equation of the line shown in each graph. Write the equation in slope-intercept form.

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$y = 3x - 5$

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$y = -43x + 3$

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$y = -2$

In the following exercises, find the equation of a line with given slope and containing the given point. Write the equation in slope-intercept form.

$m=58$, point $(8,3)$

$y=58x-2$

$m=-7$, point $(-1,-3)$

$y=-7x-10$

Horizontal line containing $(-2,5)$

$y=5$

In the following exercises, find the equation of a line containing the given points. Write the equation in slope-intercept form.

$(2,6)$ and $(5,3)$

$y=-x+8$

$(-3,-4)$ and $(5-2)$.

$y=14x-134$

$(7,2)$ and $(7,-2)$.

$x=7$

Glossary

point-slope form
: The point-slope form of an equation of a line with slope *m* and containing the point $(x1,y1)$ is $y-y1=m(x-x1)$.

More on Slope (2.4)
By the end of this section, you will be able to:

- Use slopes to identify parallel and perpendicular lines
- Find an equation of a line parallel to a given line
- Find an equation of a line perpendicular to a given line
- Find average rate of change

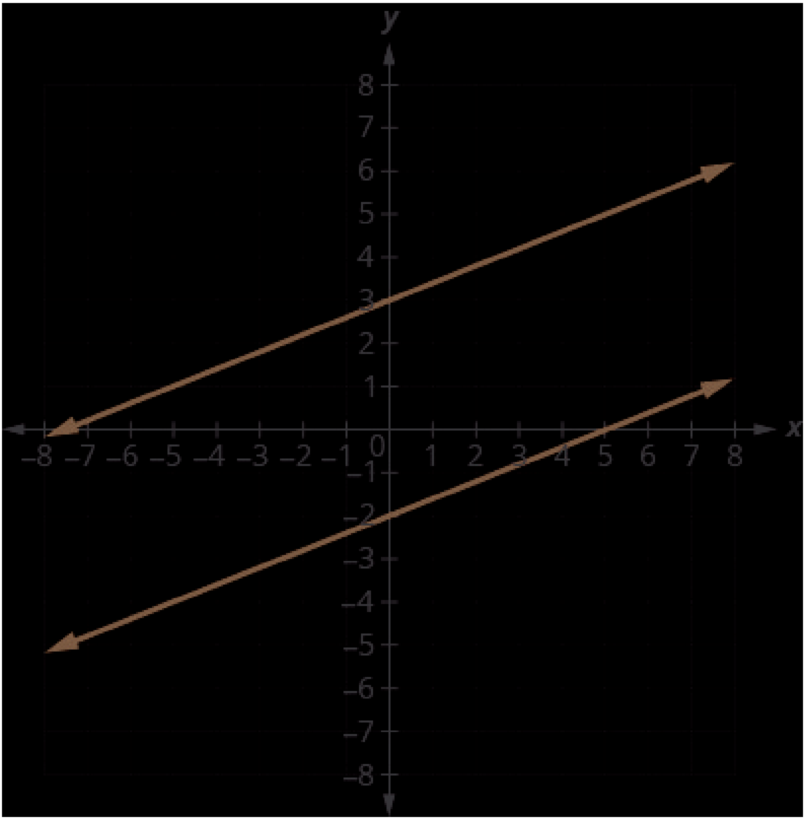
This Module supports section **2.4 of Mat 1023**.
In case you missed something in class, or just want to review a specific topic covered in this Module, here is a list of topics covered:

Identify and Graph Parallel Lines

Two lines that have the same slope are called

parallel lines. Parallel lines have the same steepness and never intersect.

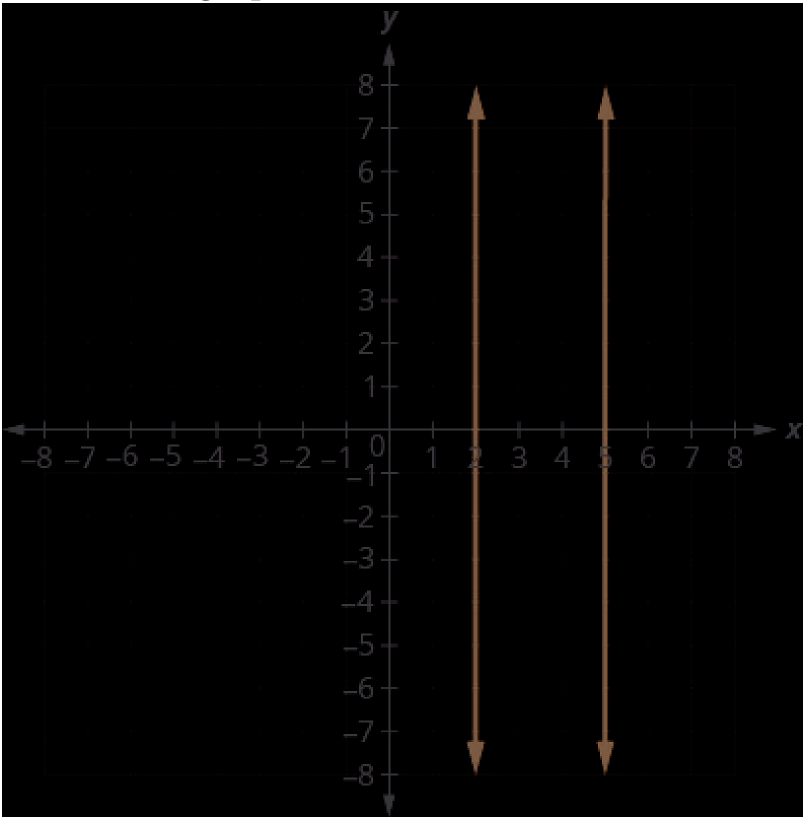
We say this more formally in terms of the rectangular coordinate system. Two lines that have the same slope and different *y*-intercepts are called parallel lines. See [link].



Verify that both lines have the same slope, $m = 25$, and different *y*-intercepts.

What about vertical lines? The slope of a vertical line is undefined, so vertical lines don't fit in the

definition above. We say that vertical lines that have different *x*-intercepts are parallel, like the lines shown in this graph.



Parallel Lines

Parallel lines are lines in the same plane that do not intersect.

- Parallel lines have the same slope and different *y*-intercepts.
- If m1 and m2 are the slopes of two parallel

lines then m1 = m2.

- Parallel vertical lines have different *x*-intercepts.

Since parallel lines have the same slope and different *y*-intercepts, we can now just look at the slope–intercept form of the equations of lines and decide if the lines are parallel.

Use slopes and *y*-intercepts to determine if the lines are parallel:

ⓐ $3x-2y=6$ and $y=32x+1$ ⓑ $y=2x-3$ and $-6x+3y=-9$.

ⓐ

$3x-2y=6$ and $y=32x+1$ Solve the first equation for y. $-2y=-3x+6$ $-2y-2=-3x+6-2$ The equation is now in slope–intercept form. $y=32x-3$ The equation of the second line is already in slope–intercept form. $y=32x+1$ Identify the slope and y-intercept of both lines. $y=32x-3$ $y=mx+b$ $m=32$ $y=32x+1$ $y=mx+b$ $y=32$ y-intercept is $(0,-3)$ y-intercept is $(0,1)$

The lines have the same slope and different *y*-intercepts and so they are parallel.
You may want to graph the lines to confirm whether they are parallel.

ⓑ
$y=2x-3$and$-6x+3y=-9$The first equation is already in slope–intercept form.$y=2x-3$Solve the second equation for$y$.$-6x+3y=-9$$3y=6x-9$$\frac{3y}{3}=\frac{6x-9}{3}$$y=2x-3$The second equation is now in slope–intercept form.$y=2x-3$Identify the slope and$y$-intercept of both lines.$y=2x-3$$y=mx+b$$m=2$$y=2x-3$$y=mx+b$$m=2$$y$-intercept is$(0,-3)$$y$-intercept is$(0,-3)$
The lines have the same slope, but they also have the same *y*-intercepts. Their equations represent the same line and we say the lines are coincident. They are not parallel; they are the same line.

Use slopes and *y*-intercepts to determine if the lines are parallel:

ⓐ $2x+5y=5$ and $y=-\frac{2}{5}x-4$ ⓑ $y=-\frac{1}{2}x-1$ and $x+2y=-2$.

ⓐ parallel ⓑ not parallel; same line

Use slopes and *y*-intercepts to determine if the lines are parallel:

ⓐ $y=-4$ and $y=3$ ⓑ $x=-2$ and $x=-5$.

ⓐ $y=-4$ and $y=3$
We recognize right away from the equations that these are horizontal lines, and so we know their slopes are both 0.
Since the horizontal lines cross the *y*-axis at $y=-4$ and at $y=3$, we know the *y*-intercepts are $(0,-4)$ and $(0,3)$.
The lines have the same slope and different *y*-intercepts and so they are parallel.
ⓑ $x=-2$ and $x=-5$
We recognize right away from the equations that these are vertical lines, and so we know their slopes are undefined.
Since the vertical lines cross the *x*-axis at $x=-2$ and $x=-5$, we know the *y*-intercepts are $(-2,0)$ and $(-5,0)$.
The lines are vertical and have different *x*-intercepts and so they are parallel.

Use slopes and *y*-intercepts to determine if the lines are parallel:

ⓐ $y=8$ and $y=-6$ ⓑ $x=1$ and $x=-5$.

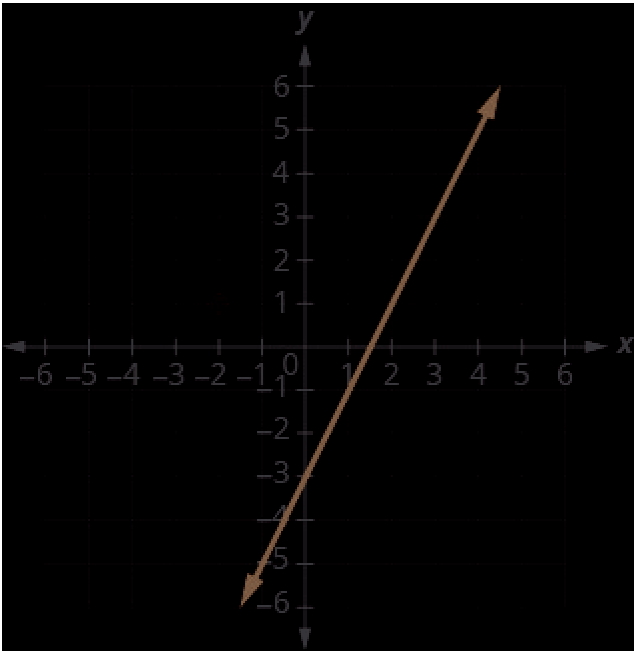
ⓐ parallel ⓑ parallel

Find an Equation of a Line Parallel to a Given Line

Suppose we need to find an equation of a line that passes through a specific point and is parallel to a given line. We can use the fact that parallel lines have the same slope. So we will have a point and the slope—just what we need to use the point-slope equation.

First, let's look at this graphically.

This graph shows $y=2x-3$. We want to graph a line parallel to this line and passing through the point $(-2,1)$.

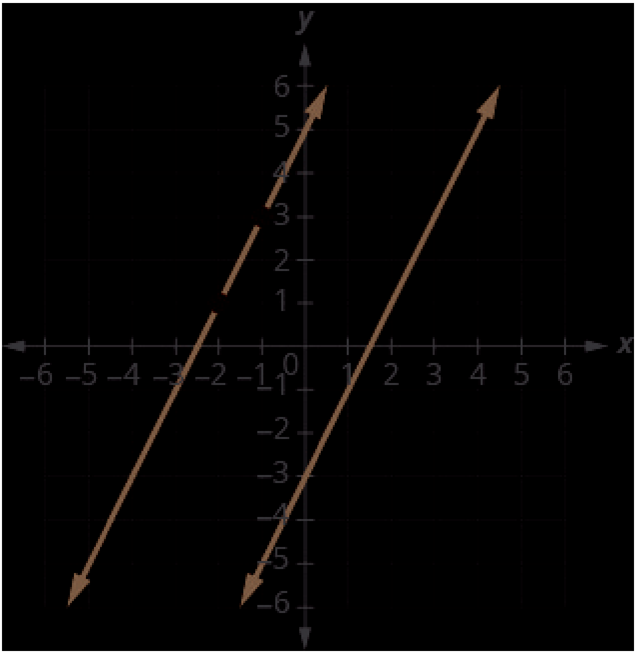


We know that parallel lines have the same slope. So the second line will have the same slope as $y=2x-3$. That slope is $m_{\|}=2$. We'll use the notation $m_{\|}$ to represent the slope of a line parallel to a line with slope *m*. (Notice that the subscript || looks like two parallel lines.)

The second line will pass through $(-2,1)$ and have $m=2$.

To graph the line, we start at$(-2,1)$ and count out the rise and run.

With $m=2$ (or $m=21$), we count out the rise 2 and the run 1. We draw the line, as shown in the graph.



Do the lines appear parallel? Does the second line pass through$(-2,1)$?

We were asked to graph the line, now let’s see how to do this algebraically.

We can use either the slope-intercept form or the point-slope form to find an equation of a line. Here we know one point and can find the slope. So we will use the point-slope form.

Find an equation of a line parallel to a given line.

Find the slope of the given line. Find the slope of

the parallel line. Identify the point. Substitute the values into the point-slope form: $y - y1 = m(x - x1)$. Write the equation in slope-intercept form.

How to Find the Equation of a Line Parallel to a Given Line and a Point

Find an equation of a line parallel to $y = 2x - 3$ that contains the point $(-2,1)$. Write the equation in slope-intercept form.

Step 1. Find the slope of the given line.	The line is in slope–intercept form, $y = 2x - 3$.	$m = 2$

Step 2. Find the slope of the parallel line.	Parallel lines have the same slope.	$m_{\parallel} = 2$

Step 3. Identify the point.	The given point is $(-2, 1)$.	$\begin{pmatrix} x_1, & y_1 \\ -2, & 1 \end{pmatrix}$

Step 4. Substitute the values into the point–slope form, $y - y_1 = m(x - x_1)$.	Simplify.	$y - y_1 = m(x - x_1)$ $y - 1 = 2(x - (-2))$ $y - 1 = 2(x + 2)$ $y - 1 = 2x + 4$

Step 5. Write the equation in slope–intercept form.	$y = 2x + 5$

Look at graph with the parallel lines shown previously. Does this equation make sense? What is the *y*-intercept of the line? What is the slope?

Find an equation of a line parallel to the line $y = 3x + 1$ that contains the point (4,2). Write the equation in slope-intercept form.

$y = 3x - 10$

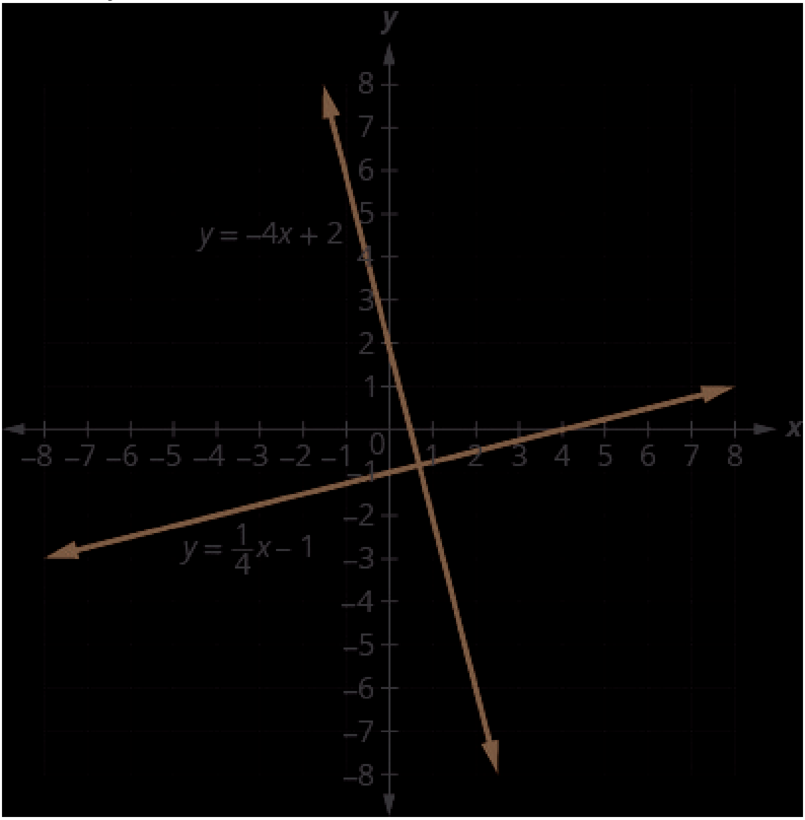
Find an equation of a line parallel to the line $y = 12x - 3$ that contains the point (6,4).

Write the equation in slope-intercept form.

$y = 12x + 1$

Identify and Graph Perpendicular Lines

Let’s look at the lines whose equations are $y=14x-1$ and $y=-4x+2$, shown in [link].



These lines lie in the same plane and intersect in right angles. We call these lines perpendicular.

If we look at the slope of the first line, $m1=14$, and the slope of the second line, $m2=-4$, we can see that they are *negative reciprocals* of each other. If we

multiply them, their product is -1.
$m_1 \cdot m_2 \frac{1}{4}(-4) -1$

This is always true for **perpendicular lines** and leads us to this definition.

Perpendicular Lines

Perpendicular lines are lines in the same plane that form a right angle.

- If m_1 and m_2 are the slopes of two perpendicular lines, then:
  - their slopes are negative reciprocals of each other, $m_1 = -\frac{1}{m_2}$.
  - the product of their slopes is -1, $m_1 \cdot m_2 = -1$.
- A vertical line and a horizontal line are always perpendicular to each other.

We were able to look at the slope–intercept form of linear equations and determine whether or not the lines were parallel. We can do the same thing for perpendicular lines.

We find the slope–intercept form of the equation,

and then see if the slopes are opposite reciprocals. If the product of the slopes is -1, the lines are perpendicular.

Use slopes to determine if the lines are perpendicular:

ⓐ $y=-5x-4$ and $x-5y=5$ ⓑ $7x+2y=3$ and $2x+7y=5$

ⓐ
The first equation is in slope–intercept form.$y=-5x-4$Solve the second equation for$y$.$x-5y=5-5y=-x+5-5y-5=-x+5-5y=15x-1$Identify the slope of each line.$y=-5x-4y=mx+bm1=-5y=15x-1y=mx+bm2=15$
The slopes are negative reciprocals of each other, so the lines are perpendicular. We check by multiplying the slopes, Since $-5(15)=-1$, it checks.

ⓑ
Solve the equations for$y$.$7x+2y=32y=-7x+32y2=-7x+32y=-72x+322x+7y=57y=-2x+57y7=-2x+57y=-27x+57$Identify the slope of each line.$y=mx$

$+bm1=-72y=mx+bm1=-27$
The slopes are reciprocals of each other, but they have the same sign. Since they are not negative reciprocals, the lines are not perpendicular.

Use slopes to determine if the lines are perpendicular:

ⓐ $y=-3x+2$ and $x-3y=4$ ⓑ $5x+4y=1$ and $4x+5y=3$.

ⓐ perpendicular ⓑ not perpendicular

Use slopes to determine if the lines are perpendicular:

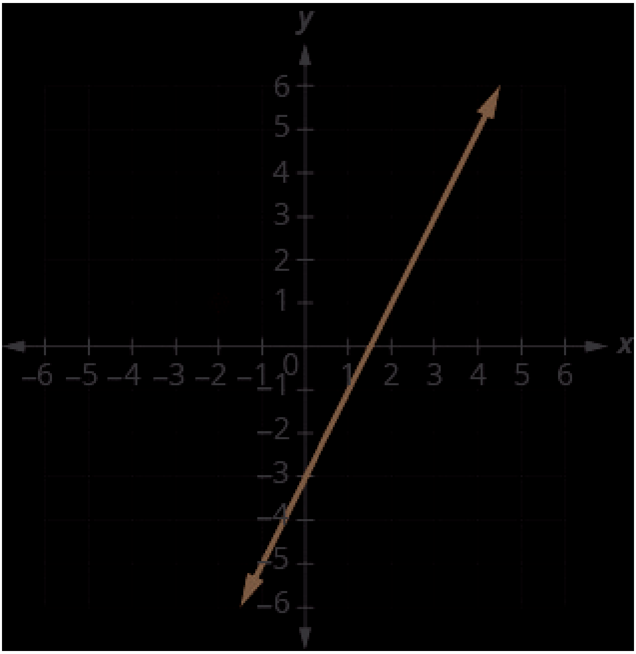
ⓐ $y=2x-5$ and $x+2y=-6$ ⓑ $2x-9y=3$ and $9x-2y=1$.

ⓐ perpendicular ⓑ not perpendicular

Find an Equation of a Line Perpendicular to a Given Line

Suppose we need to find a line passing through a specific point and which is perpendicular to a given line. We can use the fact that perpendicular lines have slopes that are negative reciprocals. We will again use the point-slope equation, like we did with parallel lines.

This graph shows $y = 2x - 3$. Now, we want to graph a line perpendicular to this line and passing through $(-2, 1)$.



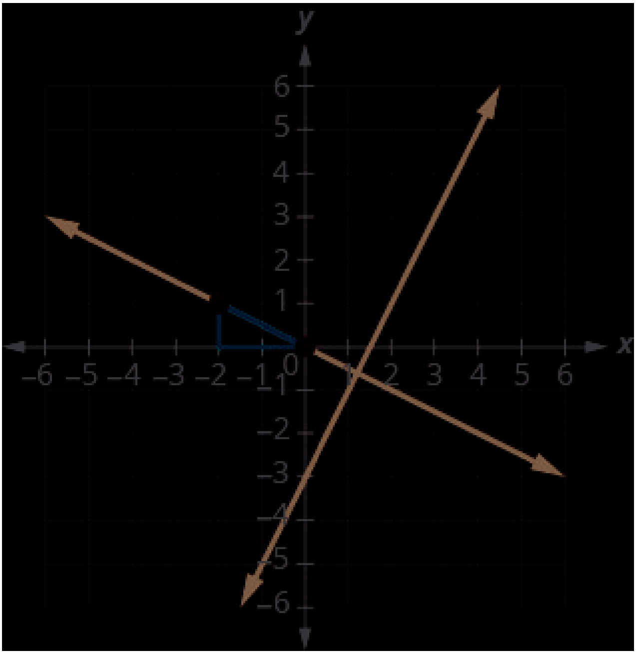
We know that perpendicular lines have slopes that are negative reciprocals.

We’ll use the notation $m\perp$ to represent the slope of a line perpendicular to a line with slope *m*. (Notice that the subscript $\perp$ looks like the right angles made by two perpendicular lines.)

$y=2x-3$perpendicular line $m=2$$m\perp=-12$

We now know the perpendicular line will pass through $(-2,1)$ with $m\perp=-12$.

To graph the line, we will start at $(-2,1)$ and count out the rise -1 and the run 2. Then we draw the line.



Do the lines appear perpendicular? Does the second line pass through$(-2,1)$?

We were asked to graph the line, now, let’s see how to do this algebraically.

We can use either the slope-intercept form or the point-slope form to find an equation of a line. In this example we know one point, and can find the slope, so we will use the point-slope form.

How to Find the Equation of a Line Perpendicular to a Given Line and a Point

Find an equation of a line perpendicular to $y = 2x - 3$ that contains the point $(-2,1)$. Write the equation in slope-intercept form.

Step 1. Find the slope of the given line.	The line is in slope–intercept form, $y = 2x - 3$.	$m = 2$
Step 2. Find the slope of the perpendicular line.	The slopes of perpendicular lines are negative reciprocals.	$m_{\perp} = -\frac{1}{2}$
Step 3. Identify the point.	The given point is $(-2, 1)$.	$\begin{pmatrix} x_1, & y_1 \\ -2, & 1 \end{pmatrix}$

Step 4. Substitute the values into the point–slope form, $y-y_1=m(x-x_1)$.	Simplify.	$y-y_1=m(x-x_1)$ $y-1=-\frac{1}{2}(x-(-2))$ $y-1=-\frac{1}{2}(x+2)$ $y-1=-\frac{1}{2}x-1$
Step 5. Write the equation in slope–intercept form.		$y=-\frac{1}{2}x$

Find an equation of a line perpendicular to the line $y=3x+1$ that contains the point $(4,2)$. Write the equation in slope-intercept form.

$y=-13x+103$

Find an equation of a line perpendicular to the line $y=12x-3$ that contains the point $(6,4)$.

Write the equation in slope-intercept form.

$y = -2x + 16$

Find an equation of a line perpendicular to a given line.

Find the slope of the given line. Find the slope of the perpendicular line. Identify the point. Substitute the values into the point-slope form, $y - y1 = m(x - x1)$. Write the equation in slope-intercept form.

Find an equation of a line perpendicular to $x = 5$ that contains the point $(3, -2)$. Write the equation in slope-intercept form.

Again, since we know one point, the point-slope option seems more promising than the slope-intercept option. We need the slope to use this form, and we know the new line will be perpendicular to $x = 5$. This line is vertical, so its perpendicular will be horizontal. This

tells us the $m_{\perp} = 0$.

Identify the point. $(3, -2)$ Identify the slope of the perpendicular line. Substitute the values into $y - y_1 = m(x - x_1)$. Simplify. $m_{\perp} = 0$ $y - y_1 = m(x - x_1)$ $y - (-2) = 0(x - 3)$ $y + 2 = 0$ $y = -2$

Sketch the graph of both lines. On your graph, do the lines appear to be perpendicular?

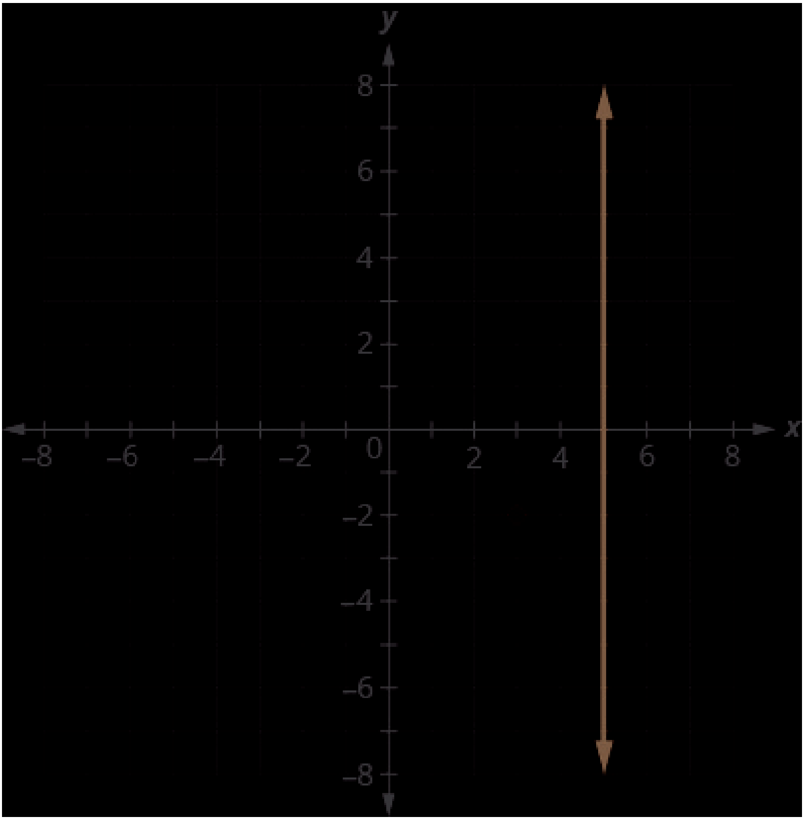
Find an equation of a line that is perpendicular to the line $x = 4$ that contains the point $(4, -5)$.. Write the equation in slope-intercept form.

$y = -5$

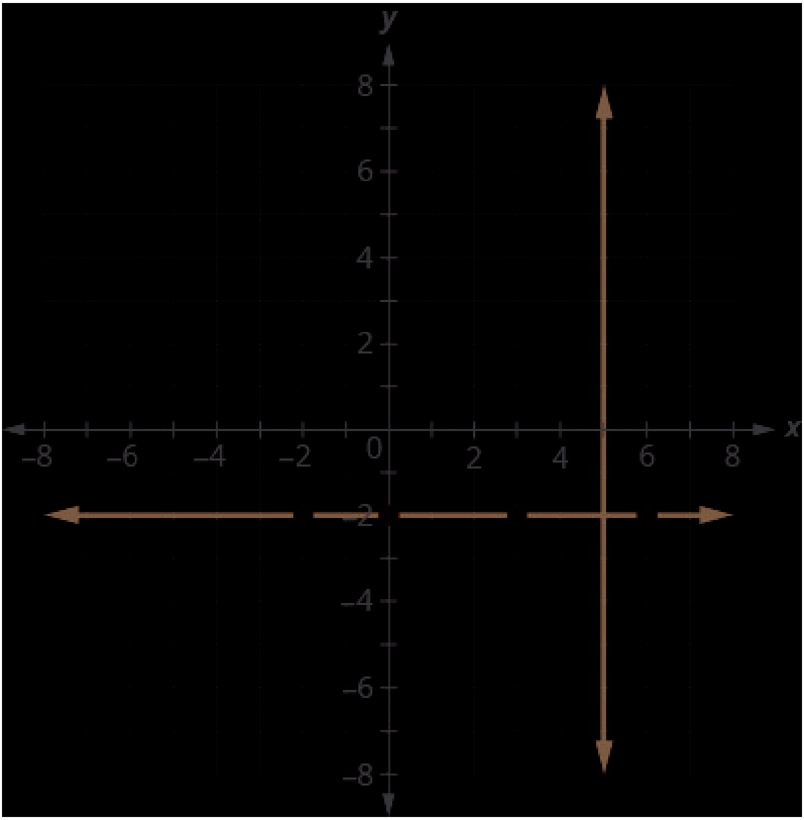
In [link], we used the point-slope form to find the equation. We could have looked at this in a different way.

We want to find a line that is perpendicular to $x = 5$ that contains the point $(3, -2)$. This graph shows us

the line$x=5$ and the point $(3,-2)$.



We know every line perpendicular to a vertical line is horizontal, so we will sketch the horizontal line through $(3,-2)$.



Do the lines appear perpendicular?

If we look at a few points on this horizontal line, we notice they all have *y*-coordinates of -2. So, the equation of the line perpendicular to the vertical line $x=5$ is $y=-2$.

Find an equation of a line that is perpendicular to $y=-3$ that contains the point $(-3,5)$.

Write the equation in slope-intercept form.

The line $y=-3$ is a horizontal line. Any line perpendicular to it must be vertical, in the form $x=a$. Since the perpendicular line is vertical and passes through $(-3,5)$, every point on it has an *x*-coordinate of -3. The equation of the perpendicular line is $x=-3$

You may want to sketch the lines. Do they appear perpendicular?

Find an equation of a line that is perpendicular to the line $y=-5$ that contains the point $(-4,-5)$. Write the equation in slope-intercept form.

$x=-4$

Access these online resources for additional instruction and practice with finding the equation of a line.

- Write an Equation of Line Given its slope and Y-Intercept
- Using Point Slope Form to Write the Equation of a Line, Find the equation given slope and point
- Find the equation given two points
- Find the equation of perpendicular and parallel lines

Rate of Change

A **rate of change** is a relationship or unit rate that shows how one variable or quantity changes with another.

Hours Spent Working	Batches of Cookies Made
0	0
1	6
2	12
3	18
4	24
5	30

The most common way to investigate rates of

change is in numerical or tabular form of data. How the values of x change compared to how the values of y change is called the rate of change. From hour 1 to hour 2, 6 batches of cookies were made. From hour 2 to hour 3, 6 more batches were made. This relationship of change is often represented with a ratio where the dependent variable is in the numerator and the independent variable is in the denominator. The rate of change here would be:

$$\frac{6\ batches\ of\ cookies}{1\ hour\ spent\ working}$$

Recall from the previous chapter, that slope is sometimes described as rise over run. We can assign a numerical value to the slope of a line by finding the ratio of the rise and run. The *rise* is the amount the vertical distance changes while the *run* measures the horizontal change, as shown in this illustration. **Slope is a rate of change.**

Note on Slope

The slope of a line, *m*, represents the veritcal change in *y* over the horizontal change in *x*. Given two points, (x_1, y_1) and (x_2, y_2), the following formula determines the slope of a line containing these points:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Slope also indicates the direction in which a line slants as well as its steepness.

So **rate of change** describes how an output quantity changes relative to the change in the input quantity. To find the **average rate of change** over the specified period of time, we divide the change in the output value by the change in the input value.

$$\text{Average rate of change} = \frac{\text{Change in output}}{\text{Change in input}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

The Greek letter Δ (delta) signifies the change in a quantity; we read the ratio as "delta-*y* over delta-*x*" or "the change in y divided by the change in x." Occasionally we write Δf instead of Δy, which still represents the change in the function's output value resulting from a change to its input value. It does not mean we are changing the function into some other function.

Gasoline costs have experienced some wild fluctuations over the last several decades. [link] [footnote] lists the average cost, in dollars, of a gallon of gasoline for the years 2005–2012. The cost of gasoline can be considered as a function of

year.
http://www.eia.gov/totalenergy/data/annual/showtext.cfm?t=ptb0524. Accessed 3/5/2014.

y	2005	2006	2007	2008	2009	2010	2011	2012
C(y)	2.31	2.62	2.84	3.30	2.41	2.84	3.58	3.68

If we were interested only in how the gasoline prices changed between 2005 and 2012, we could compute that the cost per gallon had increased from \$2.31 to \$3.68, an increase of \$1.37. While this is interesting, it might be more useful to look at how much the price changed *per year*.

In our example, the gasoline price increased by \$1.37 from 2005 to 2012. Over 7 years, the average rate of change was

$$\frac{\Delta y}{\Delta x} = \frac{\$1.37}{7 \text{ years}} \approx 0.196 \text{ dollars per year}$$

On average, the price of gas increased by about 19.6¢ each year.

Other examples of rates of change include:

- A population of rats increasing by 40 rats per week
- A car traveling 68 miles per hour (distance traveled changes by 68 miles each hour as time passes)

- A car driving 27 miles per gallon (distance traveled changes by 27 miles for each gallon)
- The current through an electrical circuit increasing by 0.125 amperes for every volt of increased voltage
- The amount of money in a college account decreasing by $4,000 per quarter

Rate of Change

A rate of change describes how an output quantity changes relative to the change in the input quantity. The units on a rate of change are “output units per input units.”

The average rate of change between two input values is the total change of the function values (output values) divided by the change in the input values.

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Given the value of a function at different points, calculate the average rate of change of a function for the interval between two values x_1 and x_2.

1. Calculate the difference $y_2 - y_1 = \Delta y$.
2. Calculate the difference $x_2 - x_1 = \Delta x$.
3. Find the ratio $\frac{\Delta y}{\Delta x}$.

Computing an Average Rate of Change

Using the data in [link], find the average rate of change of the price of gasoline between 2007 and 2009.

In 2007, the price of gasoline was \$2.84. In 2009, the cost was \$2.41. The average rate of change is

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\$2.41 - \$2.84}{2009 - 2007} = \frac{-\$0.43}{2 \text{ years}} = -\$0.22 \text{ per year}$$

Analysis

Note that a decrease is expressed by a negative change or “negative increase.” A rate of change is negative when the output decreases as the input increases or when the output increases as the input decreases.

Computing Average Rate of Change for a Function Expressed as a Formula

Compute the average rate of change of $f(x) = x^2 - \frac{1}{x}$ on the interval [2, 4].

We can start by computing the function values at each endpoint of the interval.

$$\begin{aligned} f(2) &= 2^2 - \frac{1}{2} & f(4) &= 4^2 - \frac{1}{4} \\ &= 4 - \frac{1}{2} & &= 16 - \frac{1}{4} \\ &= \frac{7}{2} & &= \frac{63}{4} \end{aligned}$$

Now we compute the average rate of change.

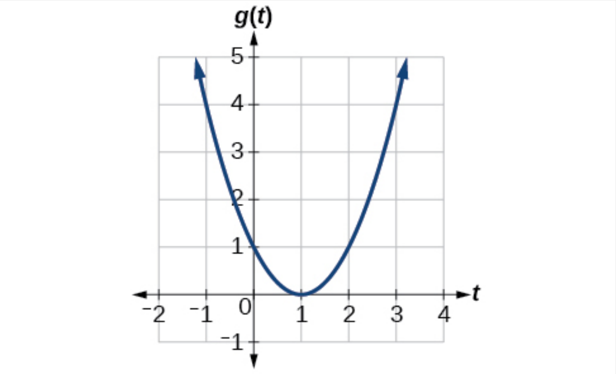
$$\begin{aligned} \text{Average rate of change} &= \frac{f(4)-f(2)}{4-2} \\ &= \frac{\frac{63}{4} - \frac{7}{2}}{4-2} \\ &= \frac{\frac{49}{4}}{2} \\ &= \frac{49}{8} \end{aligned}$$

Find the average rate of change of $f(x)=x-2\ x$ on the interval $[1, 9]$.

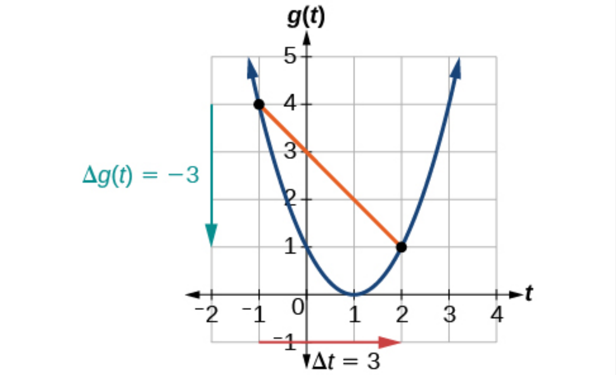
$\frac{1}{2}$

Computing Average Rate of Change from a Graph

Given the function $g(t)$ shown in [link], find the average rate of change on the interval $[-1,2]$.



At $t=-1$, [link] shows $g(-1)=4$. At $t=2$, the graph shows $g(2)=1$.



The horizontal change $\Delta t=3$ is shown by the red arrow, and the vertical change $\Delta g(t)=-3$ is shown by the turquoise arrow. The average rate of change is shown by the slope of the orange line segment. The output changes by –3 while the input changes by 3, giving an average rate of change of

$$\frac{1-4}{2-(-1)}=\frac{-3}{3}=-1$$

Analysis

Note that the order we choose is very important. If, for example, we use $\frac{y_2 - y_1}{x_1 - x_2}$, we will not get the correct answer. Decide which point will be 1 and which point will be 2, and keep the coordinates fixed as (x_1, y_1) and (x_2, y_2).

Computing Average Rate of Change from a Table

After picking up a friend who lives 10 miles away and leaving on a trip, Anna records her distance from home over time. The values are shown in [link]. Find her average speed over the first 6 hours.

***t* (hours)**	0	1	2	3	4	5	6	7
***D*(*t*) (miles)**	10	55	90	153	214	240	292	300

Here, the average speed is the average rate of

change. She traveled 282 miles in 6 hours.
$\frac{292-10}{6-0} = \frac{282}{6} = 47$

The average speed is 47 miles per hour.

Analysis

Because the speed is not constant, the average speed depends on the interval chosen. For the interval [2,3], the average speed is 63 miles per hour.

Finding an Average Rate of Change as an Expression

Find the average rate of change of $g(t) = t^2 + 3t + 1$ on the interval $[0, a]$. The answer will be an expression involving a in simplest form.

We use the average rate of change formula.

$$\begin{aligned} \text{Average rate of change} &= \frac{g(a)-g(0)}{a-0} && \\ &= \frac{(a^2+3a+1)-(0^2+3(0)+1)}{a-0} && \text{Evaluate.} \\ &= \frac{a^2+3a+1-1}{a} && \text{Simplify.} \\ &= \frac{a(a+3)}{a} && \text{Simplify and factor.} \\ &= a+3 && \text{Divide by the common factor } a. \end{aligned}$$

This result tells us the average rate of change in terms of a between $t=0$ and any other point $t=a$. For example, on the interval

$[0,5]$, the average rate of change would be $5+3=8$.

Find the average rate of change of $f(x)=x^2+2x-8$ on the interval $[5,a]$ in simplest forms in terms of a.

$a+7$

Access this online resource for additional instruction and practice with rates of change.

- Average Rate of Change

Key Concepts

- **Parallel Lines**

- Parallel lines are lines in the same plane that do not intersect.
  Parallel lines have the same slope and different *y*-intercepts.
  If m1 and m2 are the slopes of two parallel lines then $m1 = m2$.
  Parallel vertical lines have different *x*-intercepts.

- **Perpendicular Lines**

  - Perpendicular lines are lines in the same plane that form a right angle.
  - If m1 and m2 are the slopes of two perpendicular lines, then:
    their slopes are negative reciprocals of each other, $m1 = -1m2$.
    the product of their slopes is $-1, m1 \cdot m2 = -1$.
  - A vertical line and a horizontal line are always perpendicular to each other.

- **How to find an equation of a line parallel to a given line.**

  Find the slope of the given line. Find the slope of the parallel line. Identify the point. Substitute the values into the point-slope form: $y - y1 = m(x - x1)$. Write the equation in slope-intercept form

- **How to find an equation of a line**

perpendicular to a given line.

Find the slope of the given line. Find the slope of the perpendicular line. Identify the point. Substitute the values into the point-slope form, $y - y_1 = m(x - x_1)$ Write the equation in slope-intercept form.

Average rate of change	$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$

- A rate of change relates a change in an output quantity to a change in an input quantity. The average rate of change is determined using only the beginning and ending data. See [link].
- Identifying points that mark the interval on a graph can be used to find the average rate of change. See [link].
- Comparing pairs of input and output values in a table can also be used to find the average rate of change. See [link].
- An average rate of change can also be computed by determining the function values at the endpoints of an interval described by a formula. See [link] and [link].
- The average rate of change can sometimes be

determined as an expression. See [link].

Practice Makes Perfect

Use Slopes to Identify Parallel and Perpendicular Lines In the following exercises, use slopes and *y*-intercepts to determine if the lines are parallel, perpendicular, or neither.

$y = 34x - 3; 3x - 4y = -2$

---

parallel

$2x - 4y = 6; x - 2y = 3$

---

neither

$4x - 2y = 5; 3x + 6y = 8$

---

perpendicular

$3x - 6y = 12; 6x - 3y = 3$

---

neither

$3x-2y=8; 2x+3y=6$

---

perpendicular

Find an Equation of a Line Parallel to a Given Line

In the following exercises, find an equation of a line parallel to the given line and contains the given point. Write the equation in slope-intercept form.

line $y=4x+2$,
point $(1,2)$

$y=4x-2$

line $2x-y=6$,
point $(3,0)$.

$y=2x-6$

line $x=-4$,
point $(-3,-5)$.

$x = -3$

line $y = 5$,
point $(2, -2)$

$y = -2$

Find an Equation of a Line Perpendicular to a Given Line

In the following exercises, find an equation of a line perpendicular to the given line and contains the given point. Write the equation in slope-intercept form.

line $y = -2x + 3$,
point $(2,2)$

$y = 12x + 1$

line $y = 34x - 2$,
point $(-3,4)$

$y = -43x$

line $2x+5y=6$, point $(0,0)$	$y=52x$
line $x=3$, point $(3,4)$	$y=4$
line $x=7$, point $(-3,-4)$	$y=-4$
line $y-3=0$, point $(-2,-4)$	$x=-2$
line y-axis, point $(3,4)$	$y=4$

Mixed Practice

In the following exercises, find the equation of each line. Write the equation in slope-intercept form.

Containing the points $(4,3)$ and $(8,1)$

$y = -12x + 5$

$m = 16$, containing point $(6,1)$

$y = 16x$

Parallel to the line $4x + 3y = 6$, containing point $(0, -3)$

$y = -43x - 3$

$m = -34$, containing point $(8, -5)$

$y = -34x + 1$

Perpendicular to the line $y - 1 = 0$, point $(-2,6)$

$x=-2$

Containing the points $(-3,-4)$ and $(2,-5)$

$y=-15x-235$

Perpendicular to the line $x-2y=5$, point $(-2,2)$

$y=-2x-2$

Rates of Change
For the following exercises, find the average rate of change of each function on the interval specified for real numbers b or h in simplest form.

$f(x)=4 x 2 -7$ on $[1, b]$

$4(b+1)$

$f(x)=2 x 2 +1$ on $[x,x+h]$

$4x+2h$

For the following exercises, find the average rate of change of each function on the interval specified.

$h(x)=5-2x^2$ on $[-2,4]$

-4

$g(x)=3x^3-1$ on $[-3,3]$

27

$p(t)=\frac{(t^2-4)(t+1)}{t^2+3}$ on $[-3,1]$

–0.167

Glossary

average rate of change
: the difference in the output values of a function found for two values of the input divided by the difference between the inputs

parallel lines
: Parallel lines are lines in the same plane that do not intersect.

perpendicular lines

Perpendicular lines are lines in the same plane that form a right angle.

rate of change

the change of an output quantity relative to the change of the input quantity

Transformations of Functions (2.5)
By the end of this section, you will be able to:

- Identify graphs of basic functions
- Identify and graph various transformations of functions

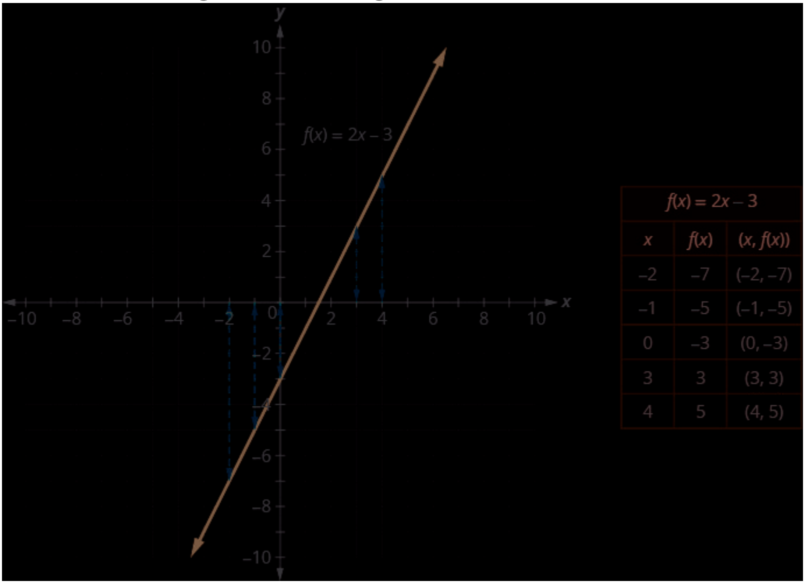
This Module supports section **2.5 of Mat 1023**.
In case you missed something in class, or just want to review a specific topic covered in this Module, here is a list of topics covered:

Identify Graphs of Basic Functions

We can write $y=2x-3$ as in function notation as $f(x)=2x-3$. It still means the same thing. The graph

of the function is the graph of all ordered pairs (x,y) where y = f(x). So we can write the ordered pairs as (x,f(x)). It looks different but the graph will be the same.

Compare the graph of y = 2x − 3 previously shown in [link] with the graph of f(x) = 2x − 3 shown in [link]. Nothing has changed but the notation.



Graph of a Function

The graph of a function is the graph of all its ordered pairs, (x,y) or using function notation, (x,f(x)) where y = f(x).

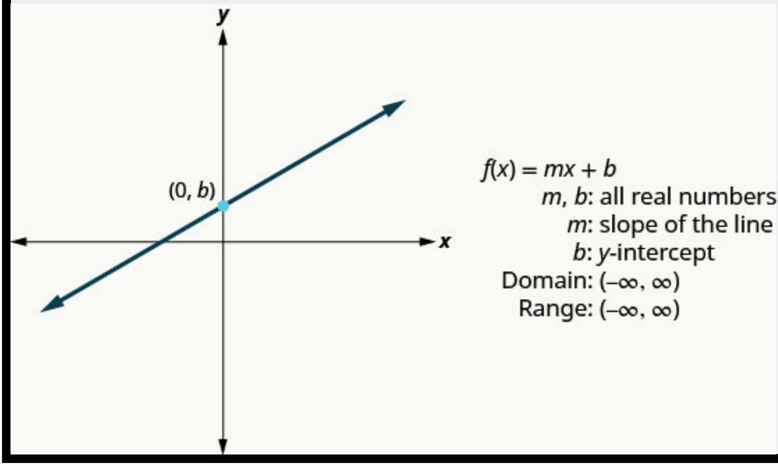
fname of functionxx-coordinate of the ordered pairf(x)y-coordinate of the ordered pair

As we move forward in our study, it is helpful to be familiar with the graphs of several basic functions and be able to identify them.

We wrote linear equations in several forms, but it will be most helpful for us here to use the slope-intercept form of the linear equation. The slope-intercept form of a linear equation is $y=mx+b$. In function notation, this linear function becomes $f(x)=mx+b$ where *m* is the slope of the line and *b* is the *y*-intercept.

The domain is the set of all real numbers, and the range is also the set of all real numbers.

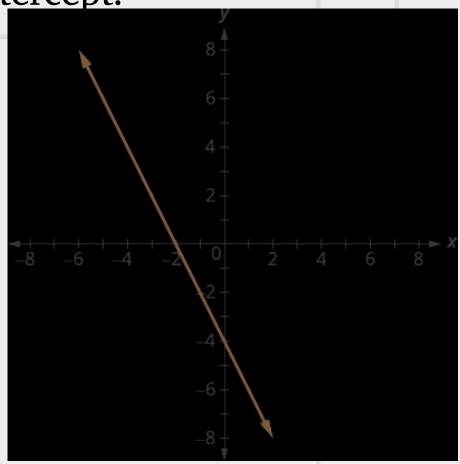
Linear Function



We will use the graphing techniques we used earlier, to graph the basic functions.

Graph: $f(x) = -2x - 4$.

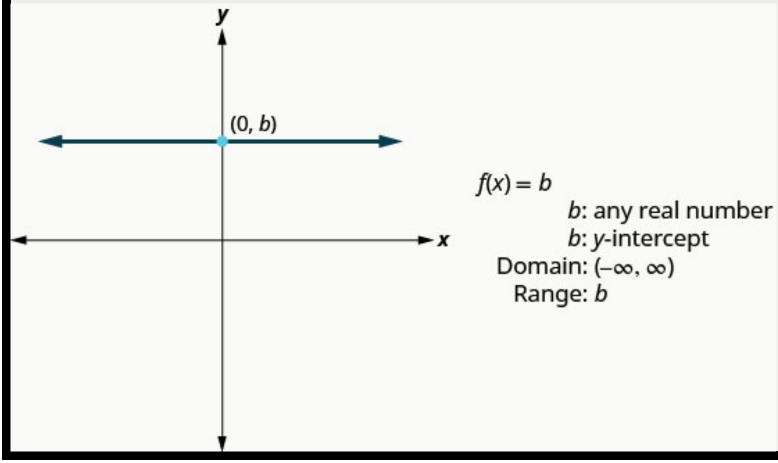
	$f(x) = -2x - 4$
We recognize this as a linear function.	
Find the slope and *y*-intercept.	$m = -2$ $b = -4$
Graph using the slope intercept.	



The next function whose graph we will look at is called the constant function and its equation is of the form $f(x)=b$, where *b* is any real number. If we replace the $f(x)$ with y, we get $y=b$. We recognize this as the horizontal line whose *y*-intercept is *b*. The graph of the function $f(x)=b$, is also the horizontal line whose *y*-intercept is *b*.

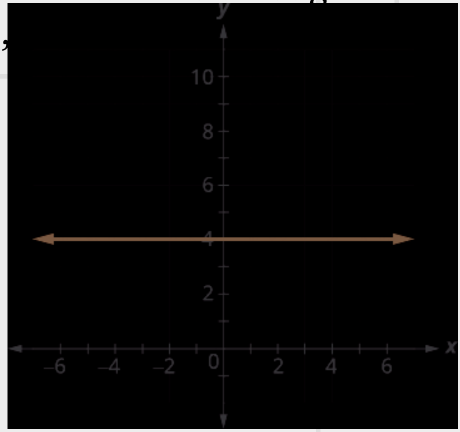
Notice that for any real number we put in the function, the function value will be *b*. This tells us the range has only one value, *b*.

Constant Function

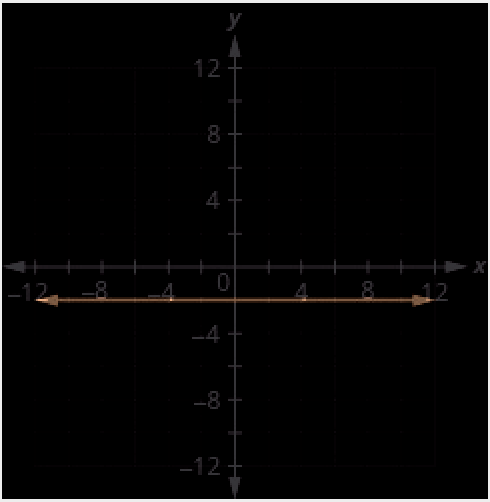


Graph: $f(x) = 4$.

	$f(x) = 4$
We recognize this as a constant function.	
The graph will be a horizontal line through (0,	

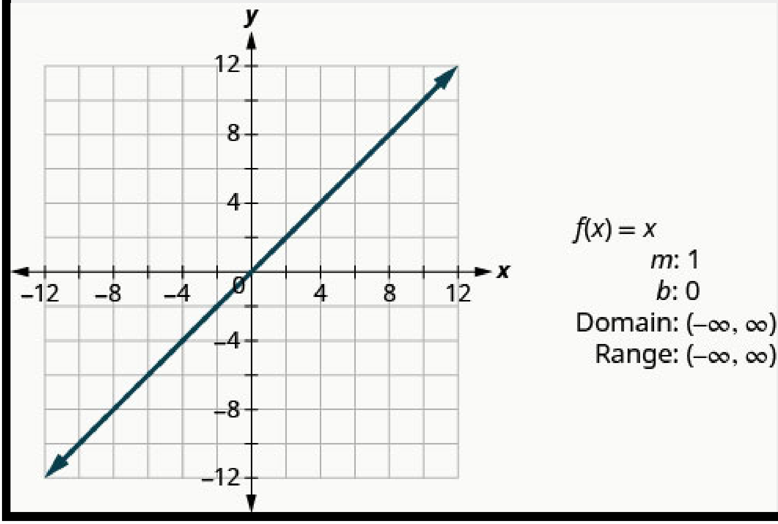


Graph: $f(x) = -2$.



The identity function, $f(x) = x$ is a special case of the linear function. If we write it in linear function form, $f(x) = 1x + 0$, we see the slope is 1 and the *y*-intercept is 0.

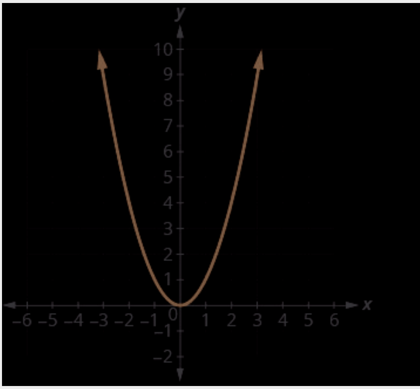
Identity Function



The next function we will look at is not a linear function. So the graph will not be a line. The only method we have to graph this function is point plotting. Because this is an unfamiliar function, we make sure to choose several positive and negative values as well as 0 for our x-values.

Graph: f(x) = x2.

We choose *x*-values. We substitute them in and then create a chart as shown.

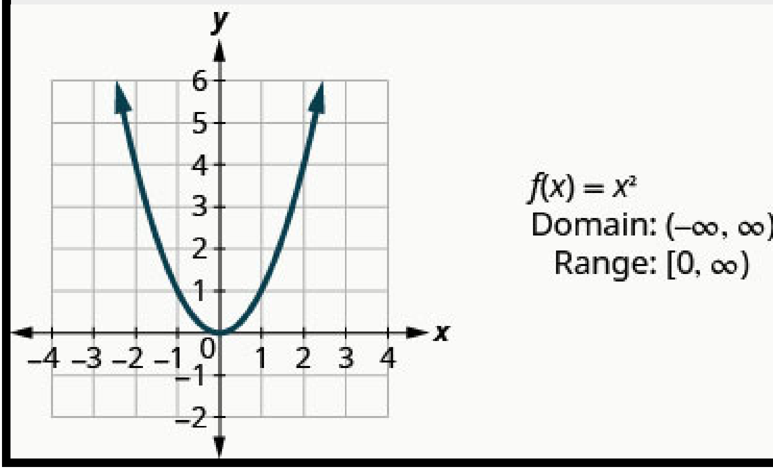


x	$f(x) = x^2$	$(x, f(x))$
–3	9	(–3, 9)
–2	4	(–2, 4)
–1	1	(–1, 1)
0	0	(0, 0)
1	1	(1, 1)
2	4	(2, 4)
3	9	(3, 9)

Looking at the result in [link], we can summarize the features of the square function. We call this graph a parabola. As we consider the domain, notice any real number can be used as an x-value. The domain is all real numbers.

The range is not all real numbers. Notice the graph consists of values of y never go below zero. This makes sense as the square of any number cannot be negative. So, the range of the square function is all non-negative real numbers.

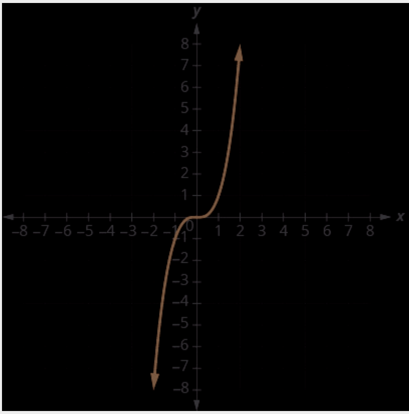
Square Function



The next function we will look at is also not a linear function so the graph will not be a line. Again we will use point plotting, and make sure to choose several positive and negative values as well as 0 for our *x*-values.

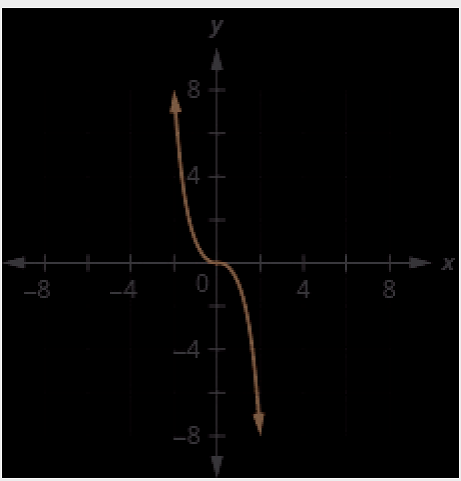
Graph: f(x) = x3.

We choose *x*-values. We substitute them in and then create a chart.



x	$f(x) = x^3$	$(x, f(x))$
–2	–8	(–2, –8)
–1	–1	(–1, –1)
0	0	(0, 0)
1	1	(1, 1)
2	8	(2, 8)

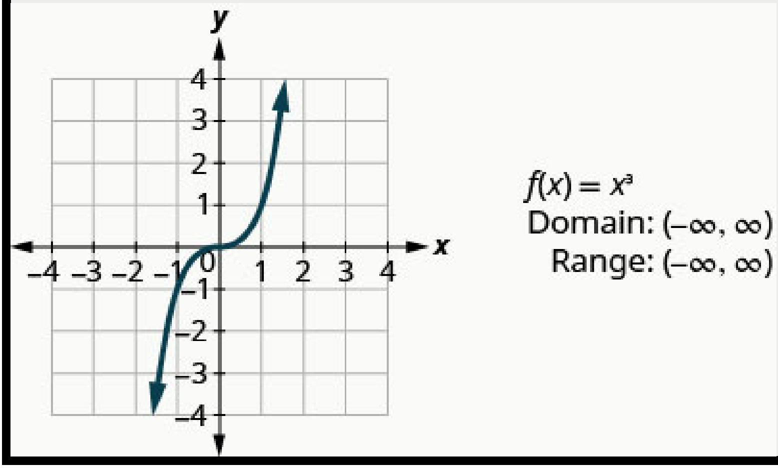
Graph: $f(x) = -x3$.



Looking at the result in [link], we can summarize the features of the cube function. As we consider the domain, notice any real number can be used as an *x*-value. The domain is all real numbers.

The range is all real numbers. This makes sense as the cube of any non-zero number can be positive or negative. So, the range of the cube function is all real numbers.

Cube Function

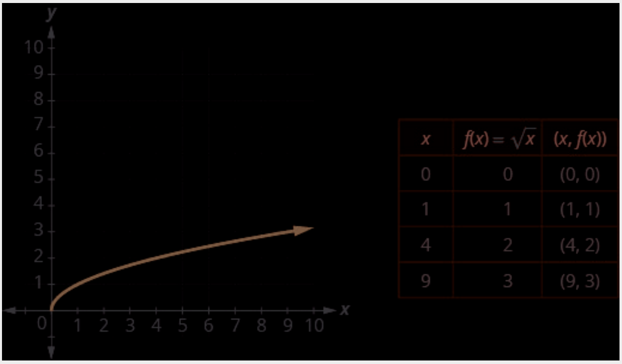


The next function we will look at does not square or cube the input values, but rather takes the square root of those values.

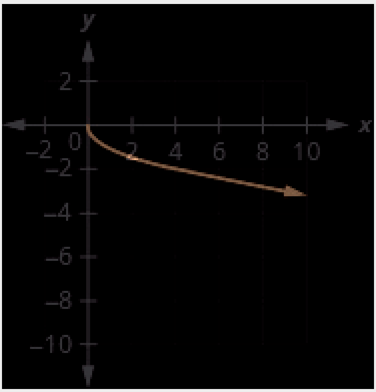
Let’s graph the function f(x) = x and then summarize the features of the function. Remember, we can only take the square root of non-negative real numbers, so our domain will be the non-negative real numbers.

f(x) = x

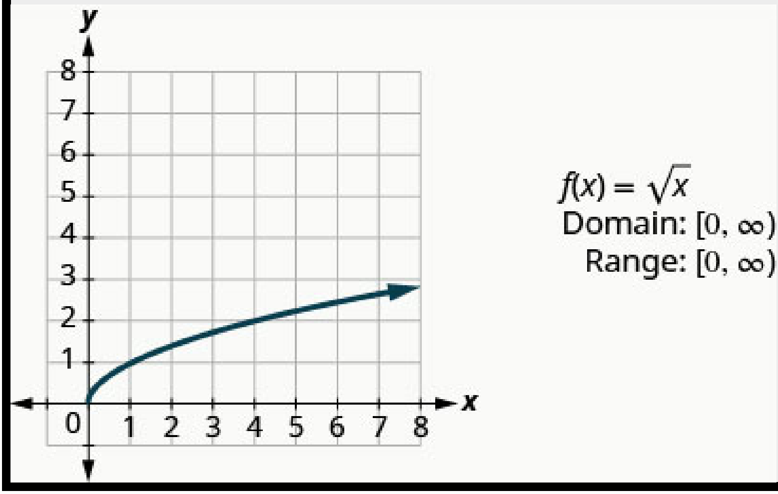
We choose *x*-values. Since we will be taking the square root, we choose numbers that are perfect squares, to make our work easier. We substitute them in and then create a chart.



Graph: $f(x) = -x$.



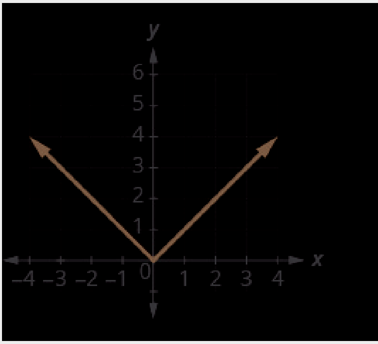
Square Root Function



Our last basic function is the absolute value function, $f(x)=|x|$. Keep in mind that the absolute value of a number is its distance from zero. Since we never measure distance as a negative number, we will never get a negative number in the range.

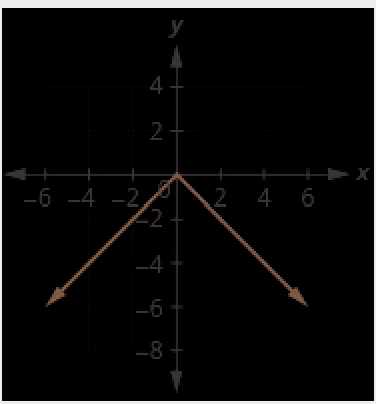
Graph: $f(x)=|x|$.

We choose *x*-values. We substitute them in and then create a chart.

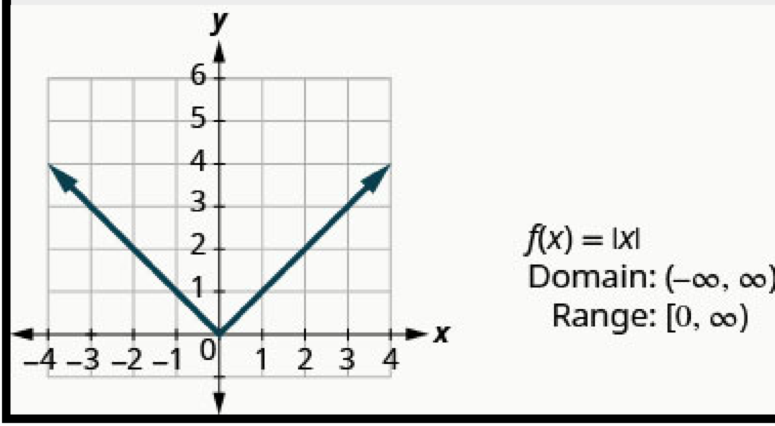


x	$f(x) = \lvert x \rvert$	$(x, f(x))$
−3	3	(−3, 3)
−2	2	(−2, 2)
−1	1	(−1, 1)
0	0	(0, 0)
1	1	(1, 1)
2	2	(2, 2)
3	3	(3, 3)

Graph: $f(x) = -\lvert x \rvert$.



Absolute Value Function

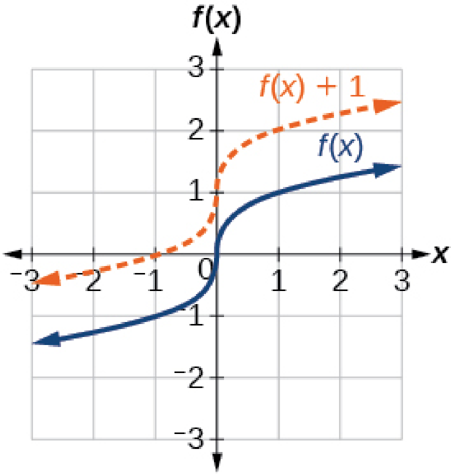


Vertical shift by $k=1$ of the cube root function $f(x)= x 3$ .Horizontal shift of the function $f(x)= x 3$. Note that $h= +1$ shifts the graph to the left, that is, towards *negative* values of x.

Vertical and Horizontal Transformations

One simple kind of transformation involves shifting the entire graph of a function up, down, right, or left. The simplest shift is a **vertical shift**, moving the graph up or down, because this transformation involves adding a positive or negative constant to the function. In other words, we add the same constant to the output value of the function regardless of the input. For a function $g(x)=f(x)+k,$ the function $f(x)$ is shifted

vertically k units. See [link] for an example.



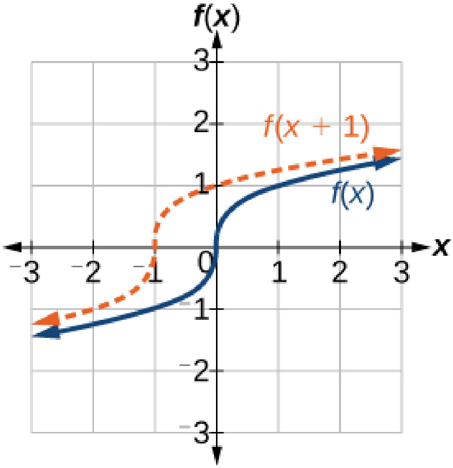
To help you visualize the concept of a vertical shift, consider that $y=f(x)$. Therefore, $f(x)+k$ is equivalent to $y+k$. Every unit of y is replaced by $y+k$, so the *y*-value increases or decreases depending on the value of k. The result is a shift upward or downward.

Vertical Shift

Given a function $f(x)$, a new function $g(x)=f(x)+k$, where k is a constant, is a **vertical shift** of the function $f(x)$. All the output values change by k units. If k is positive, the graph will shift up. If k is negative, the graph will shift down.

Horizontal Shifts

We just saw that the vertical shift is a change to the output, or outside, of the function. We will now look at how changes to input, on the inside of the function, change its graph and meaning. A shift to the input results in a movement of the graph of the function left or right in what is known as a **horizontal shift**, shown in [link].



For example, if $f(x) = x^2$, then $g(x) = (x-2)^2$ is a new function. Each input is reduced by 2 prior to squaring the function. The result is that the graph is shifted 2 units to the right, because we would need to increase the prior input by 2 units to yield the same output value as given in f.

Horizontal Shift

Given a function f, a new function $g(x) = f(x-h)$, where h is a constant, is a **horizontal shift** of the function f. If h is positive, the graph will shift

right. If h is negative, the graph will shift left.

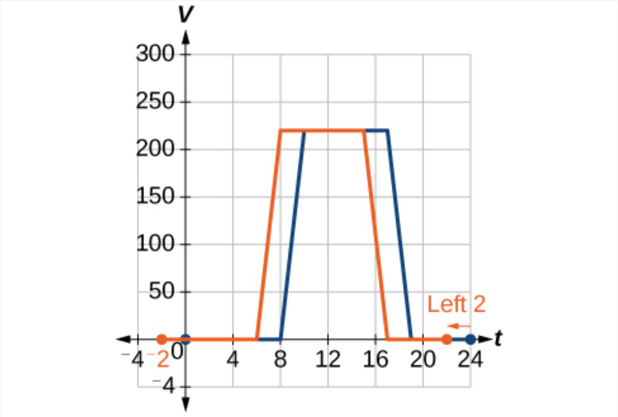
Adding a Constant to an Input

Returning to our building airflow example from [link], suppose that in autumn the facilities manager decides that the original venting plan starts too late, and wants to begin the entire venting program 2 hours earlier. Sketch a graph of the new function.

We can set $V(t)$ to be the original program and $F(t)$ to be the revised program.
$V(t) =$ the original venting plan $F(t) =$ starting 2 hrs sooner

In the new graph, at each time, the airflow is the same as the original function V was 2 hours later. For example, in the original function V, the airflow starts to change at 8 a.m., whereas for the function F, the airflow starts to change at 6 a.m. The comparable function values are $V(8)=F(6)$. See [link]. Notice also that the vents first opened to $220\ \text{ft}^2$ at 10 a.m. under the original plan, while under the new plan the vents reach $220\ \text{ft}^2$ at
8 a.m., so $V(10)=F(8)$.

In both cases, we see that, because $F(t)$ starts 2 hours sooner, $h = -2$. That means that the same output values are reached when $F(t) = V(t - (-2)) = V(t + 2)$.



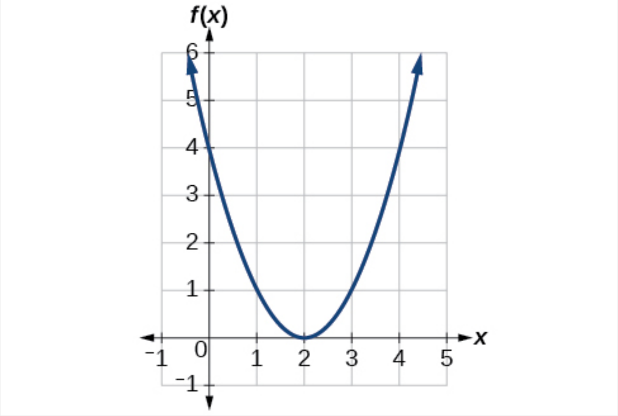
Analysis

Note that $V(t+2)$ has the effect of shifting the graph to the *left*.

Horizontal changes or "inside changes" affect the domain of a function (the input) instead of the range and often seem counterintuitive. The new function $F(t)$ uses the same outputs as $V(t)$, but matches those outputs to inputs 2 hours earlier than those of $V(t)$. Said another way, we must add 2 hours to the input of V to find the corresponding output for $F: F(t) = V(t+2)$.

Identifying a Horizontal Shift of a Toolkit Function

[link] represents a transformation of the toolkit function $f(x)= x^2$. Relate this new function $g(x)$ to $f(x)$, and then find a formula for $g(x)$.



Notice that the graph is identical in shape to the $f(x)= x^2$ function, but the *x*-values are shifted to the right 2 units. The vertex used to be at (0,0), but now the vertex is at (2,0). The graph is the basic quadratic function shifted 2 units to the right, so

$$g(x)=f(x-2)$$

Notice how we must input the value $x=2$ to get the output value $y=0$; the *x*-values must be 2 units larger because of the shift to the right by 2 units. We can then use the

definition of the $f(x)$ function to write a formula for $g(x)$ by evaluating $f(x-2)$.
$f(x) = x 2$ $g(x) = f(x-2)$ $g(x) = f(x-2) = (x-2) 2$

Analysis
To determine whether the shift is $+2$ or -2, consider a single reference point on the graph. For a quadratic, looking at the vertex point is convenient. In the original function, $f(0)=0$. In our shifted function, $g(2)=0$. To obtain the output value of 0 from the function f, we need to decide whether a plus or a minus sign will work to satisfy $g(2)=f(x-2)=f(0)=0$. For this to work, we will need to *subtract* 2 units from our input values.

Interpreting Horizontal versus Vertical Shifts

The function $G(m)$ gives the number of gallons of gas required to drive m miles. Interpret $G(m)+10$ and $G(m+10)$.

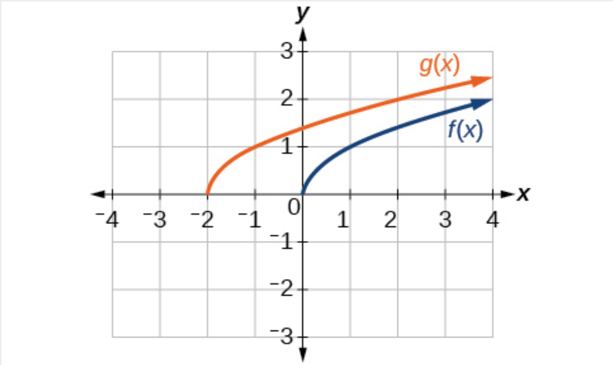
$G(m)+10$ can be interpreted as adding 10 to the output, gallons. This is the gas required to drive m miles, plus another 10 gallons of gas.

The graph would indicate a vertical shift.

$G(m+10)$ can be interpreted as adding 10 to the input, miles. So this is the number of gallons of gas required to drive 10 miles more than m miles. The graph would indicate a horizontal shift.

Given the function $f(x) = x$, graph the original function $f(x)$ and the transformation $g(x) = f(x+2)$ on the same axes. Is this a horizontal or a vertical shift? Which way is the graph shifted and by how many units?

The graphs of $f(x)$ and $g(x)$ are shown below. The transformation is a horizontal shift. The function is shifted to the left by 2 units.



Combining Vertical and Horizontal Shifts

Now that we have two transformations, we can combine them. Vertical shifts are outside changes that affect the output (y-) values and shift the function up or down. Horizontal shifts are inside changes that affect the input (x-) values and shift the function left or right. Combining the two types of shifts will cause the graph of a function to shift up or down *and* left or right.

Given a function and both a vertical and a horizontal shift, sketch the graph.

1. Identify the vertical and horizontal shifts from the formula.
2. The vertical shift results from a constant

added to the output. Move the graph up for a positive constant and down for a negative constant.
3. The horizontal shift results from a constant added to the input. Move the graph left for a positive constant and right for a negative constant.
4. Apply the shifts to the graph in either order.

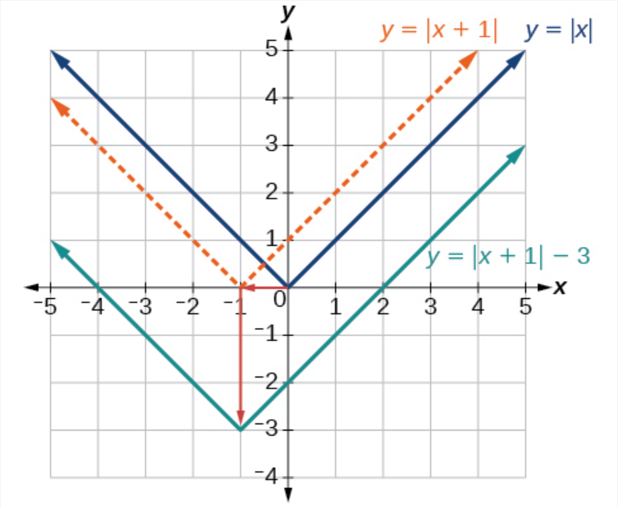
Graphing Combined Vertical and Horizontal Shifts

Given $f(x) = |x|$, sketch a graph of $h(x) = f(x+1) - 3$.

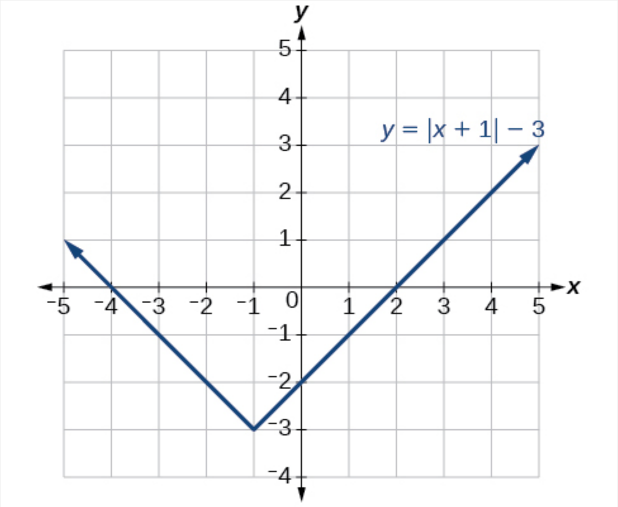
The function f is our toolkit absolute value function. We know that this graph has a V shape, with the point at the origin. The graph of h has transformed f in two ways: $f(x+1)$ is a change on the inside of the function, giving a horizontal shift left by 1, and the subtraction by 3 in $f(x+1) - 3$ is a change to the outside of the function, giving a vertical shift down by 3. The transformation of the graph is illustrated in [link].

Let us follow one point of the graph of $f(x) = |x|$.

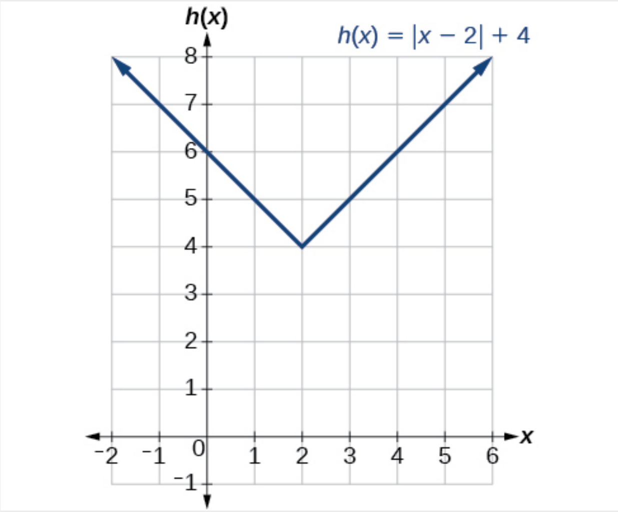
- The point (0,0) is transformed first by shifting left 1 unit: $(0,0) \to (-1,0)$
- The point $(-1,0)$ is transformed next by shifting down 3 units: $(-1,0) \to (-1,-3)$



[link] shows the graph of h.

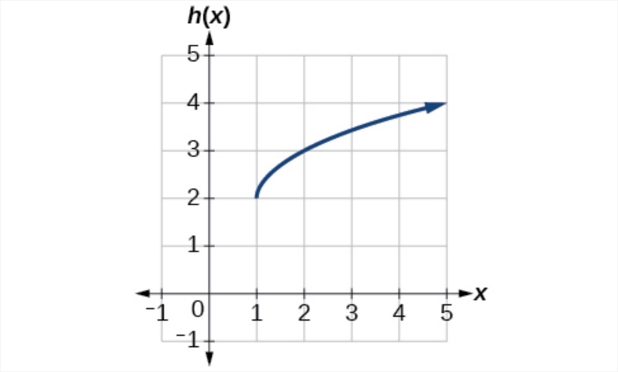


Given $f(x) = |\,x\,|$, sketch a graph of $h(x) = f(x - 2) + 4$.



Identifying Combined Vertical and Horizontal Shifts

Write a formula for the graph shown in [link], which is a transformation of the toolkit square root function.



The graph of the toolkit function starts at the origin, so this graph has been shifted 1 to the right and up 2. In function notation, we could write that as

$$h(x)=f(x-1)+2$$

Using the formula for the square root function, we can write

$$h(x)=\sqrt{x-1}+2$$

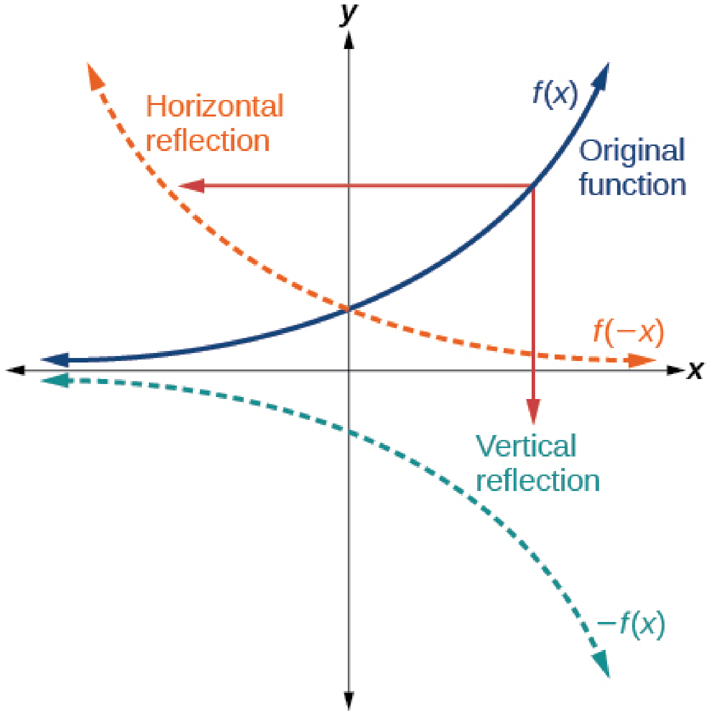
Analysis

Note that this transformation has changed the domain and range of the function. This new graph has domain $[1,\infty)$ and range $[2,\infty)$.

Vertical and horizontal reflections of a function.

Graphing Functions Using Reflections about the Axes

Another transformation that can be applied to a function is a reflection over the *x*- or *y*-axis. A **vertical reflection** reflects a graph vertically across the *x*-axis, while a **horizontal reflection** reflects a graph horizontally across the *y*-axis. The reflections are shown in [link].



Notice that the vertical reflection produces a new graph that is a mirror image of the base or original graph about the *x*-axis. The horizontal reflection produces a new graph that is a mirror image of the base or original graph about the *y*-axis.

Reflections

Given a function $f(x)$, a new function $g(x) = -f(x)$ is a **vertical reflection** of the function $f(x)$, sometimes called a reflection about (or over, or through) the *x*-axis.

Given a function $f(x)$, a new function $g(x) = f(-x)$ is a **horizontal reflection** of the function $f(x)$, sometimes called a reflection about the *y*-axis.

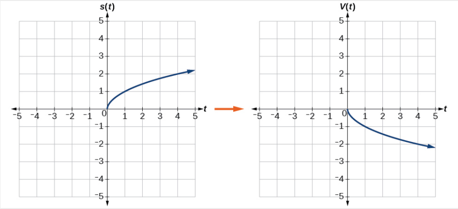
Given a function, reflect the graph both vertically and horizontally.

1. Multiply all outputs by –1 for a vertical reflection. The new graph is a reflection of the original graph about the *x*-axis.
2. Multiply all inputs by –1 for a horizontal reflection. The new graph is a reflection of the original graph about the *y*-axis.

Reflecting a Graph Horizontally and Vertically

Reflect the graph of $s(t) = \sqrt{t}$ (a) vertically and (b) horizontally.

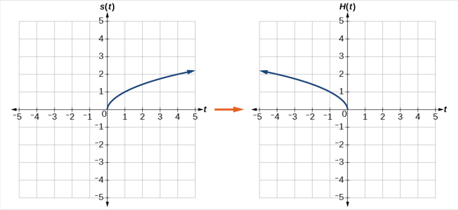
1. Reflecting the graph vertically means that each output value will be reflected over the horizontal *t*-axis as shown in [link].
   Vertical reflection of the square root function

   Because each output value is the opposite of the original output value, we can write $V(t) = -s(t)$ or $V(t) = -\ t$

   Notice that this is an outside change, or vertical shift, that affects the output $s(t)$ values, so the negative sign belongs outside of the function.

2. Reflecting horizontally means that each input value will be reflected over the vertical axis as shown in [link].
   Horizontal reflection of the square root function

   Because each input value is the opposite

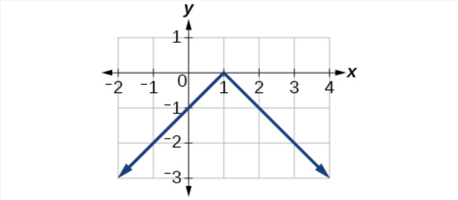
of the original input value, we can write $H(t)=s(-t)$ or $H(t)= \ -t$

Notice that this is an inside change or horizontal change that affects the input values, so the negative sign is on the inside of the function.

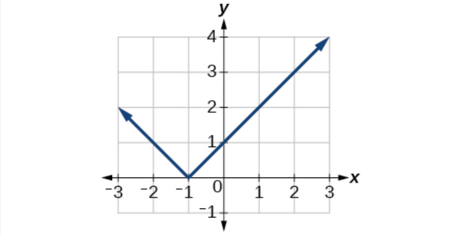
Note that these transformations can affect the domain and range of the functions. While the original square root function has domain $[0,\infty)$ and range $[0,\infty)$, the vertical reflection gives the $V(t)$ function the range $(-\infty,0]$ and the horizontal reflection gives the $H(t)$ function the domain $(-\infty,0]$.

Reflect the graph of $f(x)=|x-1|$ (a) vertically and (b) horizontally.

1.



2.



Vertical stretch and compression

Stretching and Shrinking Transformations

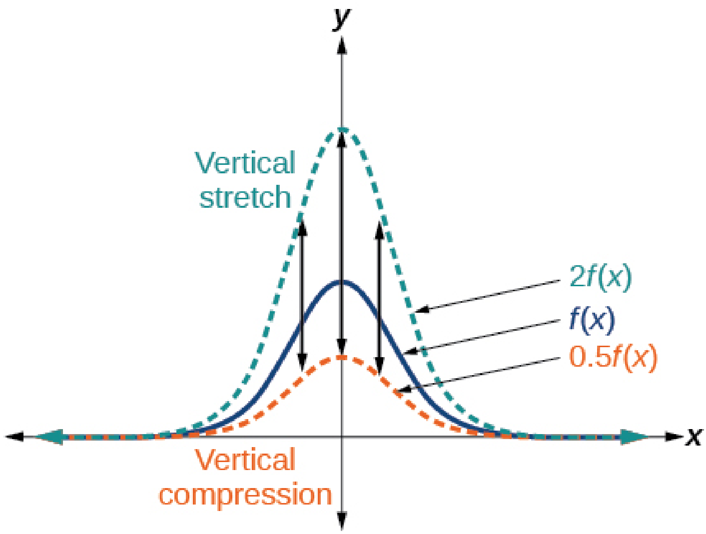
Adding a constant to the inputs or outputs of a function changed the position of a graph with respect to the axes, but it did not affect the shape of a graph. We now explore the effects of multiplying the inputs or outputs by some quantity.

We can transform the inside (input values) of a function or we can transform the outside (output values) of a function. Each change has a specific

effect that can be seen graphically.

Vertical Stretches and Compressions

When we multiply a function by a positive constant, we get a function whose graph is stretched or compressed vertically in relation to the graph of the original function. If the constant is greater than 1, we get a **vertical stretch**; if the constant is between 0 and 1, we get a **vertical compression**. [link] shows a function multiplied by constant factors 2 and 0.5 and the resulting vertical stretch and compression.



Vertical Stretches and Compressions
Given a function $f(x)$, a new function $g(x) = af(x)$, where a is a constant, is a **vertical stretch** or

vertical compression of the function $f(x)$.

- If $a > 1$, then the graph will be stretched.
- If $0 < a < 1$, then the graph will be compressed.
- If $a < 0$, then there will be combination of a vertical stretch or compression with a vertical reflection.

Given a function, graph its vertical stretch.

1. Identify the value of a.
2. Multiply all range values by a.

3. If $a > 1$, the graph is stretched by a factor of a.

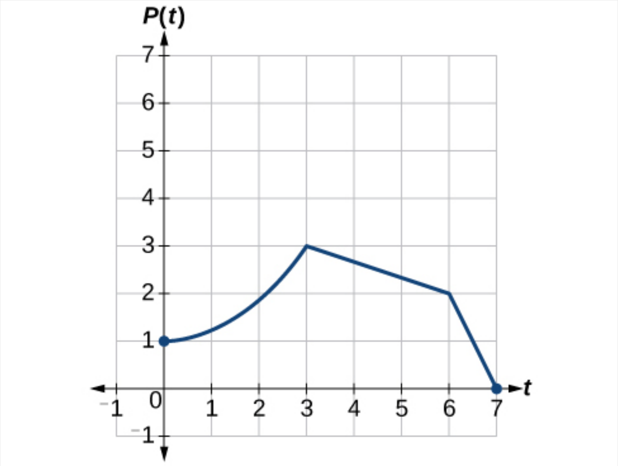
   If $0 < a < 1$, the graph is compressed by a factor of a.

   If $a < 0$, the graph is either stretched or compressed and also reflected about the *x*-axis.

Graphing a Vertical Stretch

A function $P(t)$ models the population of

fruit flies. The graph is shown in [link].



A scientist is comparing this population to another population, Q, whose growth follows the same pattern, but is twice as large. Sketch a graph of this population.

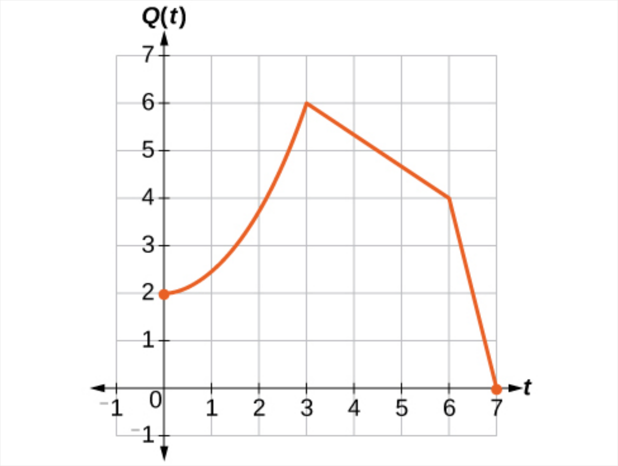
Because the population is always twice as large, the new population's output values are always twice the original function's output values. Graphically, this is shown in [link].

If we choose four reference points, (0, 1), (3, 3), (6, 2) and (7, 0) we will multiply all of the outputs by 2.

The following shows where the new points for the new graph will be located.

$(0,1) \rightarrow (0,2)$ $(3,3) \rightarrow (3,6)$ $(6,2) \rightarrow (6,$

$4)\ (7, 0) \rightarrow (7, 0)$



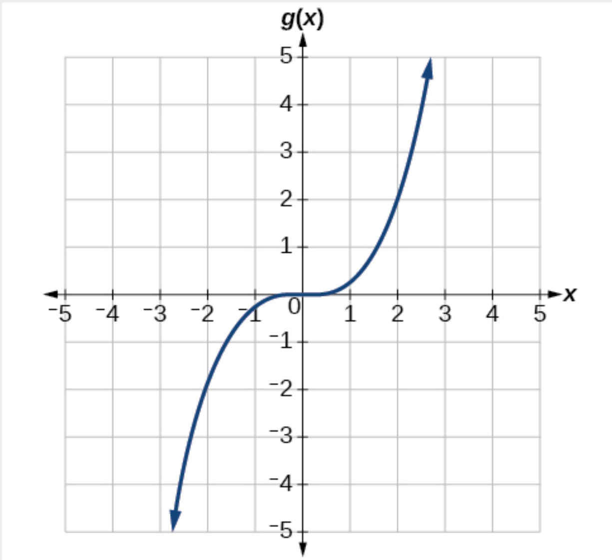
Symbolically, the relationship is written as
$Q(t) = 2P(t)$

This means that for any input t, the value of the function Q is twice the value of the function P. Notice that the effect on the graph is a vertical stretching of the graph, where every point doubles its distance from the horizontal axis. The input values, t, stay the same while the output values are twice as large as before.

Recognizing a Vertical Stretch

The graph in [link] is a transformation of the toolkit function $f(x) = x^3$. Relate this new

function $g(x)$ to $f(x)$, and then find a formula for $g(x)$.



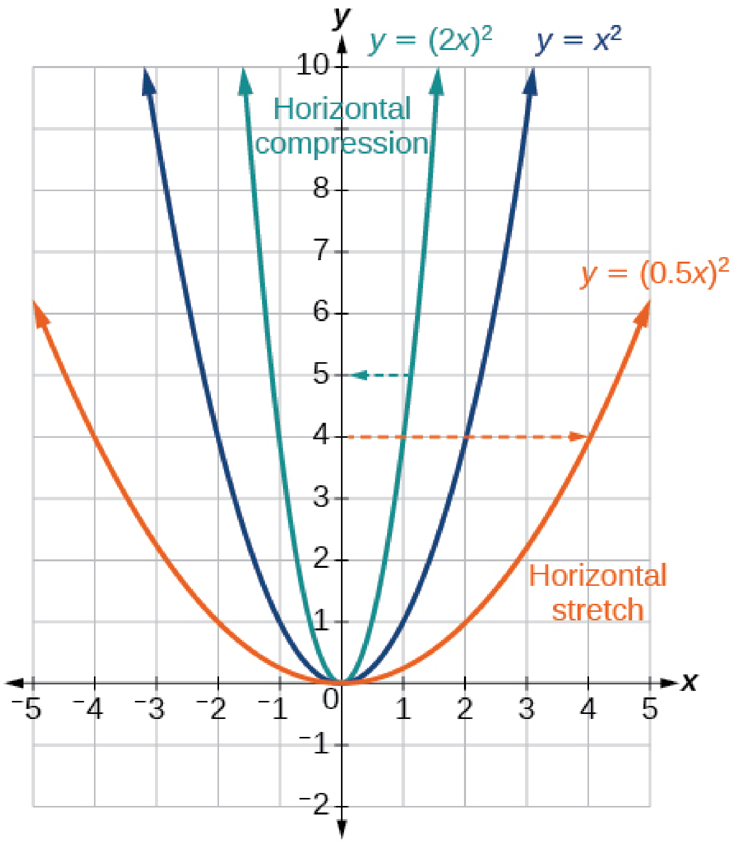
When trying to determine a vertical stretch or shift, it is helpful to look for a point on the graph that is relatively clear. In this graph, it appears that $g(2)=2$. With the basic cubic function at the same input, $f(2)=(2)^3=8$. Based on that, it appears that the outputs of g are $\frac{1}{4}$ the outputs of the function f because $g(2)=\frac{1}{4}f(2)$. From this we can fairly safely conclude that $g(x)=\frac{1}{4}f(x)$.

We can write a formula for g by using the definition of the function f.

$g(x)=\frac{1}{4}f(x)=\frac{1}{4}x^3$

Horizontal Stretches and Compressions

Now we consider changes to the inside of a function. When we multiply a function's input by a positive constant, we get a function whose graph is stretched or compressed horizontally in relation to the graph of the original function. If the constant is between 0 and 1, we get a **horizontal stretch**; if the constant is greater than 1, we get a **horizontal compression** of the function.



Given a function $y = f(x)$, the form $y = f(bx)$ results in a horizontal stretch or compression. Consider the

function $y=x^2$. Observe [link]. The graph of $y=(0.5x)^2$ is a horizontal stretch of the graph of the function $y=x^2$ by a factor of 2. The graph of $y=(2x)^2$ is a horizontal compression of the graph of the function $y=x^2$ by a factor of 2.

Horizontal Stretches and Compressions

Given a function $f(x)$, a new function $g(x)=f(bx)$, where b is a constant, is a **horizontal stretch** or **horizontal compression** of the function $f(x)$.

- If $b>1$, then the graph will be compressed by $\frac{1}{b}$.
- If $0<b<1$, then the graph will be stretched by $\frac{1}{b}$.
- If $b<0$, then there will be combination of a horizontal stretch or compression with a horizontal reflection.

Given a description of a function, sketch a horizontal compression or stretch.

1. Write a formula to represent the function.
2. Set $g(x)=f(bx)$ where $b>1$ for a compression or $0<b<1$ for a stretch.

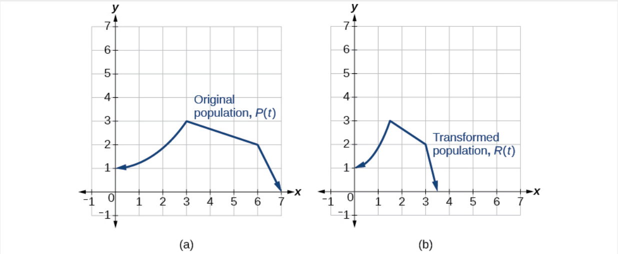
Graphing a Horizontal Compression

Suppose a scientist is comparing a population of fruit flies to a population that progresses through its lifespan twice as fast as the original population. In other words, this new population, R, will progress in 1 hour the same amount as the original population does in 2 hours, and in 2 hours, it will progress as much as the original population does in 4 hours. Sketch a graph of this population.

Symbolically, we could write $R(1) = P(2)$, $R(2) = P(4)$, and in general, $R(t) = P(2t)$.

See [link] for a graphical comparison of the original population and the compressed population.

(a) Original population graph (b) Compressed population graph



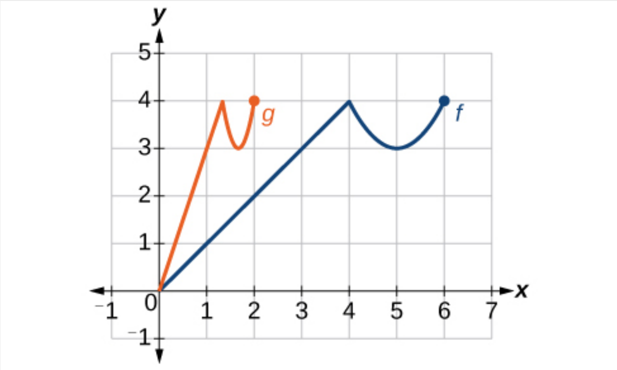
Analysis

Note that the effect on the graph is a horizontal

compression where all input values are half of their original distance from the vertical axis.

Recognizing a Horizontal Compression on a Graph

Relate the function $g(x)$ to $f(x)$ in [link].



The graph of $g(x)$ looks like the graph of $f(x)$ horizontally compressed. Because $f(x)$ ends at $(6,4)$ and $g(x)$ ends at $(2,4)$, we can see that the x- values have been compressed by $\frac{1}{3}$, because $6\left(\frac{1}{3}\right)=2$. We might also notice that $g(2)=f(6)$ and $g(1)=f(3)$. Either way, we can describe this relationship as $g(x)=f(3x)$. This is a horizontal compression by $\frac{1}{3}$.

Analysis

Notice that the coefficient needed for a horizontal stretch or compression is the reciprocal of the stretch or compression. So to stretch the graph horizontally by a scale factor of 4, we need a coefficient of $\frac{1}{4}$ in our function: $f(\frac{1}{4}x)$. This means that the input values must be four times larger to produce the same result, requiring the input to be larger, causing the horizontal stretching.

Write a formula for the toolkit square root function horizontally stretched by a factor of 3.

$g(x)=f(\frac{1}{3}x)$ so using the square root function we get $g(x)=\frac{1}{3}x$

Performing a Sequence of Transformations

When combining transformations, it is very important to consider the order of the

transformations. For example, vertically shifting by 3 and then vertically stretching by 2 does not create the same graph as vertically stretching by 2 and then vertically shifting by 3, because when we shift first, both the original function and the shift get stretched, while only the original function gets stretched when we stretch first.

When we see an expression such as $2f(x)+3$, which transformation should we start with? The answer here follows nicely from the order of operations. Given the output value of $f(x)$, we first multiply by 2, causing the vertical stretch, and then add 3, causing the vertical shift. In other words, multiplication before addition.

Horizontal transformations are a little trickier to think about. When we write $g(x)=f(2x+3)$, for example, we have to think about how the inputs to the function g relate to the inputs to the function f. Suppose we know $f(7)=12$. What input to g would produce that output? In other words, what value of x will allow $g(x)=f(2x+3)=12$? We would need $2x+3=7$. To solve for x, we would first subtract 3, resulting in a horizontal shift, and then divide by 2, causing a horizontal compression.

This format ends up being very difficult to work with, because it is usually much easier to horizontally stretch a graph before shifting. We can work around this by factoring inside the function.

$$f(bx+p)=f\left(b\left(x+\frac{p}{b}\right)\right)$$

Let’s work through an example.

$$f(x)=(2x+4)^2$$

We can factor out a 2.

$$f(x)=(2(x+2))^2$$

Now we can more clearly observe a horizontal shift to the left 2 units and a horizontal compression. Factoring in this way allows us to horizontally stretch first and then shift horizontally.

Combining Transformations

When combining vertical transformations written in the form $af(x)+k$, first vertically stretch by a and then vertically shift by k.

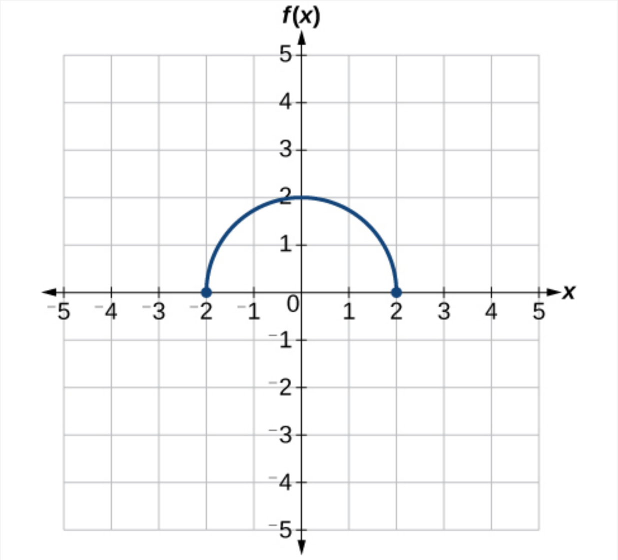
When combining horizontal transformations written in the form $f(bx-h)$, first horizontally shift by h and then horizontally stretch by $\frac{1}{b}$.

When combining horizontal transformations written in the form $f(b(x-h))$, first horizontally stretch by $\frac{1}{b}$ and then horizontally shift by h.

Horizontal and vertical transformations are independent. It does not matter whether horizontal or vertical transformations are performed first.

Finding a Triple Transformation of a Graph

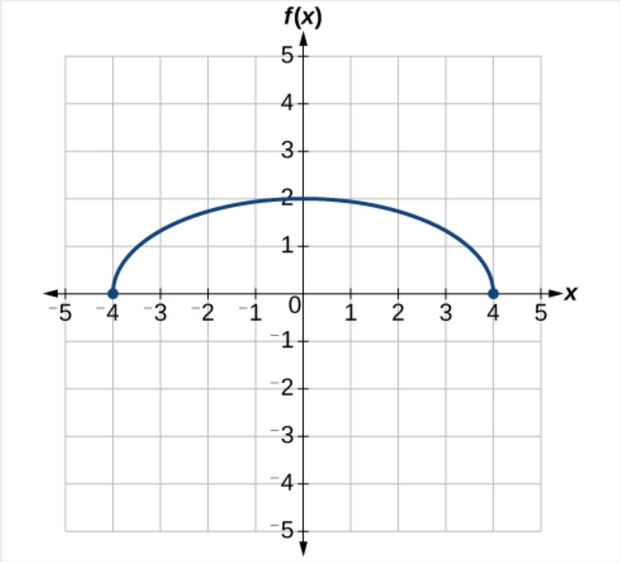
Use the graph of $f(x)$ in [link] to sketch a graph of $k(x)=f\left(\frac{1}{2}x+1\right)-3$.



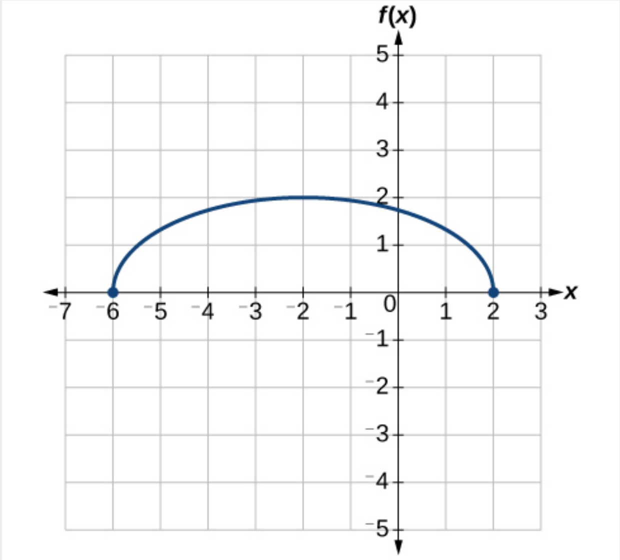
To simplify, let’s start by factoring out the inside of the function.

$$f\left(\frac{1}{2}x+1\right)-3=f\left(\frac{1}{2}(x+2)\right)-3$$

By factoring the inside, we can first horizontally stretch by 2, as indicated by the $\frac{1}{2}$ on the inside of the function. Remember that twice the size of 0 is still 0, so the point (0,2) remains at (0,2) while the point (2,0) will stretch to (4,0). See [link].

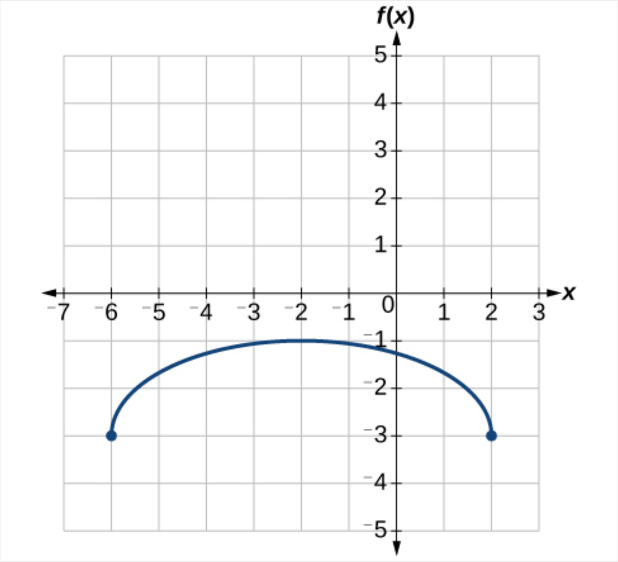


Next, we horizontally shift left by 2 units, as indicated by $x+2$. See [link].



Last, we vertically shift down by 3 to complete

our sketch, as indicated by the -3 on the outside of the function. See [link].



Access this online resource for additional instruction and practice with transformation of functions.

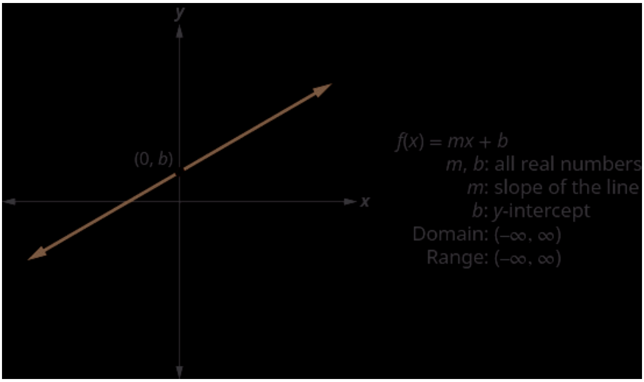
- Function Transformations

Key Concepts

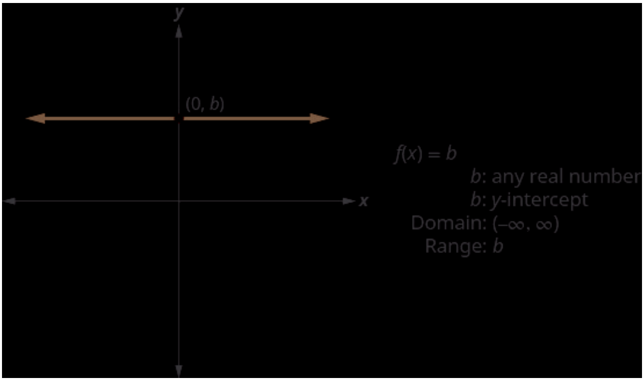
- **Graph of a Function**

    - The graph of a function is the graph of all its ordered pairs, (x,y) or using function notation, (x,f(x)) where y = f(x).
      fname of functionxx-coordinate of the ordered pairf(x)y-coordinate of the ordered pair

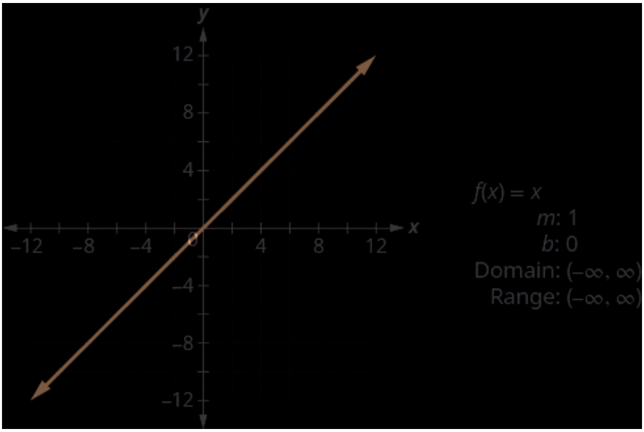
- **Linear Function**



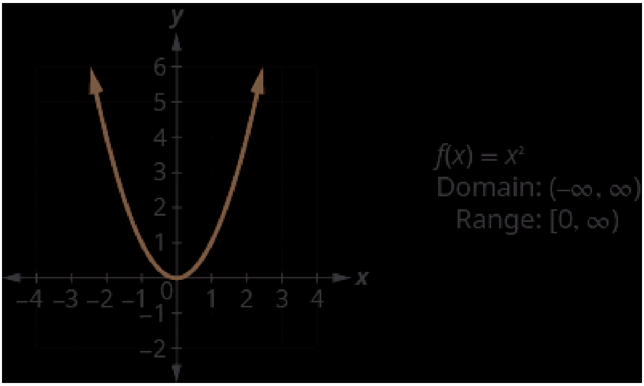
- **Constant Function**



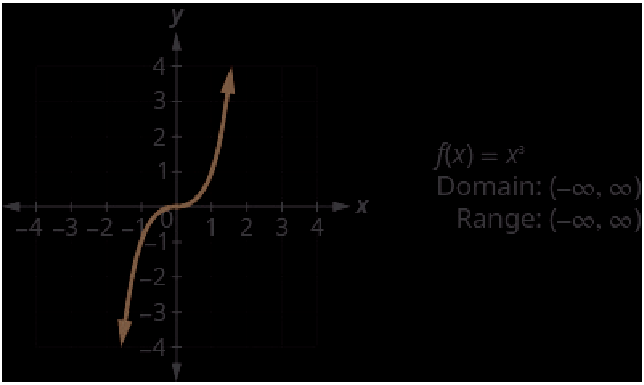
- **Identity Function**



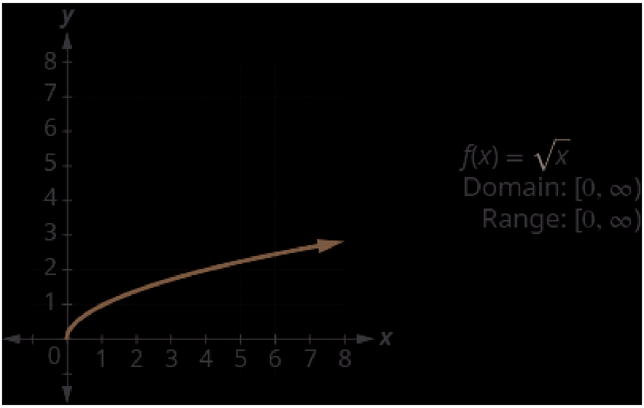
- **Square Function**



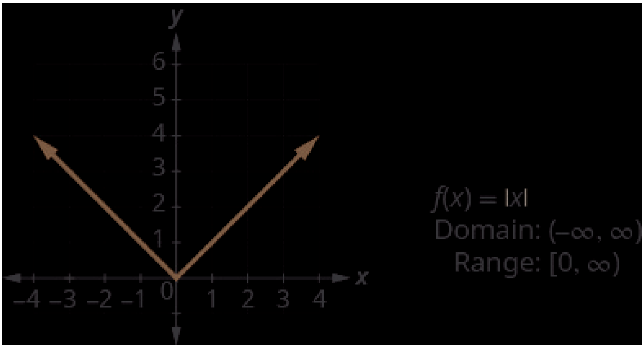
- **Cube Function**



- **Square Root Function**



- **Absolute Value Function**



Constant function	$f(x) = c$, where c is a constant
Identity function	$f(x) = x$
Absolute value function	$f(x) = \lvert x \rvert$
Quadratic function	$f(x) = x^2$

Cubic function	$f(x)=x^3$
Reciprocal function	$f(x)=\frac{1}{x}$
Reciprocal squared function	$f(x)=\frac{1}{x^2}$
Square root function	$f(x)=\sqrt{x}$
Cube root function	$f(x)=\sqrt[3]{x}$

Parent Functions

Transformations

Vertical shift	$g(x)=f(x)+k$ (up for $k>0$)
Horizontal shift	$g(x)=f(x-h)$ (right for $h>0$)
Vertical reflection	$g(x)=-f(x)$
Horizontal reflection	$g(x)=f(-x)$
Vertical stretch	$g(x)=af(x)$ ($a>0$)
Vertical compression	$g(x)=af(x)$ ($0<a<1$)
Horizontal stretch	$g(x)=f(bx)$ ($0<b<1$)
Horizontal compression	$g(x)=f(bx)$ ($b>1$)

Section Exercises

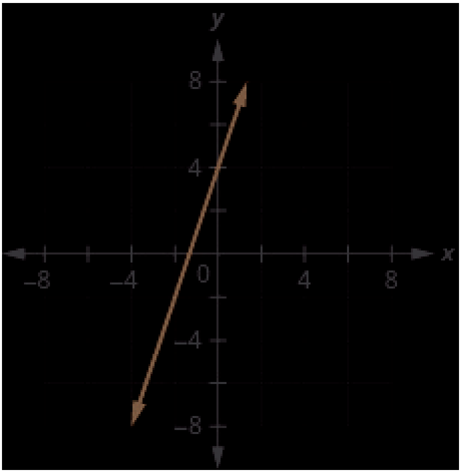
Practice Makes Perfect

Identify Graphs of Basic Functions

In the following exercises, ⓐ graph each function ⓑ state its domain and range. Write the domain and range in interval notation.

$f(x)=3x+4$

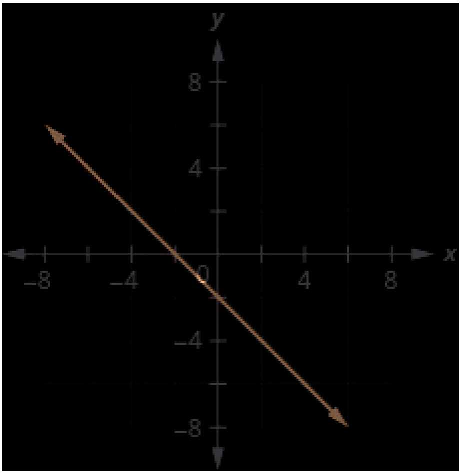
ⓐ



ⓑ D:(-∞,∞), R:(-∞,∞)

$f(x)=-x-2$

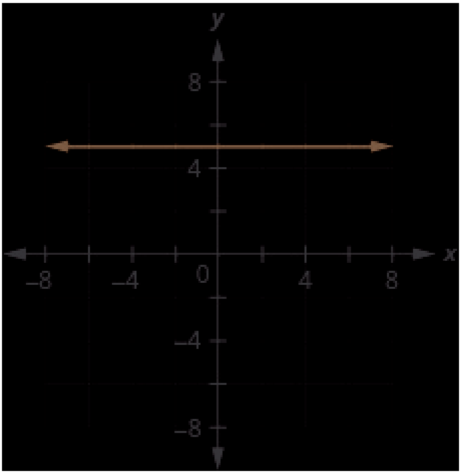
ⓐ



ⓑ D:(-∞, ∞), R:(-∞, ∞)

$f(x) = 5$

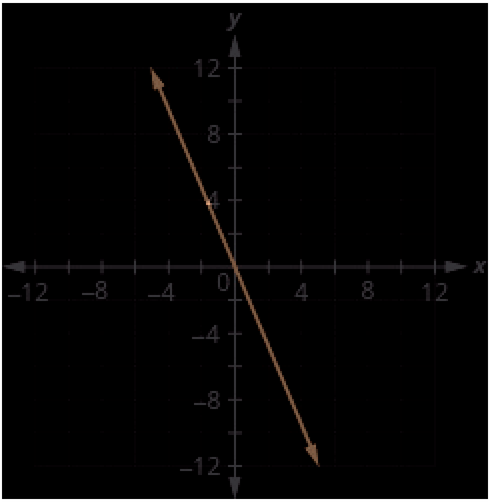
ⓐ



ⓑ D:(-∞,∞), R:{5}

$f(x) = -2x$

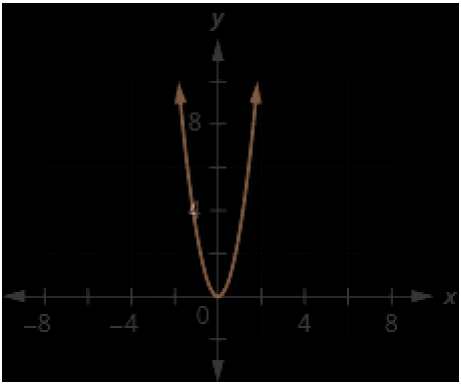
ⓐ



ⓑ D:(-∞,∞), R:(-∞,∞)

f(x) = 3x2

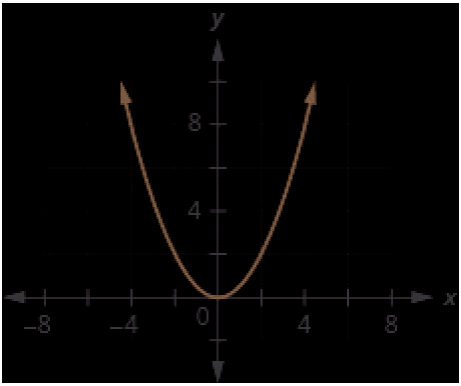
ⓐ



ⓑ D:(-∞,∞), R:[0,∞)

f(x)=12x2

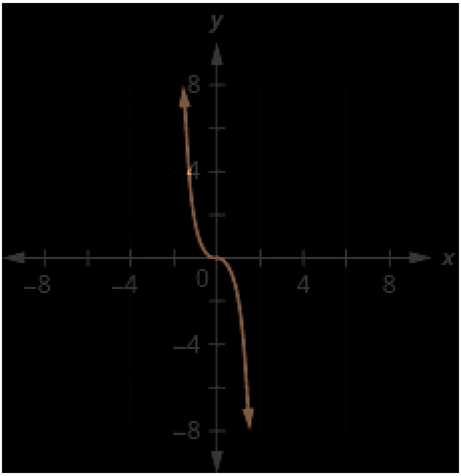
ⓐ



ⓑ (-∞,∞), R:[-∞,0)

f(x)= −2x3

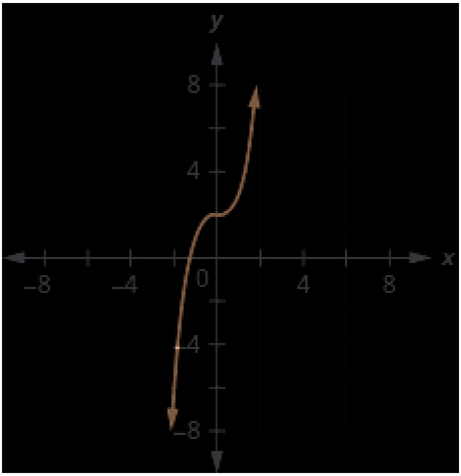
ⓐ



ⓑ D:(-∞,∞), R:(-∞,∞)

f(x)=x3+2

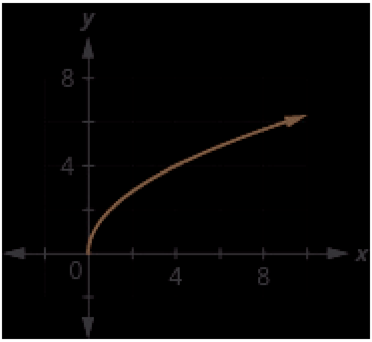
ⓐ



ⓑ D:(-∞,∞), R:(-∞,∞)

f(x) = 2x

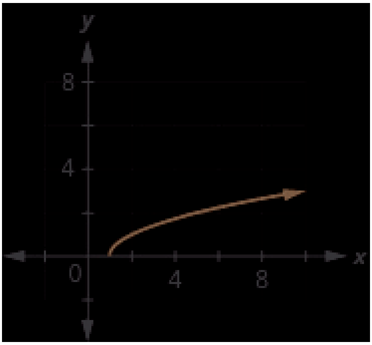
ⓐ



ⓑ $D:[0,\infty)$, $R:[0,\infty)$

$f(x)=x-1$

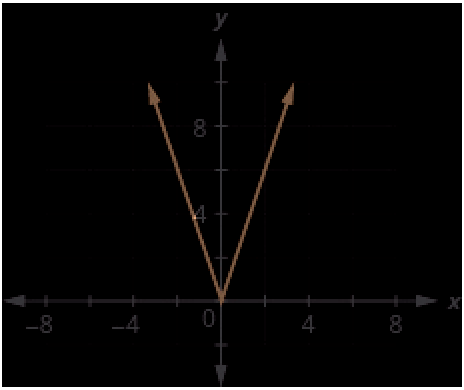
ⓐ



ⓑ $D:[1,\infty)$, $R:[0,\infty)$

$f(x)=3|x|$

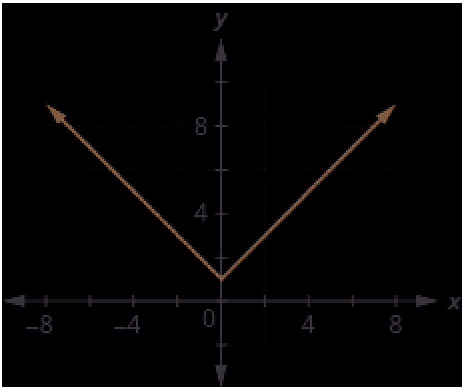
ⓐ



ⓑ D:[-1, ∞), R:[$-\infty$, ∞)

$f(x) = |x| + 1$

ⓐ



ⓑ D:$(-\infty,\infty)$, R:$[1,\infty)$

For the following exercises, write a formula for the function obtained when the graph is shifted as described.

$f(x)=|x|$ is shifted down 3 units and to the right 1 unit.

$g(x)=|x\text{-}1|-3$

$f(x)=\ 1\ x\ 2$ is shifted up 2 units and to the left 4 units.

$g(x)=\ 1\ (x+4)\ 2\ +2$

For the following exercises, describe how the graph of the function is a transformation of the graph of the original function f.

$y=f(x+43)$

The graph of $f(x+43)$ is a horizontal shift to the left 43 units of the graph of f.

$y=f(x)+8$

The graph of $f(x)+8$ is a vertical shift up 8 units of the graph of f.

$y=f(x+4)-1$

The graph of $f(x+4)-1$ is a horizontal shift to the left 4 units and a vertical shift down 1 unit of the graph of f.

For the following exercises, describe how the graph of each function is a transformation of the graph of the original function f.

$g(x)=-f(x)$

The graph of g is a vertical reflection (across the x -axis) of the graph of f.

$g(x)=4f(x)$

The graph of g is a vertical stretch by a factor of 4 of the graph of f.

$g(x)=f(5x)$

The graph of g is a horizontal compression by a factor of $\frac{1}{5}$ of the graph of f.

$g(x)=3f(-x)$

The graph of g is a horizontal reflection across the y-axis and a vertical stretch by a factor of 3 of the graph of f.

For the following exercises, write a formula for the function g that results when the graph of a given toolkit function is transformed as described.

The graph of $f(x)=|x|$ is reflected over the y-axis and horizontally compressed by a factor of $\frac{1}{4}$.

$g(x)=|-4x|$

The graph of $f(x)=\frac{1}{x^2}$ is vertically compressed by a factor of $\frac{1}{3}$, then shifted to the left 2 units and down 3 units.

$g(x)=\frac{1}{3(x+2)^2}-3$

Glossary

horizontal compression
: a transformation that compresses a function’s graph horizontally, by multiplying the input by a constant $b > 1$

horizontal reflection
: a transformation that reflects a function’s graph across the *y*-axis by multiplying the input by -1

horizontal shift
: a transformation that shifts a function’s graph left or right by adding a positive or negative constant to the input

horizontal stretch
: a transformation that stretches a function’s graph horizontally by multiplying the input by a constant $0 < b < 1$

vertical compression
: a function transformation that compresses the function’s graph vertically by multiplying the output by a constant $0 < a < 1$

vertical reflection
: a transformation that reflects a function’s graph across the *x*-axis by multiplying the output by -1

vertical shift

a transformation that shifts a function’s graph up or down by adding a positive or negative constant to the output

vertical stretch

a transformation that stretches a function’s graph vertically by multiplying the output by a constant $a > 1$

Composite Functions and Domain (2.6)
By the end of this section, you will be able to:

- Find and evaluate composite functions
- Combine functions using algebraic operations.
- Create a new function by composition of functions.
- Evaluate composite functions.
- Find the domain of a composite function.
- Decompose a composite function into its component functions.

This Module supports section **2.6 of Mat 1023**.
In case you missed something in class, or just want to review a specific topic covered in this Module, here is a list of topics covered:

Domain Review

In this section, we will practice determining domains and ranges for specific functions. We can write the domain and range in **interval notation**, which uses values within brackets to describe a set of numbers. In interval notation, we use a square bracket [when the set includes the endpoint and a parenthesis (to indicate that the endpoint is either not included or the interval is unbounded. For example, if a person has $100 to spend, he or she would need to express the interval that is more than 0 and less than or equal to 100 and write $(0, 100]$. We will discuss interval notation in greater detail later.

Let's turn our attention to finding the domain of a function whose equation is provided. Oftentimes, finding the domain of such functions involves remembering three different forms. First, if the function has no denominator or an even root, consider whether the domain could be all real numbers. Second, if there is a denominator in the function's equation, exclude values in the domain that force the denominator to be zero. Third, if there is an even root, consider excluding values that would make the radicand negative.

Before we begin, let us review the conventions of interval notation:

- The smallest number from the interval is

written first.

- The largest number in the interval is written second, following a comma.
- Parentheses, (or), are used to signify that an endpoint value is not included, called exclusive. They represent solutions greater or less than the number.
- Brackets, [or], are used to indicate that an endpoint value is included, called inclusive. They also represent solutions that are greater than or equal to or less than or equal to the number

Set Indicated	Set-Builder Notation	Interval Notation
All real numbers between *a* and *b*, but not including *a* or *b*	$\{ x \mid a < x < b \}$	(a,b)
All real numbers greater than *a*, but not including *a*	$\{ x \mid x > a \}$	(a, ∞)
All real numbers less than *b*, but not including *b*	$\{ x \mid x < b \}$	$(-\infty, b)$
All real numbers	$\{ x \mid x \geq a \}$	$[a, \infty)$

greater than *a*, including *a*		
All real numbers less than *b*, including *b*	$\{ x \mid x \leq b \}$	$(-\infty, b]$
All real numbers between *a* and *b*, including *a*	$\{ x \mid a \leq x < b \}$	$[a, b)$
All real numbers between *a* and *b*, including *b*	$\{ x \mid a < x \leq b \}$	$(a, b]$
All real numbers between *a* and *b*, including *a* and *b*	$\{ x \mid a \leq x \leq b \}$	$[a, b]$
All real numbers less than *a* or greater than *b*	$\{ x \mid x < a \text{ or } x > b \}$	$(-\infty, a) \cup (b, \infty)$
All real numbers	$\{ x \mid x \text{ is a real number} \}$	$(-\infty, \infty)$

This format **set-builder notation**: $\{ x \mid 10 \leq x < 30 \}$ describes the behavior of x in set-builder notation. The braces $\{\}$ are read as “the set of,” and the vertical bar $\mid$ is read as “such that,” so we would read $\{ x \mid 10 \leq x < 30 \}$ as “the set of *x*-values such that 10 is less than or equal to x, and x is less than 30.”

Finding the Domain of a Function as a Set

of Ordered Pairs

Find the domain of the following function: $\{(2, 10),(3, 10),(4, 20),(5, 30),(6, 40)\}$.

First identify the input values. The input value is the first coordinate in an ordered pair. There are no restrictions, as the ordered pairs are simply listed. The domain is the set of the first coordinates of the ordered pairs.
$\{2,3,4,5,6\}$

How To
Given a function written in equation form, find the domain.

1. Identify the input values.
2. Identify any restrictions on the input and exclude those values from the domain.
3. Write the domain in interval form, if possible.

Finding the Domain of a Function

Find the domain of the function $f(x)=x^2-1$.

The input value, shown by the variable x in the equation, is squared and then the result is lowered by one. Any real number may be squared and then be lowered by one, so there are no restrictions on the domain of this function. The domain is the set of real numbers.

In interval form, the domain of f is $(-\infty, \infty)$.

Given a function written in an equation form that includes a fraction, find the domain.

1. Identify the input values.
2. Identify any restrictions on the input. If there is a denominator in the function's formula, set the denominator equal to zero and solve for x. If the function's formula contains an even root, set the radicand greater than or equal to 0, and then solve.
3. Write the domain in interval form, making sure to exclude any restricted values from the domain.

Finding the Domain of a Function Involving

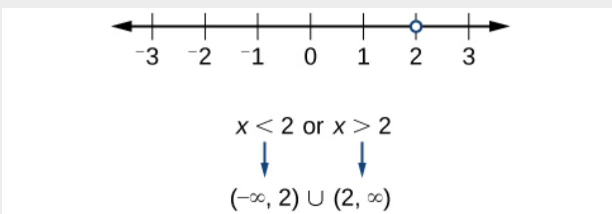
a Denominator

Find the domain of the function $f(x)=\frac{x+1}{2-x}$.

When there is a denominator, we want to include only values of the input that do not force the denominator to be zero. So, we will set the denominator equal to 0 and solve for x.

$$\begin{aligned} 2-x &= 0 \\ -x &= -2 \\ x &= 2 \end{aligned}$$

Now, we will exclude 2 from the domain. The answers are all real numbers where $x<2$ or $x>2$ as shown in [link]. We can use a symbol known as the union, $\cup$, to combine the two sets. In interval notation, we write the solution: $(-\infty,2)\cup(2,\infty)$.



Given a function written in equation form including an even root, find the domain.

1. Identify the input values.

2. Since there is an even root, exclude any real numbers that result in a negative number in the radicand. Set the radicand greater than or equal to zero and solve for x.
3. The solution(s) are the domain of the function. If possible, write the answer in interval form.

Finding the Domain of a Function with an Even Root

Find the domain of the function $f(x) = \sqrt{7-x}$.

When there is an even root in the formula, we exclude any real numbers that result in a negative number in the radicand.

Set the radicand greater than or equal to zero and solve for x.

$7-x \geq 0$ $-x \geq -7$ $x \leq 7$

Now, we will exclude any number greater than 7 from the domain. The answers are all real numbers less than or equal to 7, or $(-\infty,7]$.

Find the domain of the function $f(x) = 5+2x$.
$[-5\ 2, \infty)$

Combining Functions Using Algebraic Operations

Function composition is only one way to combine existing functions. Another way is to carry out the usual algebraic operations on functions, such as addition, subtraction, multiplication and division. We do this by performing the operations with the function outputs, defining the result as the output of our new function.

Suppose we need to add two columns of numbers that represent a husband and wife’s separate annual incomes over a period of years, with the result being their total household income. We want to do this for every year, adding only that year’s incomes and then collecting all the data in a new column. If $w(y)$ is the wife’s income and $h(y)$ is the husband’s income in year y, and we want T to represent the total income, then we can define a new function.

$$T(y) = h(y) + w(y)$$

If this holds true for every year, then we can focus on the relation between the functions without reference to a year and write
$T = h + w$

Just as for this sum of two functions, we can define difference, product, and ratio functions for any pair of functions that have the same kinds of inputs (not necessarily numbers) and also the same kinds of outputs (which do have to be numbers so that the usual operations of algebra can apply to them, and which also must have the same units or no units when we add and subtract). In this way, we can think of adding, subtracting, multiplying, and dividing functions.

For two functions $f(x)$ and $g(x)$ with real number outputs, we define new functions $f+g$, $f-g$, fg, and $\frac{f}{g}$ by the relations
Sum $(f+g)(x) = f(x)+g(x)$ Difference $(f-g)(x) = f(x)-g(x)$ Product $(fg)(x) = f(x)g(x)$ Quotient $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ where $g(x) \neq 0$

Performing Algebraic Operations on Functions

Find and simplify the functions $(g-f)(x)$ and $\left(\frac{g}{f}\right)(x)$, given $f(x) = x-1$ and $g(x) = x^2 - 1$. Are they the same function?

Begin by writing the general form, and then substitute the given functions.
$(g-f)(x) = g(x)-f(x)$ $(g-f)(x) = x\ 2\ -1-(x-1)$ $(g-f)(x) = x\ 2\ -x$ $(g-f)(x) = x(x-1)$ $(\ g\ f\ )(x) = g(x)\ f(x)$ $(\ g\ f\ )(x) = x\ 2\ -1\ x-1$ $(\ g\ f\ )(x) = (x+1)(x-1)\ x-1$ where $x \neq 1$ $(\ g\ f\ )(x) = x+1$

No, the functions are not the same.

Note: For $(\ g\ f\ )(\ x\ ),$ the condition $x \neq 1$ is necessary because when $x=1,$ the denominator is equal to 0, which makes the function undefined.

Find and simplify the functions $(\ fg\ )(\ x\ )$ and $(\ f-g\ )(\ x\ ).$
$f(\ x\ )=x-1$ and $g(\ x\ )=\ x\ 2\ -1$

Are they the same function?

$(\ fg\ )(\ x\ )=f(\ x\ )g(\ x\ )=(\ x-1\ )(\ x\ 2\ -1\ )=\ x\ 3\ -\ x\ 2\ -x+1$ $(\ f-g\ )(\ x\ )=f(\ x\ )-g(\ x\ )=(\ x-1\ )-(\ x\ 2\ -1\ )=x-\ x\ 2$

No, the functions are not the same.

Find and Evaluate Composite Functions

Before we introduce the functions, we need to look at another operation on functions called composition. In composition, the output of one function is the input of a second function. For functions f and g, the composition is written $f \circ g$ and is defined by:

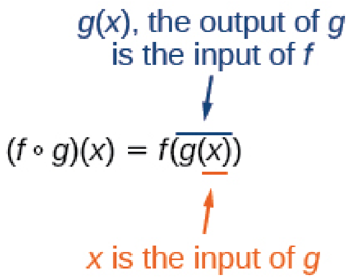
$$(f \circ g)(x) = f(g(x))$$

We read the left-hand side as "f composed with g at x," and the right-hand side as "f of g of x." The two sides of the equation have the same mathematical meaning and are equal. The open circle symbol $\circ$ is called the composition operator. We use this operator mainly when we wish to emphasize the relationship between the functions themselves without referring to any particular input value. Composition is a binary operation that takes two functions and forms a new function, much as addition or multiplication takes two numbers and gives a new number. However, it is important not to confuse function composition with multiplication because, as we learned above, in most cases $f(g(x)) \neq f(x)g(x)$.

To do a composition, the output of the first function, g(x), becomes the input of the second function, *f*, and so we must be sure that it is part of the domain

of *f*.

It is also important to understand the order of operations in evaluating a composite function. We follow the usual convention with parentheses by starting with the innermost parentheses first, and then working to the outside. In the equation above, the function g takes the input x first and yields an output $g(x)$. Then the function f takes $g(x)$ as an input and yields an output $f(g(x))$.



In general, $f \circ g$ and $g \circ f$ are different functions. In other words, in many cases $f(g(x)) \neq g(f(x))$ for all x. We will also see that sometimes two functions can be composed only in one specific order.

For example, if $f(x) = x^2$ and $g(x) = x + 2$, then
$f(g(x)) = f(x+2) = (x+2)^2 = x^2 + 4x + 4$

but
$g(f(x)) = g(x^2) = x^2 + 2$

These expressions are not equal for all values of x, so the two functions are not equal. It is irrelevant

that the expressions happen to be equal for the single input value $x=-\frac{1}{2}$.

Note that the range of the inside function (the first function to be evaluated) needs to be within the domain of the outside function. Less formally, the composition has to make sense in terms of inputs and outputs.

Composition of Functions

When the output of one function is used as the input of another, we call the entire operation a composition of functions. For any input x and functions f and g, this action defines a **composite function**, which we write as $f\circ g$ such that

$$(f\circ g)(x)=f(g(x))$$

The domain of the composite function $f\circ g$ is all x such that x is in the domain of g and $g(x)$ is in the domain of f.

It is important to realize that the product of functions fg is not the same as the function composition $f(g(x))$, because, in general, $f(x)g(x)\neq f(g(x))$.

The next example will demonstrate that $(f\circ g)(x)$, $(g\circ f)(x)$ and $(f\cdot g)(x)$ usually result in different outputs.

For functions $f(x) = 4x - 5$ and $g(x) = 2x + 3$, find: ⓐ $(f \circ g)(x)$, ⓑ $(g \circ f)(x)$, and ⓒ $(f \cdot g)(x)$.

ⓐ

Use the definition of $(f \circ g)(x)$.

[redacted]

[redacted]

[redacted]

Distribute.

[redacted]

Simplify.

[redacted]

ⓑ

Use the definition of $(f \circ g)(x)$.

[redacted]

[redacted]

[redacted]

Distribute.

[redacted]

Simplify.

[redacted]

Notice the difference in the result in part ⓐ and part ⓑ.

ⓒ Notice that $(f \cdot g)(x)$ is different than $(f \circ g)(x)$. In part ⓐ we did the composition of the functions. Now in part ⓒ we are not composing them, we are multiplying them.

Use the definition of$(f \cdot g)(x)$.$(f \cdot g)(x) = f(x) \cdot g(x)$Substitute$f(x) = 4x - 5andg(x) = 2x + 3$.$(f \cdot g)(x) = (4x - 5) \cdot (2x + 3)$Multiply.$(f \cdot g)(x) = 8x2 + 2x - 15$

For functions $f(x)=3x-2$ and $g(x)=5x+1$, find ⓐ $(f\circ g)(x)$ ⓑ $(g\circ f)(x)$ ⓒ $(f\cdot g)(x)$.

ⓐ $15x+1$ ⓑ $15x-9$
ⓒ $15x2-7x-2$

In the next example we will evaluate a composition for a specific value.

For functions $f(x)=x2-4$, and $g(x)=3x+2$, find: ⓐ $(f\circ g)(-3)$, ⓑ $(g\circ f)(-1)$, and ⓒ $(f\circ f)(2)$.

ⓐ

Use the definition of $(f\circ g)(-3)$.		
[redacted]		

[redacted]

Simplify.

[redacted]

[redacted]

Simplify.

[redacted]

ⓑ

Use the definition of $(g \circ f)(-1)$.

[redacted]

[redacted]

Simplify.

[redacted]

[redacted]

Simplify.

[redacted]

ⓒ

Use the definition of
$(f \circ f)(2)$.

[redacted]

[redacted]

Simplify.

[redacted]

[redacted]

Simplify.

[redacted]

For functions $f(x) = x2 - 9$, and $g(x) = 2x + 5$, find ⓐ $(f \circ g)(-2)$, ⓑ $(g \circ f)(-3)$, and ⓒ $(f \circ f)(4)$.

ⓐ –8 ⓑ 5 ⓒ 40

Finding the Domain of a Composite Function

As we discussed previously, the domain of a composite function such as $f \circ g$ is dependent on the domain of g and the domain of f. It is important to know when we can apply a composite function and when we cannot, that is, to know the domain of a function such as $f \circ g$. Let us assume we know the domains of the functions f and g separately. If we write the composite function for an input x as $f(g(x)),$ we can see right away that x must be a member of the domain of g in order for the expression to be meaningful, because otherwise we cannot complete the inner function evaluation. However, we also see that $g(x)$ must be a member of the domain of f, otherwise the second function evaluation in $f(g(x))$ cannot be completed, and the expression is still undefined. Thus the domain of

$f \circ g$ consists of only those inputs in the domain of g that produce outputs from g belonging to the domain of f. Note that the domain of f composed with g is the set of all x such that x is in the domain of g and $g(x)$ is in the domain of f.

Domain of a Composite Function

The domain of a composite function $f(g(x))$ is the set of those inputs x in the domain of g for which $g(x)$ is in the domain of f.

HOW TO

Given a function composition $f(g(x))$, determine its domain.

1. Find the domain of g.
2. Find the domain of f.
3. Find those inputs x in the domain of g for which $g(x)$ is in the domain of f. That is, exclude those inputs x from the domain of g for which $g(x)$ is not in the domain of f. The resulting set is the domain of $f \circ g$.

Finding the Domain of a Composite Function

Find the domain of
$(f \circ g)(x)$ where $f(x) = 5x - 1$ and $g(x) = 4$

$3x-2$

The domain of $g(x)$ consists of all real numbers except $x=\frac{2}{3}$, since that input value would cause us to divide by 0. Likewise, the domain of f consists of all real numbers except 1. So we need to exclude from the domain of $g(x)$ that value of x for which $g(x)=1$.

$$\frac{4}{3x-2}=1 \quad 4=3x-2 \quad 6=3x \quad x=2$$

So the domain of $f\circ g$ is the set of all real numbers except $\frac{2}{3}$ and 2. This means that $x\neq\frac{2}{3}$ or $x\neq 2$

We can write this in interval notation as $\left(-\infty,\frac{2}{3}\right)\cup\left(\frac{2}{3},2\right)\cup(2,\infty)$

Finding the Domain of a Composite Function Involving Radicals

Find the domain of $(f\circ g)(x)$ where $f(x)= x+2$ and $g(x)= 3-x$

Because we cannot take the square root of a negative number, the domain of g is $(-\infty,3]$. Now we check the domain of the composite

function
$(f \circ g)(x) = 3 - x + 2$

For $(f \circ g)(x) = 3 - x + 2$, $3 - x + 2 \geq 0$, since the radicand of a square root must be positive. Since square roots are positive, $3 - x \geq 0$, or, $3 - x \geq 0$, which gives a domain of $(-\infty, 3]$.

Analysis
This example shows that knowledge of the range of functions (specifically the inner function) can also be helpful in finding the domain of a composite function. It also shows that the domain of $f \circ g$ can contain values that are not in the domain of f, though they must be in the domain of g.

Find the domain of
$(f \circ g)(x)$ where $f(x) = 1 \; x - 2$ and $g(x) = x + 4$

$[-4, 0) \cup (0, \infty)$

Decomposing a Composite Function into its Component Functions

In some cases, it is necessary to decompose a complicated function. In other words, we can write it as a composition of two simpler functions. There may be more than one way to decompose a composite function, so we may choose the decomposition that appears to be most expedient.

Decomposing a Function

Write $f(x)=\sqrt{5-x^2}$ as the composition of two functions.

We are looking for two functions, g and h, so $f(x)=g(h(x))$. To do this, we look for a function inside a function in the formula for $f(x)$. As one possibility, we might notice that the expression $5-x^2$ is the inside of the square root. We could then decompose the function as

$$h(x)=5-x^2 \text{ and } g(x)=\sqrt{x}$$

We can check our answer by recomposing the functions.

$$g(h(x))=g(5-x^2)=\sqrt{5-x^2}$$

Write $f(x) = 4\ 3 - \ 4 + \ x\ 2$ as the composition of two functions.

Possible answer:

$g(x) = 4 + x\ 2$ $h(x) = 4\ 3 - x$ $f = h \circ g$

Access these online resources for additional instruction and practice with composite functions.

- Composite Functions
- Composite Function Notation Application
- Composite Functions Using Graphs
- Decompose Functions
- Composite Function Values

Key Concepts

- The domain of a function includes all real input values that would not cause us to attempt an undefined mathematical operation, such as dividing by zero or taking the square root of a

negative number.

- The domain of a function can be determined by listing the input values of a set of ordered pairs. See [link].
- The domain of a function can also be determined by identifying the input values of a function written as an equation. See [link], [link], and [link].
- Interval values represented on a number line can be described using inequality notation, set-builder notation, and interval notation. See [link].
- For many functions, the domain and range can be determined from a graph. See [link] and [link].
- An understanding of toolkit functions can be used to find the domain and range of related functions. See [link], [link], and [link].
- **Composition of Functions:** The composition of functions f and g, is written $f \circ g$ and is defined by
  $(f \circ g)(x) = f(g(x))$
  We read $f(g(x))$ as f of g of x.
- We can perform algebraic operations on functions. See [link].
- When functions are combined, the output of the first (inner) function becomes the input of the second (outer) function.
- The function produced by combining two functions is a composite function. See [link] and [link].

- The order of function composition must be considered when interpreting the meaning of composite functions. See [link].
- A composite function can be evaluated by evaluating the inner function using the given input value and then evaluating the outer function taking as its input the output of the inner function.
- The domain of a composite function consists of those inputs in the domain of the inner function that correspond to outputs of the inner function that are in the domain of the outer function. See [link] and [link].
- Just as functions can be combined to form a composite function, composite functions can be decomposed into simpler functions.
- Functions can often be decomposed in more than one way. See [link].

Practice Makes Perfect

For the following exercises, find the domain of each function using interval notation.

$f(x)=5-2x^2$

$(-\infty,\infty)$

$f(x)=3- 6-2x$

$(-\infty,3]$

$f(x)= 3x+1 4x+2$

$(-\infty,- 1 2)\cup(- 1 2 ,\infty)$

$f(x)= x-3 x 2 +9x-22$

$(-\infty,-11)\cup(-11,2)\cup(2,\infty)$

$2x+1 5-x$

$(-\infty,5)$

$f(x)= x-6 x-4$

$[6,\infty)$

$f(x)= x 2 -9x x 2 -81$

$(-\infty, -9) \cup (-9, 9) \cup (9, \infty)$

For $f(x) = 1\ x$ and $g(x) = x - 1$, write the domain of $(f \circ g)(x)$ in interval notation.

$(1, \infty)$

Given $f(x) = x\ 2 + 2x$ and $g(x) = 6 - x\ 2$, find $f + g, f - g, fg,$ and $f\ g$.

$(f+g)(x) = 2x + 6$, domain: $(-\infty, \infty)$

$(f-g)(x) = 2\ x\ 2 + 2x - 6$, domain: $(-\infty, \infty)$

$(fg)(x) = -\ x\ 4 - 2\ x\ 3 + 6\ x\ 2 + 12x$, domain: $(-\infty, \infty)$

$(f\ g)(x) = x\ 2 + 2x\ 6 - x\ 2$, domain: $(-\infty, -\ 6) \cup (-\ 6, 6) \cup (6, \infty)$

Given $f(x) = 2\ x\ 2 + 4x$ and $g(x) = 1\ 2x$, find $f + g, f - g, fg,$ and $f\ g$.

$(f+g)(x) = 4\ x\ 3 + 8\ x\ 2 + 1\ 2x$, domain: $(-\infty, 0) \cup (0, \infty)$

$(f-g)(x)= 4x3 +8x2 -1 2x$, domain: $(-\infty,0)\cup(0,\infty)$

$(fg)(x)=x+2$, domain: $(-\infty,0)\cup(0,\infty)$

$(f g)(x)=4x3 +8x2$, domain: $(-\infty,0)\cup(0,\infty)$

Given $f(x)=3x2$ and $g(x)= x-5$, find $f+g$, $f-g$, fg, and $f g$.

$(f+g)(x)=3x2 + x-5$, domain: $[5,\infty)$

$(f-g)(x)=3x2 - x-5$, domain: $[5,\infty)$

$(fg)(x)=3x2x-5$, domain: $[5,\infty)$

$(f g)(x)= 3x2x-5$, domain: $(5,\infty)$

Find and Evaluate Composite Functions

In the following exercises, find ⓐ $(f \circ g)(x)$, ⓑ $(g \circ f)(x)$, and ⓒ $(f \cdot g)(x)$.

$f(x)=4x+3$ and $g(x)=2x+5$

ⓐ $8x+23$ ⓑ $8x+11$ ⓒ $8x2+26x+15$

$f(x) = 6x - 5$ and $g(x) = 4x + 1$

ⓐ $24x + 1$ ⓑ $24x - 19$
ⓒ $24x2 + 14x - 5$

$f(x) = 3x$ and $g(x) = 2x2 - 3x$

ⓐ $6x2 - 9x$ ⓑ $18x2 - 9x$
ⓒ $6x3 - 9x2$

For the following exercises, use each pair of functions to find $f(\ g(\ x\ )\ )$ and $g(\ f(\ x\ )\ )$. Simplify your answers.

$f(x) = \ x\ + 2, g(x) = \ x\ 2\ + 3$

$f(g(x)) = \ x\ 2\ + 3\ + 2, g(f(x)) = x + 4\ x\ + 7$

For the following exercises, find functions $f(x)$ and $g(x)$ so the given function can be expressed as $h(x) = f(\ g(x)\ )$.

$h(x) = \ (x - 5)\ 3$

sample: $f(x) = \ x\ 3\ g(x) = x - 5$

$h(x) = 4 (x+2) 2$

---

sample: $f(x) = 4 x$ $g(x) = (x+2) 2$

$h(x) = 2x+6$

---

sample: $f(x) = x$ $g(x) = 2x+6$

Glossary

composite function
: the new function formed by function composition, when the output of one function is used as the input of another

Inverse Functions (2.7)
By the end of this section, you will be able to:

- Determine whether a function is one-to-one
- Find the inverse of a function

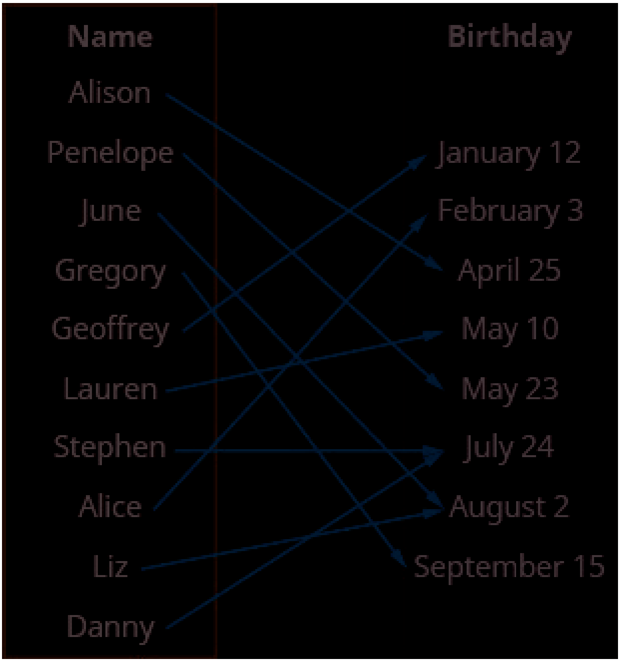
This Module supports section **2.7 of Mat 1023**.
In case you missed something in class, or just want to review a specific topic covered in this Module, here is a list of topics covered:

1. One-to-One Functions [link]
2. Define Inverse of a Function [link]
3. Find Inverse of a Function [link]
4. Key Concepts [link]

Determine Whether a Function is One-to-One

When we first introduced functions, we said a function is a relation that assigns to each element in its domain exactly one element in the range. For each ordered pair in the relation, each x-value is matched with only one y-value.

We used the birthday example to help us understand the definition. Every person has a birthday, but no one has two birthdays and it is okay for two people to share a birthday. Since each person has exactly one birthday, that relation is a function.



A function is **one-to-one** if each value in the range has exactly one element in the domain. For each ordered pair in the function, each *y*-value is matched with only one *x*-value.

Our example of the birthday relation is not a one-to-one function. Two people can share the same birthday. The range value August 2 is the birthday of Liz and June, and so one range value has two domain values. Therefore, the function is not one-to-

one.

One-to-One Function

A function is **one-to-one** if each value in the range corresponds to one element in the domain. For each ordered pair in the function, each *y*-value is matched with only one *x*-value. There are no repeated *y*-values.

For each set of ordered pairs, determine if it represents a function and, if so, if the function is one-to-one.

ⓐ $\{(-3,27),(-2,8),(-1,1),(0,0),(1,1),(2,8),(3,27)\}$ and ⓑ $\{(0,0),(1,1),(4,2),(9,3),(16,4)\}$.

ⓐ

$\{(-3,27),(-2,8),(-1,1),(0,0),(1,1),(2,8),(3,27)\}$

Each *x*-value is matched with only one *y*-value. So this relation is a function.

But each *y*-value is not paired with only one *x*-value, $(-3,27)$ and $(3,27)$, for example. So

this function is not one-to-one.

ⓑ
$\{(0,0),(1,1),(4,2),(9,3),(16,4)\}$

Each x-value is matched with only one y-value. So this relation is a function.

Since each y-value is paired with only one x-value, this function is one-to-one.

For each set of ordered pairs, determine if it represents a function and if so, is the function one-to-one.

ⓐ $\{(-3,-6),(-2,-4),(-1,-2),(0,0),(1,2),(2,4),(3,6)\}$ ⓑ $\{(-4,8),(-2,4),(-1,2),(0,0),(1,2),(2,4),(4,8)\}$

ⓐ One-to-one function
ⓑ Function; not one-to-one

To help us determine whether a relation is a function, we use the vertical line test. A set of points

in a rectangular coordinate system is the graph of a function if every vertical line intersects the graph in at most one point. Also, if any vertical line intersects the graph in more than one point, the graph does not represent a function.

The vertical line is representing an x-value and we check that it intersects the graph in only one y-value. Then it is a function.

To check if a function is one-to-one, we use a similar process. We use a horizontal line and check that each horizontal line intersects the graph in only one point. The horizontal line is representing a y-value and we check that it intersects the graph in only one x-value. If every horizontal line intersects the graph of a function in at most one point, it is a one-to-one function. This is the **horizontal line test**.

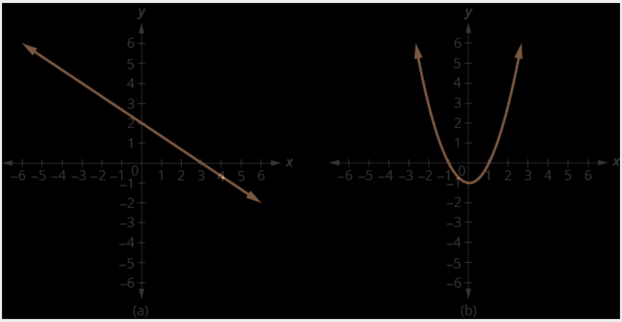
Horizontal Line Test

If every horizontal line intersects the graph of a function in at most one point, it is a one-to-one function.

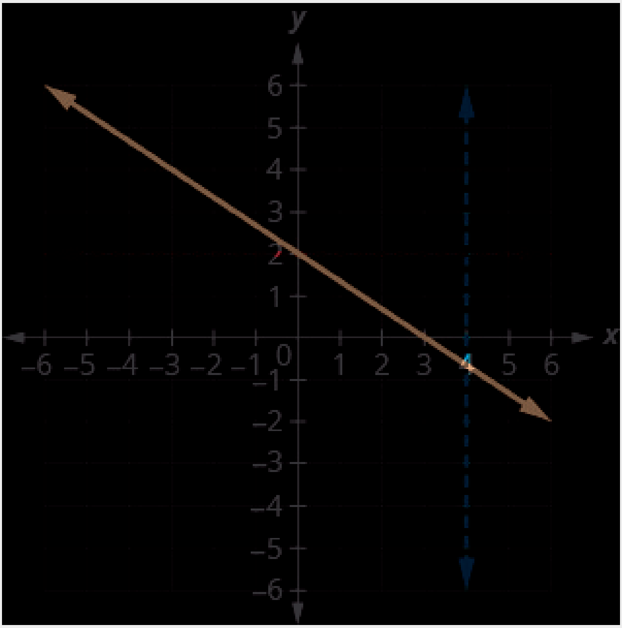
We can test whether a graph of a relation is a function by using the vertical line test. We can then tell if the function is one-to-one by applying the

horizontal line test.

Determine ⓐ whether each graph is the graph of a function and, if so, ⓑ whether it is one-to-one.

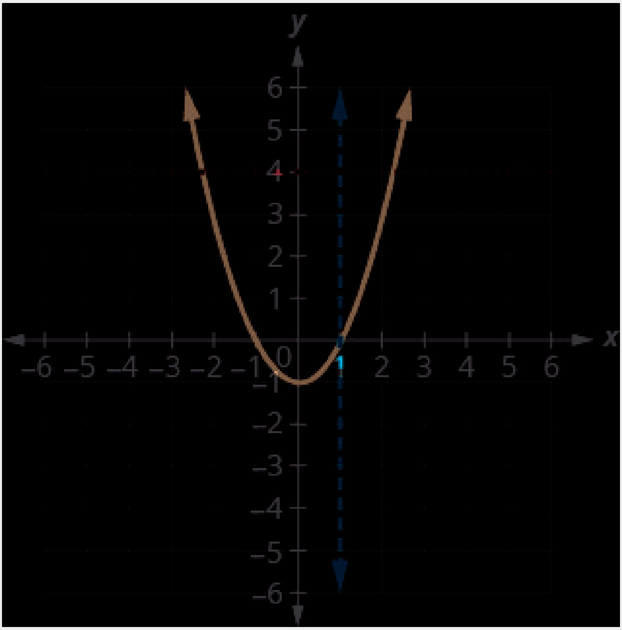


ⓐ



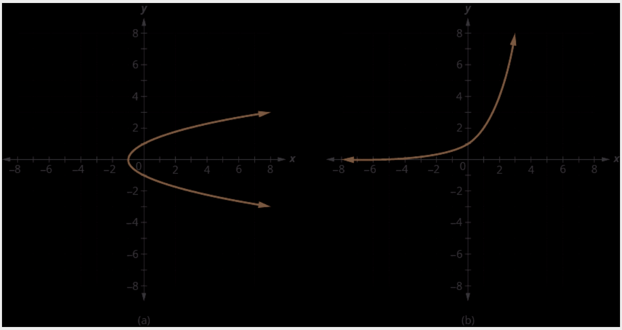
Since any vertical line intersects the graph in at most one point, the graph is the graph of a function. Since any horizontal line intersects the graph in at most one point, the graph is the graph of a one-to-one function.

ⓑ



Since any vertical line intersects the graph in at most one point, the graph is the graph of a function. The horizontal line shown on the graph intersects it in two points. This graph does not represent a one-to-one function.

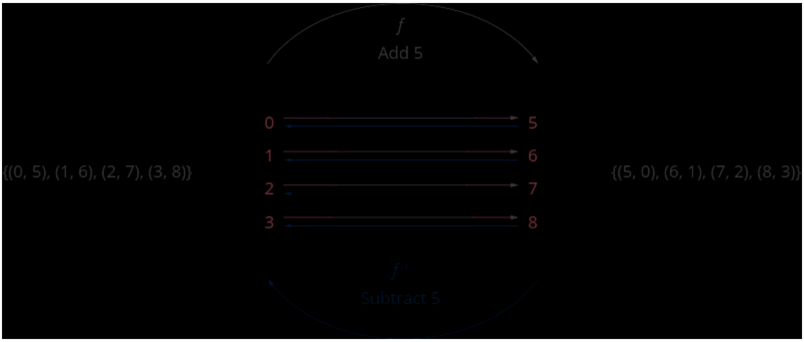
Determine ⓐ whether each graph is the graph of a function and, if so, ⓑ whether it is one-to-one.



ⓐ Not a function ⓑ One-to-one function

Defining the Inverse of a Function

Let's look at a one-to one function, f, represented by the ordered pairs $\{(0,5),(1,6),(2,7),(3,8)\}$. For each x-value, f adds 5 to get the y-value. To 'undo' the addition of 5, we subtract 5 from each y-value and get back to the original x-value. We can call this "taking the inverse of f" and name the function $f-1$.



Notice that that the ordered pairs of f and $f-1$ have their x-values and y-values reversed. The domain of f is the range of $f-1$ and the domain of $f-1$ is the range of f.

Inverse of a Function Defined by Ordered Pairs

If $f(x)$ is a one-to-one function whose ordered pairs are of the form (x,y), then its inverse function $f-1(x)$ is the set of ordered pairs (y,x).

The range of a function $f(x)$ is the domain of the inverse function $f-1(x)$.

The domain of $f(x)$ is the range of $f-1(x)$.

In the next example we will find the inverse of a function defined by ordered pairs.

Find the inverse of the function {(0,3),(1,5),(2,7),(3,9)}. Determine the domain and range of the inverse function.

This function is one-to-one since every x-value is paired with exactly one y-value.

To find the inverse we reverse the x-values and y-values in the ordered pairs of the function. Function{(0,3),(1,5),(2,7),(3,9)}Inverse Function{(3,0),(5,1),(7,2),(9,3)}Domain of Inverse Function{3,5,7,9}Range of Inverse Function{0,1,2,3}

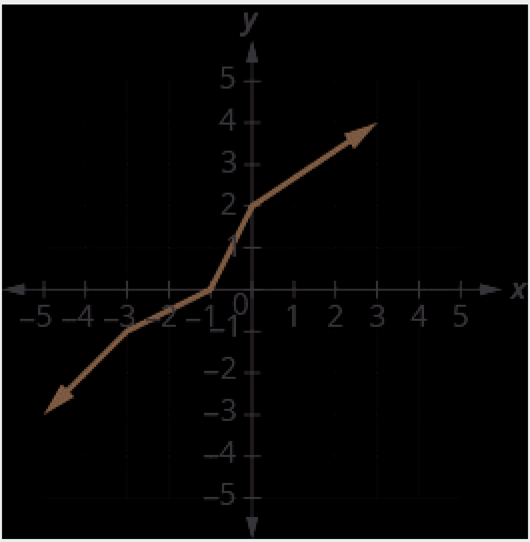
Find the inverse of {(0,4),(1,7),(2,10),(3,13)}. Determine the domain and range of the inverse function.

Inverse function: {(4,0),(7,1),(10,2),(13,3)}. Domain: {4,7,10,13}. Range: {0,1,2,3}.

Since every point on the graph of a function f(x) is a mirror image of a point on the graph of $f-1(x)$, we

say the graphs are mirror images of each other through the line $y = x$. We will use this concept to graph the inverse of a function in the next example.

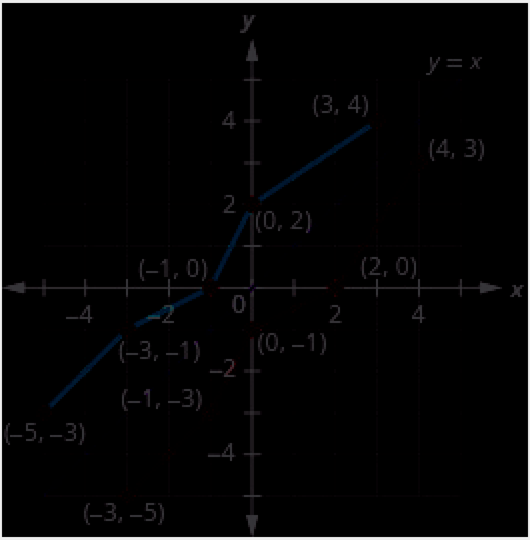
Graph, on the same coordinate system, the inverse of the one-to one function shown.



We can use points on the graph to find points on the inverse graph. Some points on the graph are: $(-5,-3),(-3,-1),(-1,0),(0,2),(3,4)$.

So, the inverse function will contain the

points: $(-3,-5),(-1,-3),(0,-1),(2,0),(4,3)$.



Notice how the graph of the original function and the graph of the inverse functions are mirror images through the line $y=x$.

Given a function $f(x)$, we represent its inverse as $f^{-1}(x)$, read as "f inverse of x." The raised -1 is part of the notation. It is not an exponent; it does not imply a power of -1. In other words, $f^{-1}(x)$ does *not* mean $\frac{1}{f(x)}$ because $\frac{1}{f(x)}$ is the reciprocal of f and not the inverse.

The "exponent-like" notation comes from an analogy between function composition and

multiplication: just as $a^{-1}a=1$ (1 is the identity element for multiplication) for any nonzero number a, so $f^{-1}\circ f$ equals the identity function, that is,

$$(f^{-1}\circ f)(x)=f^{-1}(f(x))=f^{-1}(y)=x$$

Inverse Functions

For any one-to-one function $f(x)=y$, a function $f^{-1}(x)$ is an **inverse function** of f if $f^{-1}(y)=x$. This can also be written as $f^{-1}(f(x))=x$ for all x in the domain of f. It also follows that $f(f^{-1}(x))=x$ for all x in the domain of f^{-1} if f^{-1} is the inverse of f.

The notation f^{-1} is read " f inverse." Like any other function, we can use any variable name as the input for f^{-1}, so we will often write $f^{-1}(x)$, which we read as "f inverse of x." Keep in mind that

$$f^{-1}(x)\neq\frac{1}{f(x)}$$

and not all functions have inverses.

Inverse function, which is a function for which the input of the original function becomes the output of the inverse function and the output of the original function becomes the input of the inverse function.

$$f^{-1}(f(x))=x, \text{for all } x \text{ in the domain of } f$$
$$f(f^{-1}(x))=x, \text{for all } x \text{ in the domain of } f^{-1}$$

When we began our discussion of an inverse function, we talked about how the inverse function 'undoes' what the original function did to a value in its domain in order to get back to the original *x*-value. This holds for all x in the domain of f. Informally, this means that inverse functions "undo" each other. However, just as zero does not have a reciprocal, some functions do not have inverses.

> For example, $y=4x$ and $y=\frac{1}{4}x$ are inverse functions.
> $(f^{-1}\circ f)(x)=f^{-1}(4x)=\frac{1}{4}(4x)=x$
> and
> $(f\circ f^{-1})(x)=f(\frac{1}{4}x)=4(\frac{1}{4}x)=x$

We can use this property to verify that two functions are inverses of each other.

> How To
> **Given two functions $f(x)$ and $g(x)$, test whether the functions are inverses of each other.**
>
> 1. Determine whether $f(g(x))=x$ or $g(f(x))=x$.
> 2. If either statement is true, then both are true,

and $g = f^{-1}$ and $f = g^{-1}$. If either statement is false, then both are false, and $g \neq f^{-1}$ and $f \neq g^{-1}$.

Verify that $f(x) = 5x - 1$ and $g(x) = x + 15$ are inverse functions.

The functions are inverses of each other if $g(f(x)) = x$ and $f(g(x)) = x$.

[redacted]

Substitute $5x - 1$ for $f(x)$.

[redacted]

[redacted]

Simplify.

[redacted]

Simplify.

[redacted]

[redacted]

Substitute x + 15 for g(x).

[redacted]

[redacted]

Simplify.

[redacted]

Simplify.

[redacted]

Since both $g(f(x)) = x$ and $f(g(x)) = x$ are true, the functions $f(x) = 5x - 1$ and $g(x) = x + 15$ are inverse functions. That is, they are inverses of each other.

Verify that the functions are inverse functions.

$f(x) = 4x - 3$ and $g(x) = x + 34$.

g(f(x)) = x, and f(g(x)) = x, so they are inverses.

Verify that the functions are inverse functions.

f(x) = 2x + 6 and g(x) = x − 62.

g(f(x)) = x, and f(g(x)) = x, so they are inverses.

Testing Inverse Relationships Algebraically

If f(x) = 1 x+2 and g(x) = 1 x −2, is g = f −1 ?

g(f(x)) = 1 (1 x+2) −2 = x+2−2 = x

so
g = f −1 and f = g −1

This is enough to answer yes to the question, but we can also verify the other formula.
f(g(x)) = 1 1 x −2+2 = 1 1 x = x

Analysis

Notice the inverse operations are in reverse order of the operations from the original function.

Finding the Inverse of a Function

We have found inverses of function defined by ordered pairs and from a graph. We will now look at how to find an inverse using an algebraic equation. The method uses the idea that if f(x) is a one-to-one function with ordered pairs (x,y), then its inverse function $f^{-1}(x)$ is the set of ordered pairs (y,x).

If we reverse the *x* and *y* in the function and then solve for *y*, we get our inverse function.

How To Find the Inverse

Substitute *y* for f(x). Interchange the variables *x* and *y*. Solve for *y*. Substitute $f^{-1}(x)$ for *y*. Verify that the functions are inverses.

How to Find the inverse of a One-to-One Function

Find the inverse of $f(x) = 4x + 7$.

Step 1. Substitute y for $f(x)$.	Replace $f(x)$ with y.	$f(x) = 4x + 7$ $y = 4x + 7$
Step 2. Interchange the variables x and y.	Replace x with y and then y with x.	$x = 4y + 7$
Step 3. Solve for y.	Subtract 7 from each side. Divide by 4.	$x - 7 = 4y$ $\frac{x-7}{4} = y$
Step 4. Substitute $f^{-1}(x)$ for y.	Replace y with $f^{-1}(x)$.	$\frac{x-7}{4} = f^{-1}(x)$
Step 5. Verify that the functions are inverses.	Show $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$	$f^{-1}(f(x)) \stackrel{?}{=} x$ $f^{-1}(4x + 7) \stackrel{?}{=} x$ $\frac{(4x+7)-7}{4} \stackrel{?}{=} x$ $\frac{4x}{4} \stackrel{?}{=} x$ $x = x$ ✓ $f(f^{-1}(x)) \stackrel{?}{=} x$ $f\left(\frac{x-7}{4}\right) \stackrel{?}{=} x$ $4\left(\frac{x-7}{4}\right) + 7 \stackrel{?}{=} x$ $x - 7 + 7 \stackrel{?}{=} x$ $x = x$ ✓

Find the inverse of the function $f(x)=5x-3$.

$f-1(x)=x+35$

How to Find the Inverse of a One-to-One Function

Find the inverse of $f(x)=2x-35$.

$f(x)=2x-35$Substituteyforf(x).$y=2x$
-35Interchange the variablesxandy.$x=2y$
-35Solve fory.$(x)5=(2y-35)5x5=2y$
$-3x5+3=2yx5+32=y$Substitutef
$-1(x)$fory.$f-1(x)=x5+32$

Verify that the functions are inverses.
$f-1(f(x))=?xf(f-1(x))=?xf-1(2x-35)=?$
$xf(x5+32)=?x(2x-35)5+32=?$
$x2(x5+32)-35=?x2x-3+32=?$
$xx5+3-35=?x2x2=?xx55=?xx=x✓x=x✓$

Find the inverse of the function $f(x)=6x-74$.
$f-1(x)=x4+76$

$f(x)= x x+2$
$f -1 (x)= -2x x-1$

Key Concepts

- **Composition of Functions:** The composition of functions f and g, is written $f \circ g$ and is defined by
  $(f \circ g)(x)=f(g(x))$
  We read $f(g(x))$ as f of g of x.
- **Horizontal Line Test:** If every horizontal line, intersects the graph of a function in at most one point, it is a one-to-one function.
- **Inverse of a Function Defined by Ordered

Pairs: If f(x) is a one-to-one function whose ordered pairs are of the form (x,y), then its inverse function $f-1(x)$ is the set of ordered pairs (y,x).

- **Inverse Functions:** For every x in the domain of one-to-one function f and $f-1$, $f-1(f(x))=xf(f-1(x))=x$
- **How to Find the Inverse of a One-to-One Function:**

  Substitute *y* for f(x). Interchange the variables *x* and *y*. Solve for *y*. Substitute $f-1(x)$ for y. Verify that the functions are inverses.

Practice Makes Perfect

Determine Whether a Function is One-to-One

In the following exercises, determine if the set of ordered pairs represents a function and if so, is the function one-to-one.

$\{(-3,9),(-2,4),(-1,1),(0,0),$
$(1,1),(2,4),(3,9)\}$

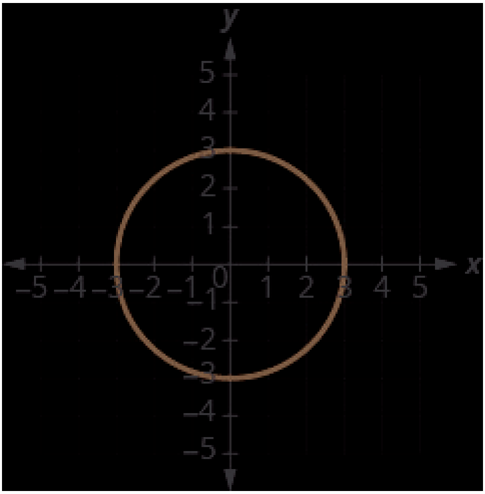
Function; not one-to-one

$\{(-3,-5),(-2,-3),(-1,-1),$
$(0,1),(1,3),(2,5),(3,7)\}$

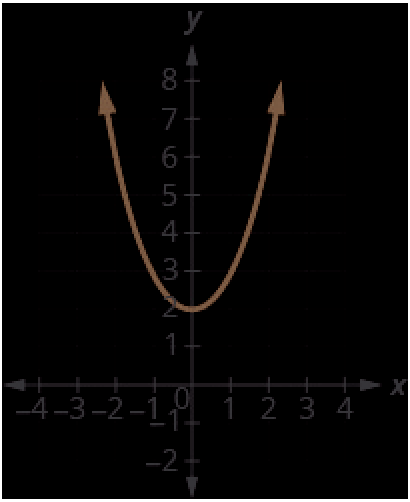
One-to-one function

In the following exercises, determine whether each graph is the graph of a function and if so, is it one-to-one.

ⓐ

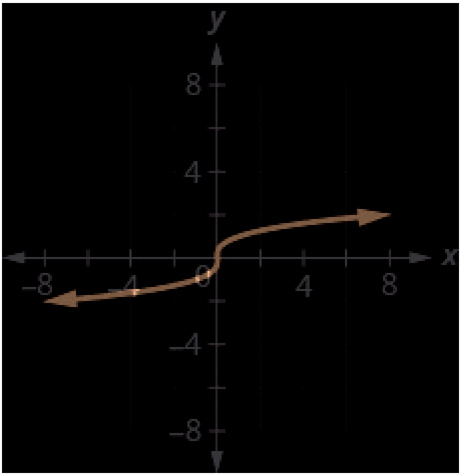


ⓑ

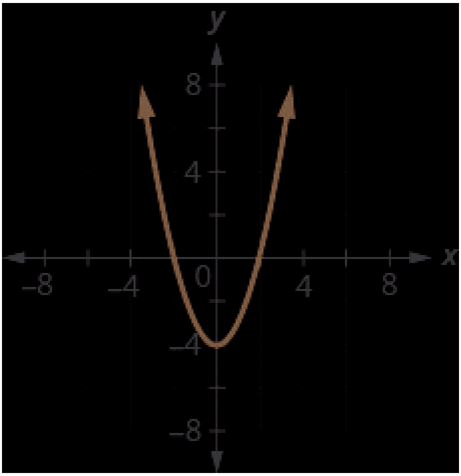


ⓐ Not a function ⓑ Function; not one-to-one

ⓐ



ⓑ



ⓐ One-to-one function
ⓑ Function; not one-to-one

In the following exercises, find the inverse of each function. Determine the domain and range of the inverse function.

$\{(2,1),(4,2),(6,3),(8,4)\}$

Inverse function: $\{(1,2),(2,4),(3,6),(4,8)\}$. Domain: $\{1,2,3,4\}$. Range: $\{2,4,6,8\}$.

$\{(0,-2),(1,3),(2,7),(3,12)\}$

Inverse function: $\{(-2,0),(3,1),(7,2),(12,3)\}$. Domain: $\{-2,3,7,12\}$. Range: $\{0,1,2,3\}$.

In the following exercises, determine whether or not the given functions are inverses.

$f(x)=x+8$ and $g(x)=x-8$

$g(f(x))=x$, and $f(g(x))=x$, so they are inverses.

$f(x)=7x$ and $g(x)=x7$

$g(f(x))=x$, and $f(g(x))=x$, so they are inverses.

$f(x)=7x+3$ and $g(x)=x-37$

$g(f(x))=x$, and $f(g(x))=x$, so they are inverses.

$f(x)=x+2$ and $g(x)=x2-2$

$g(f(x))=x$, and $f(g(x))=x$, so they are inverses (for nonnegative x).

In the following exercises, find the inverse of each

function.

$f(x)=x-12$

$f-1(x)=x+12$

$f(x)=9x$

$f-1(x)=x9$

$f(x)=6x-7$

$f-1(x)=x+76$

$f(x)=-2x+5$

$f-1(x)=x-5-2$

$f(x)=x3-4$

$f-1(x)=x+43$

$f(x)=1x+2$

$f-1(x)=1x-2$

$f(x)=x-33$

$f-1(x)=x3+3$

$f(x)=9x-54,\ x\geq 59$

$f-1(x)=x4+59,\ x\geq 0$

$f(x)=-3x+55$

$f-1(x)=x5-5-3$

Glossary

one-to-one function
: A function is one-to-one if each value in the range has exactly one element in the domain. For each ordered pair in the function, each *y*-value is matched with only one *x*-value.

Quadratic Functions (3.1)
By the end of this section, you will be able to:

- Recognize the graph of a quadratic function
- Find the axis of symmetry and vertex of a parabola
- Find the intercepts of a parabola
- Graph quadratic functions
- Solve maximum and minimum applications

This Module supports section **3.1 of Mat 1023**.
In case you missed something in class, or just want to review a specific topic covered in this Module, here is a list of topics covered:

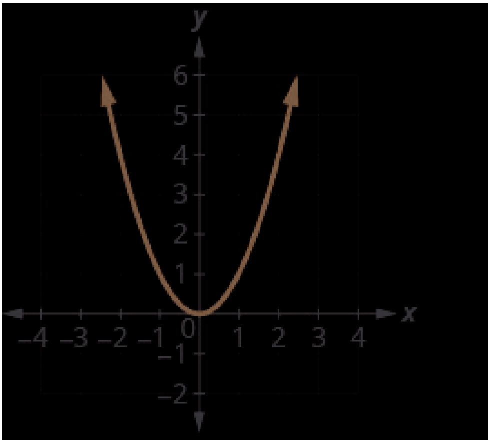
1. Recognize the Graph of a Quadratic Function [link]
2. Axis of Symmetry and Vertex of a Quadratic Function [link]
3. Intercepts of a Parabola [link]
4. Graph Quadratic Functions [link]
5. Maximum and Minimum of a Quadratic Function [link]
6. Key Concepts [link]

Recognize the Graph of a Quadratic Function

Previously we very briefly looked at the function $f(x)=x2$, which we called the square function. It was one of the first non-linear functions we looked at. Now we will graph functions of the form $f(x)=ax2+bx+c$ if $a\neq 0$. We call this kind of function a quadratic function.

Quadratic Function
A **quadratic function**, where *a*, *b*, and *c* are real numbers and $a\neq 0$, is a function of the form
$f(x)=ax2+bx+c$

We graphed the quadratic function $f(x)=x2$ by plotting points.



x	$f(x) = x^2$	$(x, f(x))$
−3	9	(−3, 9)
−2	4	(−2, 4)
−1	1	(−1, 1)
0	0	(0, 0)
1	1	(1, 1)
2	4	(2, 4)
3	9	(3, 9)

Every quadratic function has a graph that looks like this. We call this figure a **parabola**.

Let's practice graphing a parabola by plotting a few points.

Graph $f(x)=x2-1$.
We will graph the function by plotting points.

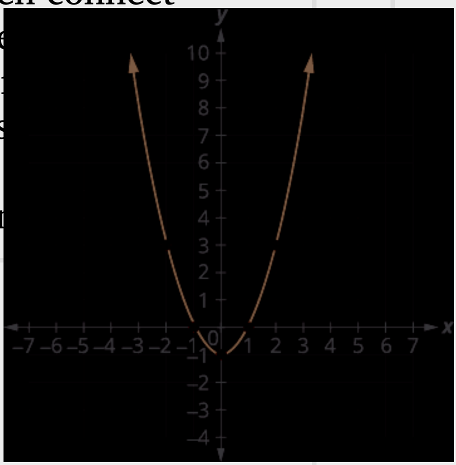
Choose integer values for x,		

su
the
an
f(x

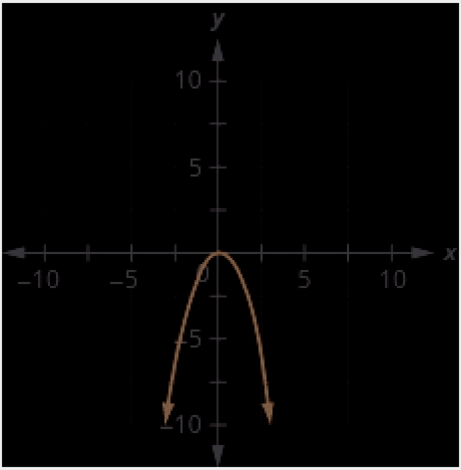
$f(x) = x^2 - 1$	
x	$f(x)$
0	–1
1	0
–1	0
2	3
–2	3

Re
the ordered pairs in the chart.

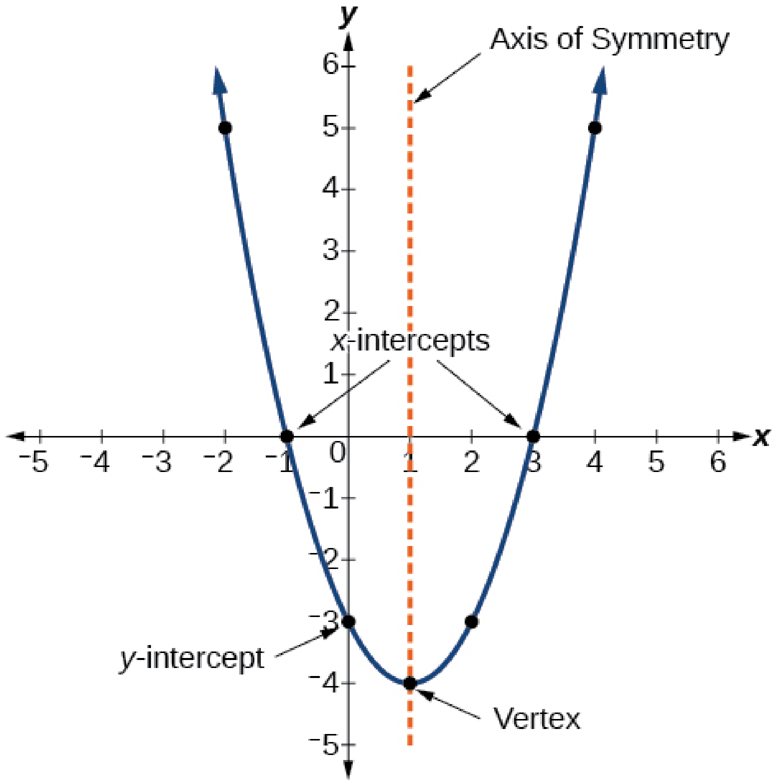
Plot the points, and then connect
the
cu
res
of
fun



Graph $f(x) = -x2$..



The graph of a quadratic function is a U-shaped curve called a parabola. One important feature of the graph is that it has an extreme point, called the **vertex**. If the parabola opens up, the vertex represents the lowest point on the graph, or the minimum value of the quadratic function. If the parabola opens down, the vertex represents the highest point on the graph, or the maximum value. In either case, the vertex is a turning point on the graph. The graph is also symmetric with a vertical line drawn through the vertex, called the **axis of symmetry**. These features are illustrated in [link].

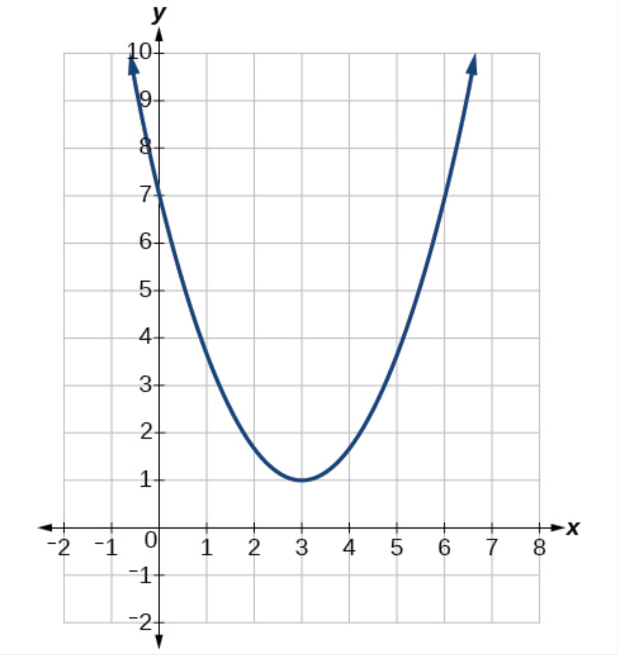


The *y*-intercept is the point at which the parabola crosses the *y*-axis. The *x*-intercepts are the points at which the parabola crosses the *x*-axis. If they exist, the *x*-intercepts represent the **zeros,** or **roots**, of the quadratic function, the values of x at which $y=0$.

Identifying the Characteristics of a Parabola

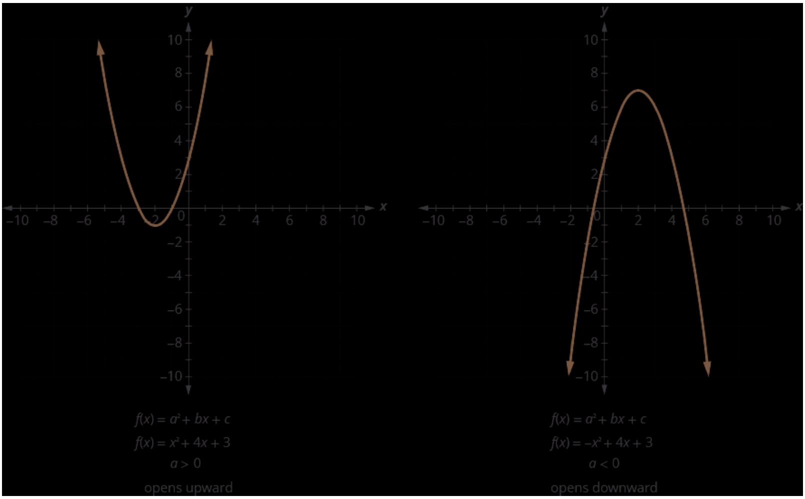
Determine the vertex, axis of symmetry, zeros,

and y- intercept of the parabola shown in [link].



The vertex is the turning point of the graph. We can see that the vertex is at $(3,1)$. Because this parabola opens upward, the axis of symmetry is the vertical line that intersects the parabola at the vertex. So the axis of symmetry is $x=3$. This parabola does not cross the x- axis, so it has no zeros. It crosses the y- axis at $(0,7)$ so this is the *y*-intercept.

All graphs of quadratic functions of the form $f(x) = ax2 + bx + c$ are parabolas that open upward or downward. See [link].



Notice that the only difference in the two functions is the negative sign before the quadratic term ($x2$ in the equation of the graph in [link]). When the quadratic term, is positive, the parabola opens upward, and when the quadratic term is negative, the parabola opens downward.

Parabola Orientation

For the graph of the quadratic function $f(x) = ax2 + bx + c$, if

- $a > 0$, the parabola opens upward
- $a < 0$, the parabola opens downward

Determine whether each parabola opens upward or downward:

ⓐ $f(x) = -3x2 + 2x - 4$ ⓑ $f(x) = 6x2 + 7x - 9$.

ⓐ

Find the value of "*a*".	
[redacted]	
	Since the "*a*" is negative, the parabola will open downward.

ⓑ

Find the value of "a".	
$f(x) = ax^2 + bx + c$ $f(x) = 6x^2 + 7x - 9$ $a = 6$	
	Since the "a" is positive, the parabola will open upward.

Determine whether the graph of each function is a parabola that opens upward or downward:

ⓐ f(x)=2x2+5x−2 ⓑ f(x)= −3x2−4x+7.

ⓐ up; ⓑ down

Find the Axis of Symmetry and Vertex of a Parabola

The **general form of a quadratic function** presents

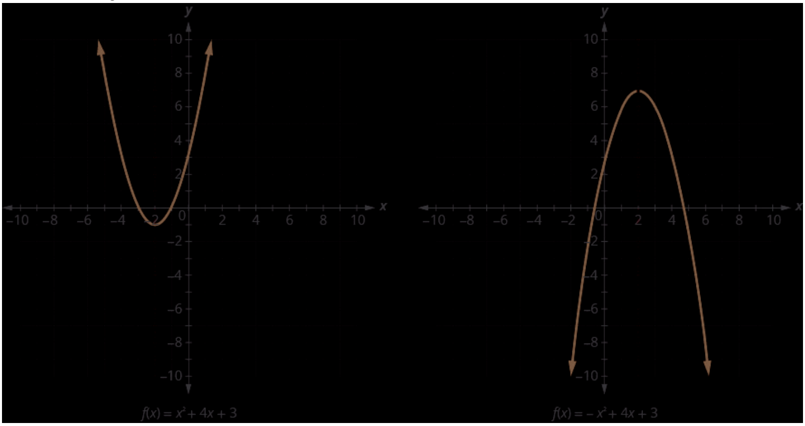
the function in the form

$$f(x) = ax^2 + bx + c$$

where $a, b,$ and c are real numbers and $a \neq 0.$ If $a > 0,$ the parabola opens upward. If $a < 0,$ the parabola opens downward. We can use the general form of a parabola to find the equation for the axis of symmetry.

Look again at [link]. Do you see that we could fold each parabola in half and then one side would lie on top of the other? The 'fold line' is a line of symmetry. We call it the **axis of symmetry** of the parabola.

We show the same two graphs again with the axis of symmetry. See [link].



The axis of symmetry is defined by $x = -\frac{b}{2a}.$ If we use the quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$ to solve $ax^2 + bx + c = 0$ for the x- intercepts, or zeros, we find the value of x halfway between

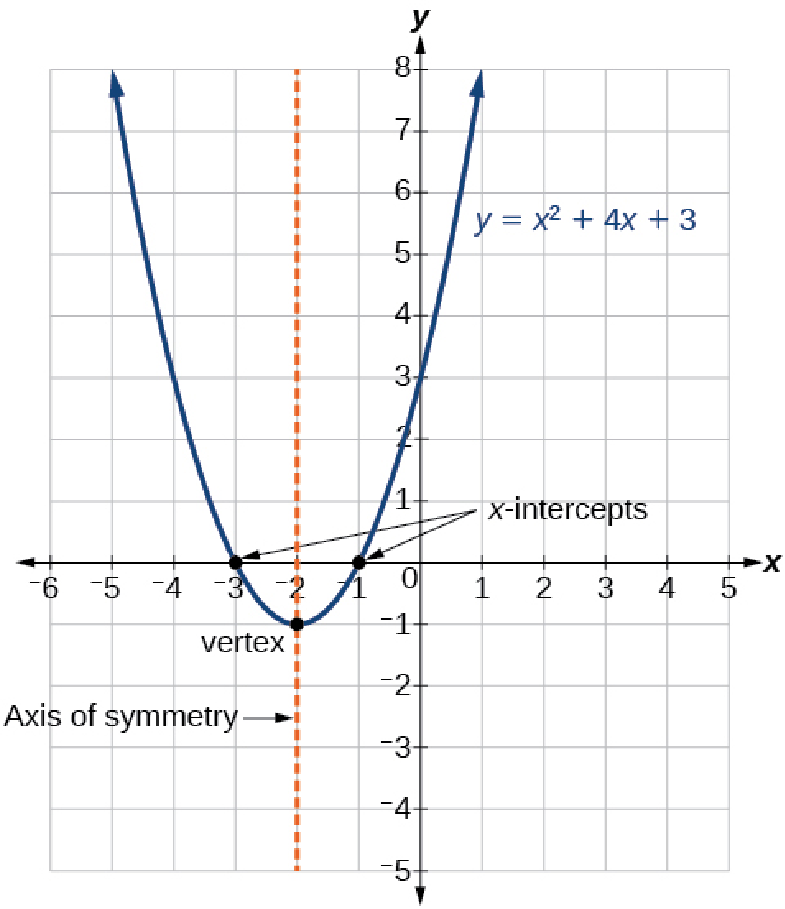
them is always $x = -\frac{b}{2a}$, the equation for the axis of symmetry.

So to find the equation of symmetry of each of the parabolas we graphed above, we will substitute into the formula $x = -\frac{b}{2a}$.

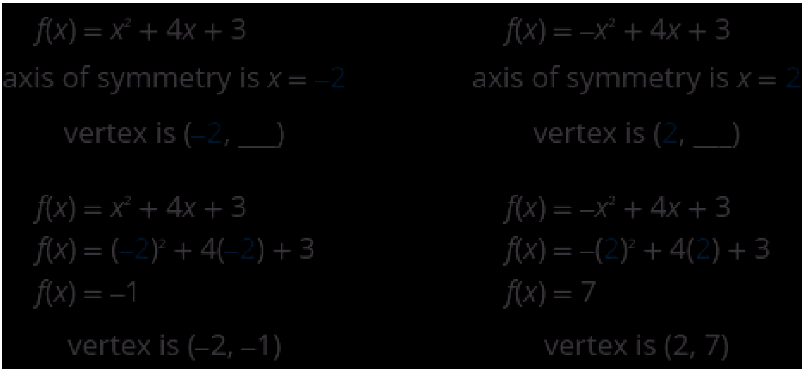
$f(x) = ax^2 + bx + c$	$f(x) = ax^2 + bx + c$
$f(x) = x^2 + 4x + 3$	$f(x) = -x^2 + 4x + 3$
axis of symmetry	axis of symmetry
$x = -\frac{b}{2a}$	$x = -\frac{b}{2a}$
$x = -\frac{4}{2 \cdot 1}$	$x = -\frac{4}{2(-1)}$
$x = -2$	$x = 2$

Notice that these are the equations of the dashed blue lines on the graphs.

The point on the parabola that is the lowest (parabola opens up), or the highest (parabola opens down), lies on the axis of symmetry. This point is called the **vertex** of the parabola.



We can easily find the coordinates of the vertex, because we know it is on the axis of symmetry. This means its
x-coordinate is −b2a. To find the *y*-coordinate of the vertex we substitute the value of the *x*-coordinate into the quadratic function.



Axis of Symmetry and Vertex of a Parabola

The graph of the function $f(x) = ax^2 + bx + c$ is a parabola where:

- the axis of symmetry is the vertical line $x = -\frac{b}{2a}$.
- the vertex is a point on the axis of symmetry, so its *x*-coordinate is $-\frac{b}{2a}$.
- the *y*-coordinate of the vertex is found by substituting $x = -\frac{b}{2a}$ into the quadratic equation.

Given a quadratic function in general form, find the vertex of the parabola.

1. Identify a, b, and c.
2. Find h, the *x*-coordinate of the vertex, by substituting a and b into $h = -\frac{b}{2a}$.

3. Find k, the y-coordinate of the vertex, by evaluating $k=f(h)=f(-\;b\;2a)$.

For the graph of $f(x)=3x2-6x+2$ find:

ⓐ the axis of symmetry ⓑ the vertex.

ⓐ

[redacted]

The axis of symmetry is the vertical line $x=-b2a$.

Substitute the values of a,b into the

eq[redacted]

Simplify.

[redacted]

The axis of symmetry

is the line $x=1$.

ⓑ

$f(x)=3x^2-6x+2$

The vertex is a point on the line of symmetry, so its x-coordinate will be $x=1$.

$f(1)=3(1)^2-6(1)+2$

Find $f(1)$.

Simplify.

$f(1)=3\cdot 1-6+2$

The result is the *y*-coordinate.

$f(1)=-1$

The vertex is $(1,-1)$.

For the graph of $f(x)=2x2-8x+1$ find:

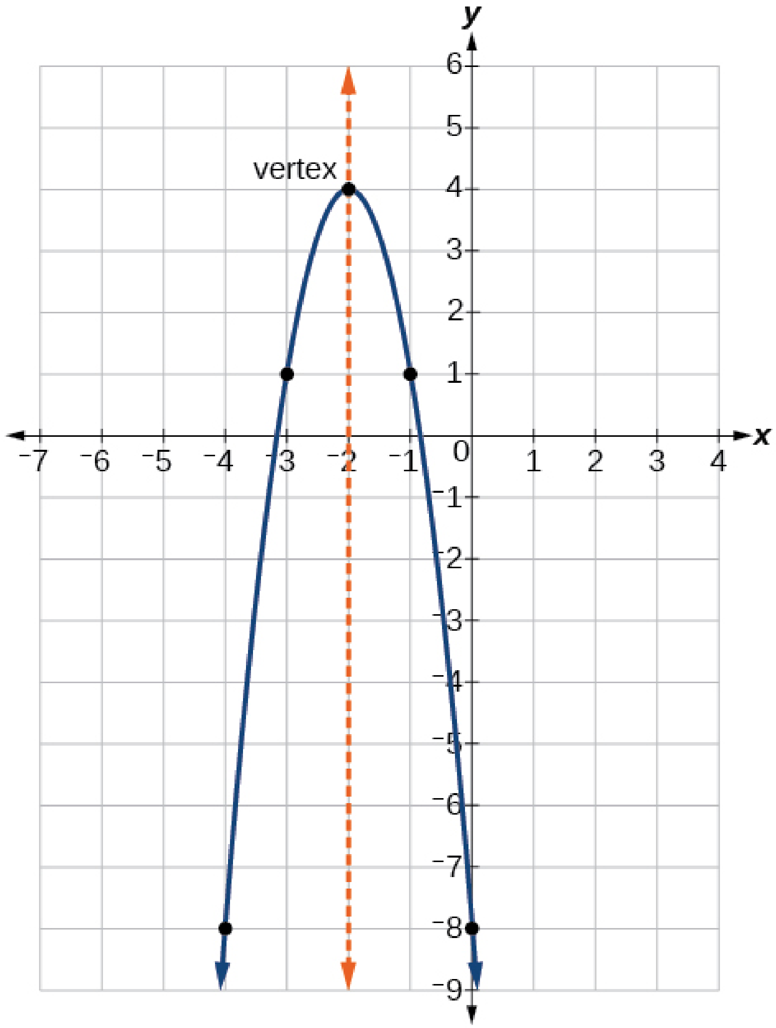
ⓐ the axis of symmetry ⓑ the vertex.
ⓐ $x=2$; ⓑ $(2, -7)$

The **standard form of a quadratic function** presents the function in the form

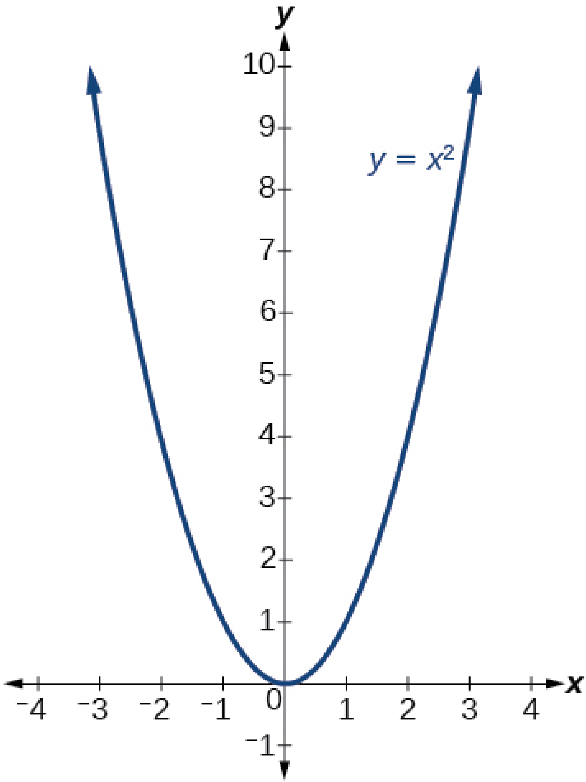
$$f(x)=a(x-h)^2+k$$

where (h, k) is the vertex. Because the vertex appears in the standard form of the quadratic function, this form is also known as the **vertex form of a quadratic function**.

As with the general form, if $a>0$, the parabola opens upward and the vertex is a minimum. If $a<0$, the parabola opens downward, and the vertex is a maximum. [link] represents the graph of the quadratic function written in standard form as $y=-3(x+2)^2+4$. Since $x–h=x+2$ in this example, $h=–2$. In this form, $a=-3, h=-2$, and $k=4$. Because $a<0$, the parabola opens downward. The vertex is at $(-2, 4)$.



The standard form is useful for determining how the graph is transformed from the graph of $y = x^2$. [link] is the graph of this basic function.



The standard form and the general form are equivalent methods of describing the same function. We can see this by expanding out the general form and setting it equal to the standard form.

$$a(x-h)^2+k = ax^2+bx+c$$
$$ax^2-2ahx+(ah^2+k) = ax^2+bx+c$$

For the linear terms to be equal, the coefficients must be equal.

$-2ah = b$, so $h = -\frac{b}{2a}$

This is the axis of symmetry we defined earlier. Setting the constant terms equal:

$$ah^2+k=c$$
$$k=c-ah^2=c-a-\left(\frac{b}{2a}\right)^2=c-\frac{b^2}{4a}$$

In practice, though, it is usually easier to remember that *k* is the output value of the function when the input is h, so $f(h)=k$.

Forms of Quadratic Functions

A quadratic function is a polynomial function of degree two. The graph of a quadratic function is a parabola.

The **general form of a quadratic function** is $f(x)=ax^2+bx+c$ where $a,b,$ and c are real numbers and $a\neq 0$.

The **standard form of a quadratic function** is $f(x)=a(x-h)^2+k$ where $a\neq 0$.

The vertex (h,k) is located at

$$h=-\frac{b}{2a},\ k=f(h)=f\left(\frac{-b}{2a}\right)$$

Finding the Vertex of a Quadratic Function

Find the vertex of the quadratic function $f(x)=2x^2-6x+7$. Rewrite the quadratic in standard form (vertex form).

The horizontal coordinate of the vertex will be at $h=-\frac{b}{2a}=-\frac{-6}{2(2)}=\frac{6}{4}=\frac{3}{2}$

The vertical coordinate of the vertex will be at $k=f(h)=f(\frac{3}{2})=2(\frac{3}{2})^2-6(\frac{3}{2})+7=\frac{5}{2}$

Rewriting into standard form, the stretch factor will be the same as the a in the original quadratic. First, find the horizontal coordinate of the vertex. Then find the vertical coordinate of the vertex. Substitute the values into standard form, using the "a" from the general form.

$f(x)=ax^2+bx+c$ $f(x)=2x^2-6x+7$

The standard form of a quadratic function prior to writing the function then becomes the following:

$f(x)=2(x-\frac{3}{2})^2+\frac{5}{2}$

Analysis

One reason we may want to identify the vertex of the parabola is that this point will inform us where the maximum or minimum value of the output occurs, k, and where it occurs, x.

Given the equation $g(x)=13+x^2-6x$, write the equation in general form and then in

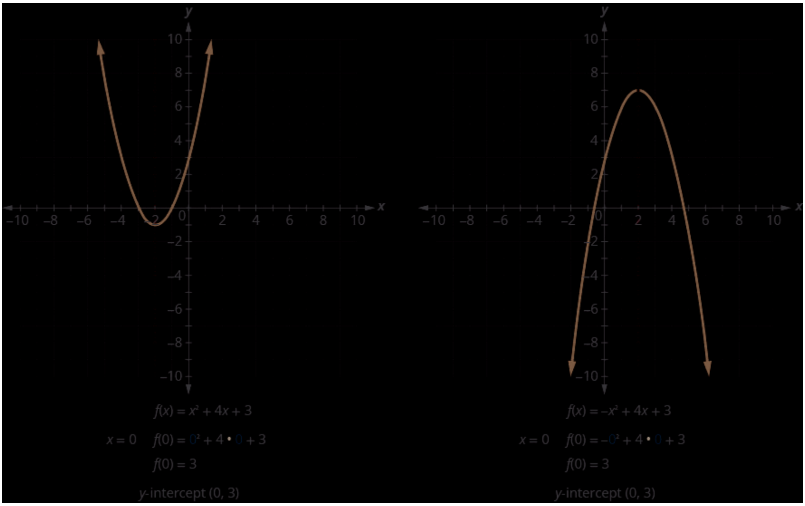
standard form.
$g(x)=x^2-6x+13$ in general form; $g(x)=(x-3)^2+4$ in standard form

Find the Intercepts of a Parabola

When we graphed linear equations, we often used the *x*- and *y*-intercepts to help us graph the lines. Finding the coordinates of the intercepts will help us to graph parabolas, too.

Remember, at the *y*-intercept the value of *x* is zero. So to find the *y*-intercept, we substitute $x=0$ into the function.

Let's find the *y*-intercepts of the two parabolas shown in [link].



An x-intercept results when the value of $f(x)$ is zero. To find an x-intercept, we let $f(x) = 0$. In other words, we will need to solve the equation $0 = ax^2 + bx + c$ for x.

$$f(x) = ax^2 + bx + c \quad 0 = ax^2 + bx + c$$

Solving quadratic equations like this is exactly what we have done earlier in this chapter!

We can now find the x-intercepts of the two parabolas we looked at. First we will find the x-intercepts of the parabola whose function is $f(x) = x^2 + 4x + 3$.

[redacted]

Let f(x) = 0.

[redacted]

Factor.

[redacted]

Use the Zero Product Property.

[redacted]

Solve.

[redacted]

The *x*-intercepts are $(-1,0)$ and $(-3,0)$.

Now we will find the *x*-intercepts of the parabola whose function is $f(x) = -x^2 + 4x + 3$.

[redacted]

Let f(x) = 0.

[redacted]

This quadratic does not factor, so

we

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Formula:

$a = -1, b = 4, c = 3$

$$x = \frac{-4 \pm \sqrt{4^2 - 4(-1)(3)}}{2(-1)}$$

Simplify.

$$x = \frac{-4 \pm \sqrt{28}}{-2}$$

$$x = \frac{-4 \pm 2\sqrt{7}}{-2}$$

$$x = \frac{-2(2 \pm \sqrt{7})}{-2}$$

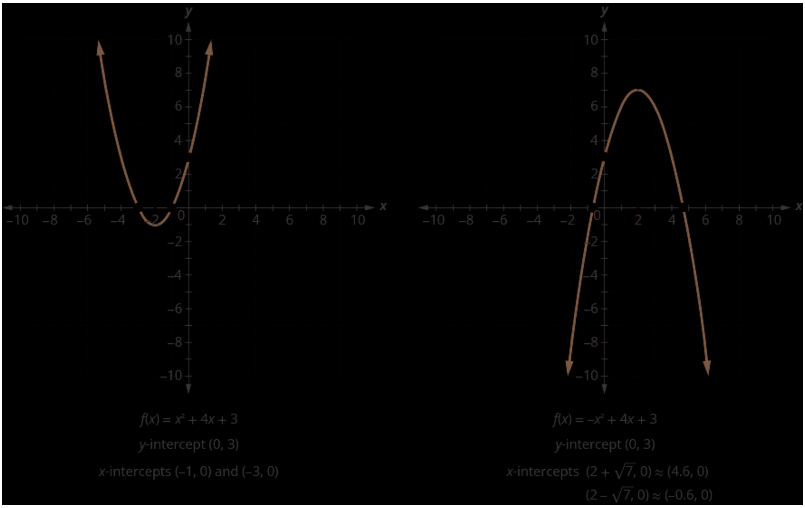
$$x = 2 \pm \sqrt{7}$$

The *x*-intercepts are $(2+7,0)$ and $(2-7,0)$.

We will use the decimal approximations of the *x*-intercepts, so that we can locate these points on the graph,

$(2+7,0) \approx (4.6,0)(2-7,0) \approx (-0.6,0)$

Do these results agree with our graphs? See [link].



Find the Intercepts of a Parabola

To find the intercepts of a parabola whose function is $f(x)=ax2+bx+c$:

y-interceptx-interceptsLet$x=0$and solve forf(x).Let$f(x)=0$and solve forx.

Find the intercepts of the parabola whose function is $f(x)=x2-2x-8$.

To find the ***y***-intercept, let x =

$f(x) = x^2 - 2x - 8$

solve for f(x).

$f(0) = 0^2 - 2 \cdot 0 - 8$

$f(0) = -8$

When x = 0, then f(0) = − 8. The *y*-intercept is the point (0, − 8).

To find the ***x***-intercept, let f(x)

$f(x) = x^2 - 2x - 8$

solve for x.

$0 = x^2 - 2x - 8$

Solve by factoring.

$0 = (x - 4)(x + 2)$

$0 = x - 4 \quad 0 = x + 2$

$4 = x \quad 2 = x$

When f(x) = 0,

then $x = 4 or x = -2$.
The x-intercepts are the points (4,0) and $(-2,0)$.

Find the intercepts of the parabola whose function is $f(x) = x2 - 4x - 12$.

y-intercept: $(0, -12)$ x-intercepts $(-2,0),(6,0)$

We are now looking at quadratic functions of the form $f(x) = ax2 + bx + c$. The graphs of these functions are parabolas. The x-intercepts of the parabolas occur where $f(x) = 0$.

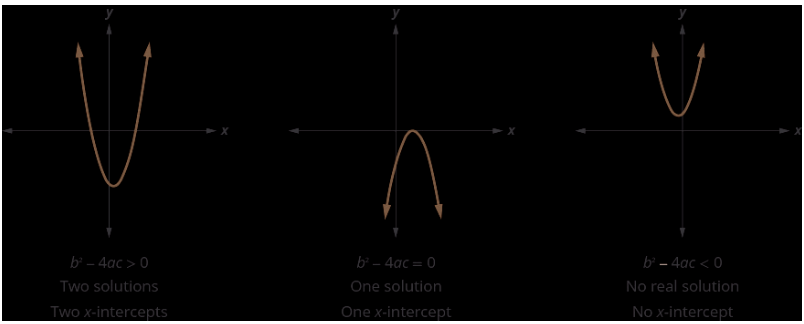
For example:
Quadratic equationQuadratic function $x2 - 2x - 15 = 0(x - 5)(x + 3) = 0x - 5 = 0x + 3 = 0x = 5x = -3Letf(x) = 0.f(x) = x2 - 2x - 150 = x2 - 2x - 150 = (x - 5)(x + 3)x - 5 = 0x + 3 = 0x = 5x =$

$-3 (5,0)$ and $(-3,0)$ x-intercepts

The solutions of the quadratic function are the *x* values of the *x*-intercepts.

Earlier, we saw that quadratic equations have 2, 1, or 0 solutions. The graphs below show examples of parabolas for these three cases. Since the solutions of the functions give the *x*-intercepts of the graphs, the number of *x*-intercepts is the same as the number of solutions.

Previously, we used the discriminant to determine the number of solutions of a quadratic function of the form $ax^2+bx+c=0$. Now we can use the discriminant to tell us how many *x*-intercepts there are on the graph.



Before you to find the values of the *x*-intercepts, you may want to evaluate the discriminant so you know how many solutions to expect.

Find the intercepts of the parabola for the function f(x) = 5x2 + x + 4.

[redacted]

To find the *y*-intercept, let x = 0 and so [redacted]

[redacted]

When x = 0, then f(0) = 4.
The *y*-intercept is the point (0, 4).

To find the *x*-intercept, let f(x) = 0 and so [redacted]

[redacted]

Find the value of the discriminant to predict the number of solutions which is

also the number of *x*-intercepts.

$b2-4ac12-4\cdot5\cdot41-80-79$

Since the value of the discriminant is negative, there is no real solution to the equation.
There are no *x*-intercepts.

Find the intercepts of the parabola whose function is $f(x)=3x2+4x+4$.

y-intercept: (0, 4) no *x*-intercept

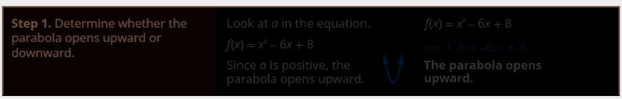
Graph Quadratic Functions Using Properties

Now we have all the pieces we need in order to

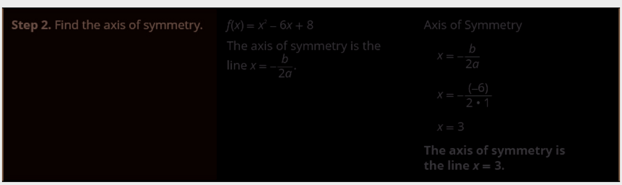
graph a quadratic function. We just need to put them together. In the next example we will see how to do this.

How to Graph a Quadratic Function Using Properties

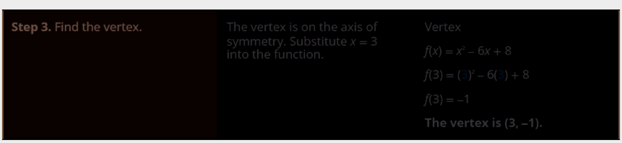
Graph $f(x) = x^2 - 6x + 8$ by using its properties.



Step 1. Determine whether the parabola opens upward or downward.	Look at a in the equation. $f(x) = x^2 - 6x + 8$ Since a is positive, the parabola opens upward.	$f(x) = x^2 - 6x + 8$ $a = 1, b = -6, c = 8$ **The parabola opens upward.**



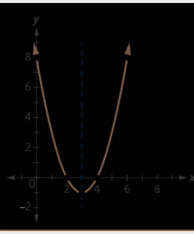
Step 2. Find the axis of symmetry.	$f(x) = x^2 - 6x + 8$ The axis of symmetry is the line $x = -\frac{b}{2a}$.	Axis of Symmetry $x = -\frac{b}{2a}$ $x = -\frac{(-6)}{2 \cdot 1}$ $x = 3$ **The axis of symmetry is the line $x = 3$.**



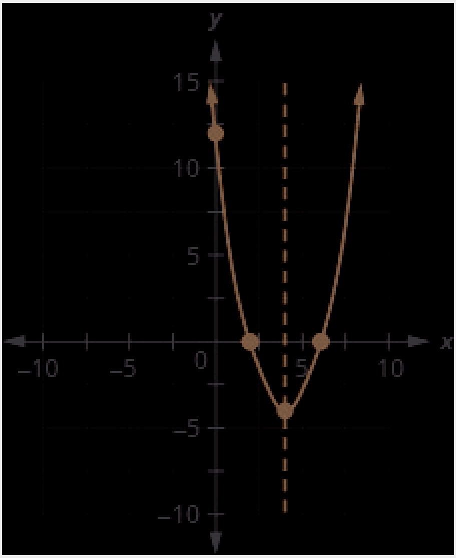
Step 3. Find the vertex.	The vertex is on the axis of symmetry. Substitute $x = 3$ into the function.	Vertex $f(x) = x^2 - 6x + 8$ $f(3) = (3)^2 - 6(3) + 8$ $f(3) = -1$ **The vertex is (3, –1).**

Step 4. Find the y-intercept. Find the point symmetric to the y-intercept across the axis of symmetry.	We find $f(0)$.	y-intercept $f(x) = x^2 - 6x + 8$ $f(0) = (0)^2 - 6(0) + 8$ $f(0) = 8$ **The y-intercept is (0, 8).**
	We use the axis of symmetry to find a point symmetric to the y-intercept. The y-intercept is 3 units left of the axis of symmetry, $x = 3$. A point 3 units to the right of the axis of symmetry has $x = 6$.	Point symmetric to y-intercept: **The point is (6, 8).**

Step 5. Find the x-intercepts. Find additional points if needed.	We solve $f(x) = 0$. We can solve this quadratic equation by factoring.	x-intercepts $f(x) = x^2 - 6x + 8$ $0 = x^2 - 6x + 8$ $0 = (x - 2)(x - 4)$ $x = 2$ or $x = 4$ **The x-intercepts are (2, 0) and (4, 0).**

Step 6. Graph the parabola.	We graph the vertex, intercepts, and the point symmetric to the y-intercept. We connect these 5 points to sketch the parabola.	

Graph $f(x) = x^2 - 8x + 12$ by using its properties.



To graph a quadratic function using properties.

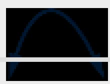
Determine whether the parabola opens upward or downward. Find the equation of the axis of symmetry. Find the vertex. Find the y-intercept. Find the point symmetric to the y-intercept across the axis of symmetry. Find the x-intercepts. Find additional points if needed. Graph the parabola.

We were able to find the x-intercepts in the last example by factoring. We find the x-intercepts in the next example by factoring, too.

Graph $f(x) = x2 + 6x - 9$ by using its properties.

[redacted]

Since a is -1, the parabola opens downward.



To find the equation of the axis of symmetry, use

[redacted]

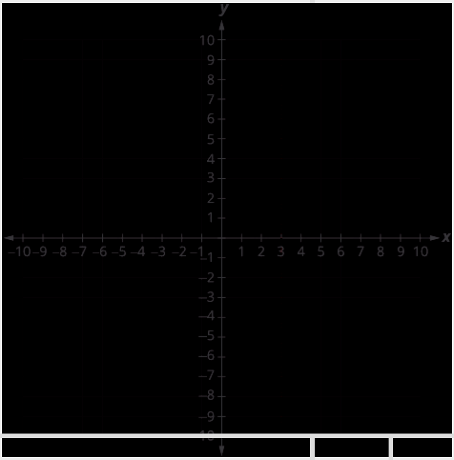
x = − b2a.

[redacted]

[redacted]

The axis of symmetry is x = 3.

The vertex is on the line $x = 3$.



Find f(3).

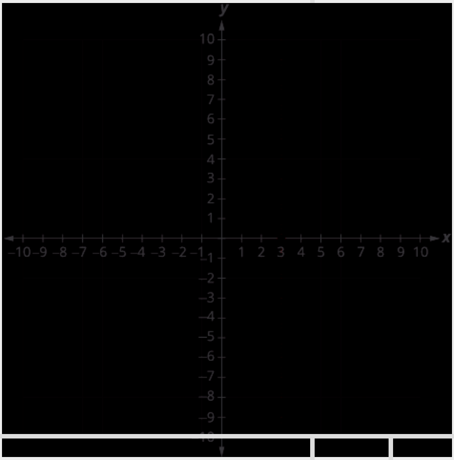
[redacted]

[redacted]

[redacted]

[redacted]

The vertex is (3,0).



The *y*-intercept occurs when x = 0. Find f(0).

[redacted]

Substitute x = 0.

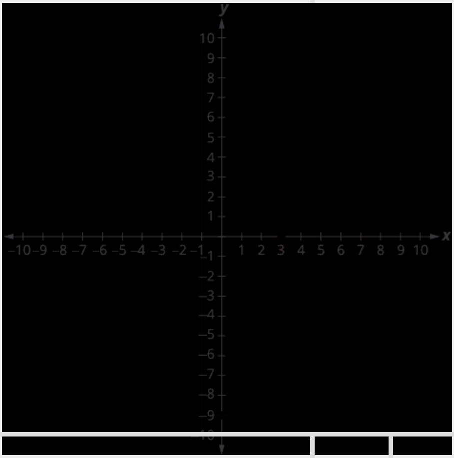
[redacted]

Simplify.

[redacted]

The *y*-intercept is (0, −9).

The point (0, −9) is three units to the left of the line of symmetry. The point three units to the right of the line of symmetry is (6, −9).



Point symmetric to the y intercept is $(6, -9)$

The x-intercept occurs when $f(x) = 0$.

[redacted]

Find $f(x) = 0$.

[redacted]

Factor the GCF.

[redacted]

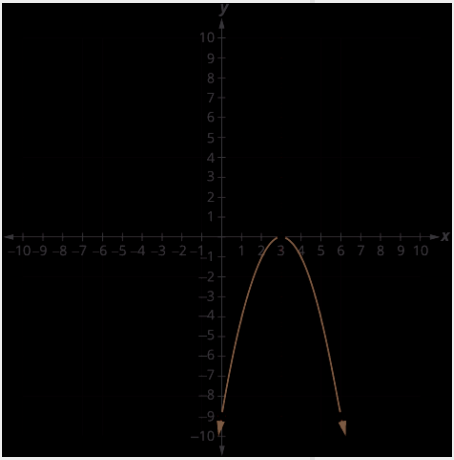
Factor the trinomial.

[redacted]

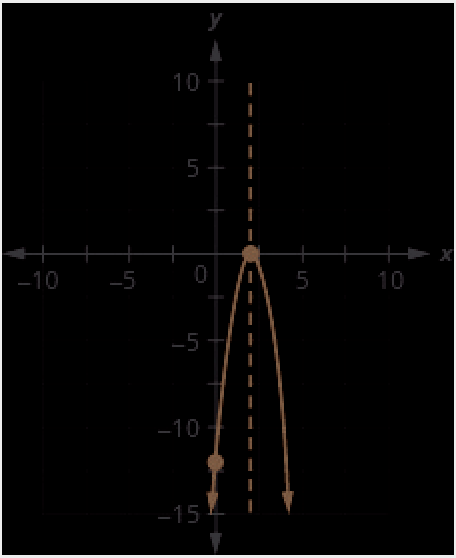
Solve for x.

[redacted]

Connect the points to graph the parabola.



Graph $f(x) = 3x^2 + 12x - 12$ by using its properties.



For the graph of $f(x) = -x^2 + 6x - 9$, the vertex and the x-intercept were the same point. Remember how the discriminant determines the number of solutions of a quadratic equation? The discriminant of the equation $0 = -x^2 + 6x - 9$ is 0, so there is only one solution. That means there is only one x-intercept, and it is the vertex of the parabola.

How many x-intercepts would you expect to see on the graph of $f(x) = x^2 + 4x + 5$?

Graph $f(x) = x^2 + 4x + 5$ by using its properties.

[redacted]

Since a is 1, the parabola opens upward.

To find the axis of symmetry, find $x = -$ [redacted]

[redacted]

[redacted]

The equation of the axis of symmetry is $x = -2$.

[redacted]

The vertex is on the line $x = -2$.

Find $f(x)$ when $x = -2$.

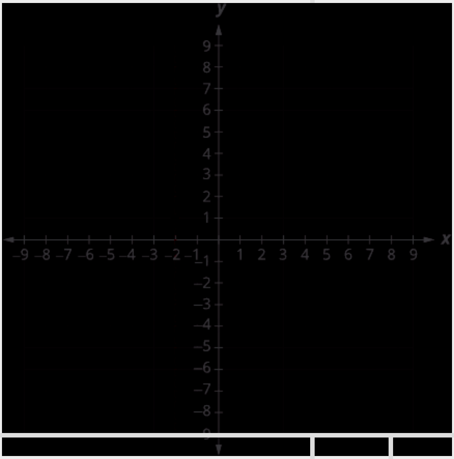
[redacted]

[redacted]

[redacted]

[redacted]

The vertex is $(-2,1)$.



The *y*-intercept occurs when $x = 0$.

[redacted]

Find f(0).

[redacted]

Simplify.

[redacted]

The *y* intercept is (0,5).

The point $(-4,5)$ is two units to the left of the

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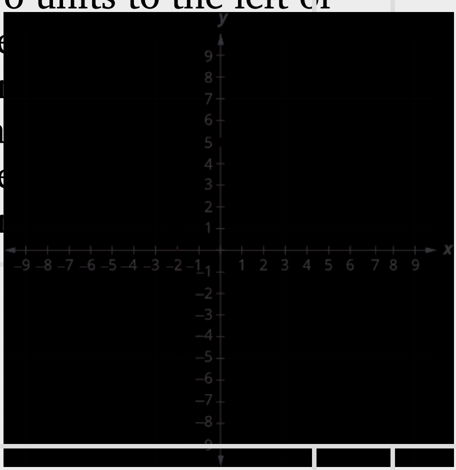
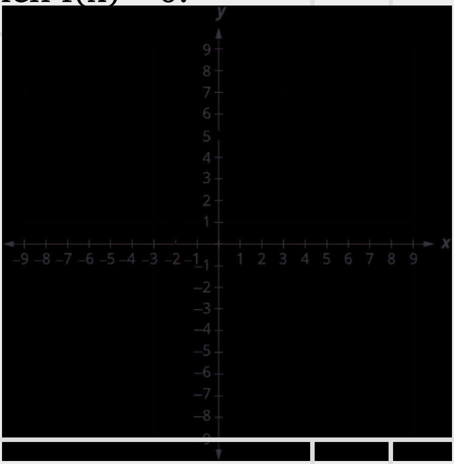


Point symmetric to the y intercept is $(-4,5)$.

The x-intercept occurs when $f(x)=0$.



Find $f(x)=0$.

[redacted]

Test the discriminant.

[redacted]

[redacted]

[redacted]

[redacted]

Since the value of the discriminant is

negative, there is
no real solution and so
no x intercept.
Connect the points to
graph the parabola.
Yo
wa
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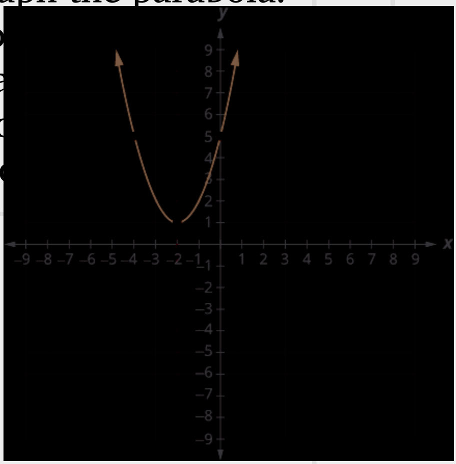
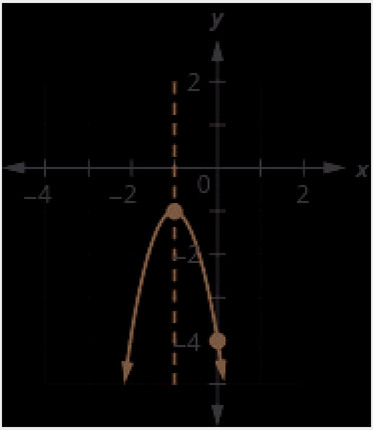


Graph $f(x) = -3x^2 - 6x - 4$ by using its properties.



Finding the y-intercept by finding $f(0)$ is easy, isn't it? Sometimes we need to use the Quadratic Formula to find the x-intercepts.

Graph $f(x) = 2x^2 - 4x - 3$ by using its properties.

[redacted]

Since a is 2, the parabola opens upward.

[redacted]

To find the equation of the axis of symmetry, use [redacted]

$x = -b2a$.

[redacted]

[redacted]

The equation of the axis of symmetry is $x = 1$.

The vertex is on the line $x = 1$.

[redacted]

Find f(1).

[redacted]

[redacted]

[redacted]

The vertex is $(1, -5)$.

The y-intercept occurs when $x = 0$.

[redacted]

Find $f(0)$.

[redacted]

Simplify.

[redacted]

The y-intercept is $(0, -3)$.

The point $(0, -3)$ is one unit to the left of the line of symmetry.

Point symmetric to the y-intercept is $(2, -3)$

The point one unit to the right of the line of symmetry is $(2, -3)$.

The x-intercept occurs when $y = 0$.

[redacted]

Find $f(x) = 0$.

[redacted]

Use the Quadratic Formula.

[redacted]

Substitute in the values of a,b, and c.

[redacted]

Simplify.

[redacted]

Simplify inside the radical.

[redacted]

Simplify the radical.

[redacted]

Factor the GCF.

[redacted]

Remove common factors.

[redacted]

Write as two equations.

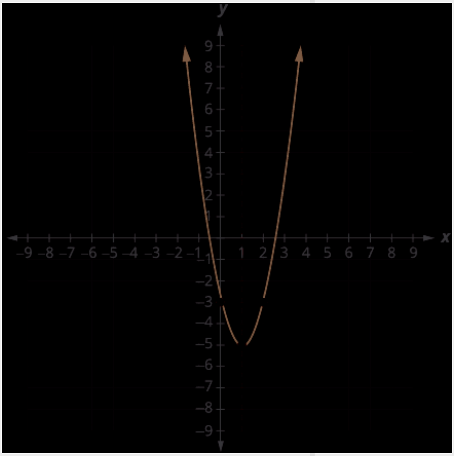
[redacted]

Approximate the values.

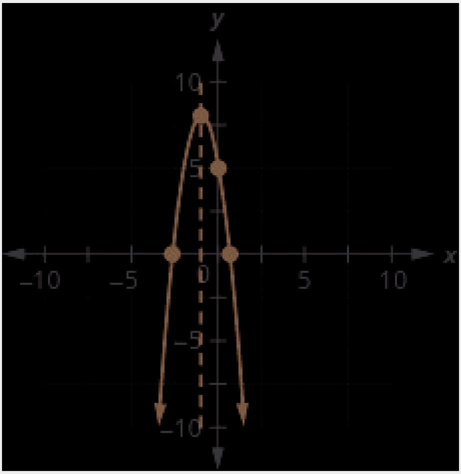
[redacted]

The approximate values of the *x*-intercepts are $(2.5,0)$ **and** $(-0.6,0)$.

Graph the parabola using the points found.

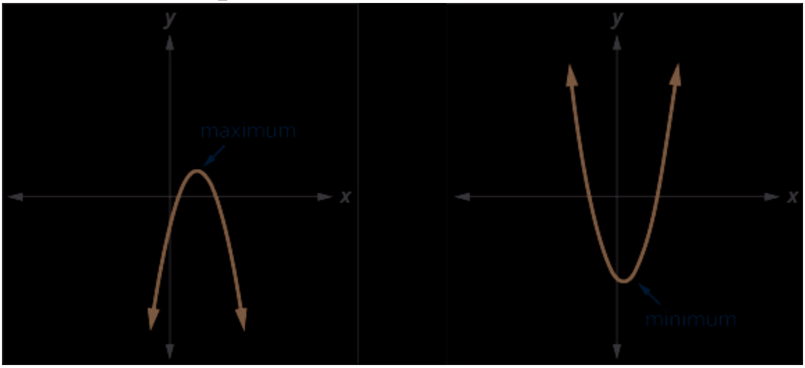


Graph $f(x) = -3x^2 - 6x + 5$ by using its properties.



Solve Maximum and Minimum Applications

Knowing that the vertex of a parabola is the lowest or highest point of the parabola gives us an easy way to determine the minimum or maximum value of a quadratic function. The *y*-coordinate of the vertex is the minimum value of a parabola that opens upward. It is the maximum value of a parabola that opens downward. See [link].



Minimum or Maximum Values of a Quadratic Function

The ***y*-coordinate of the vertex** of the graph of a quadratic function is the

- *minimum* value of the quadratic equation if the parabola opens *upward*.
- *maximum* value of the quadratic equation if the parabola opens *downward*.

So

$f(\frac{-b}{2a})$

is the Minimum or Maximum depending on the parabola's orientation.

Find the minimum or maximum value of the quadratic function $f(x) = x^2 + 2x - 8$.

[redacted]

Since *a* is positive, the parabola opens upward.

The quadratic equation has a minimum.

Find the equation of

the [redacted]

[redacted]

[redacted]

The equation of the axis of symmetry is $x = -1$.

The vertex is on the line $x = -1$.

[redacted]

Find $f(-1)$.

[redacted]

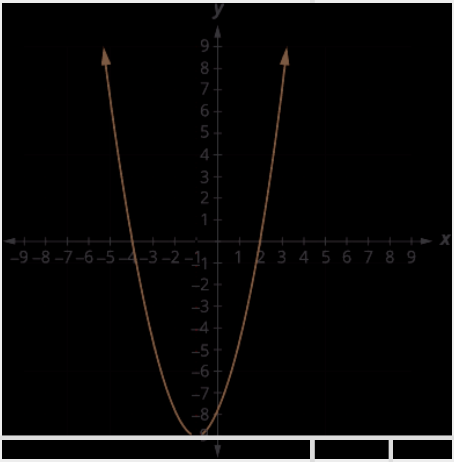
[redacted]

[redacted]

The vertex is $(-1, -9)$.

Since the parabola has a minimum, the *y*-coordinate of the vertex is the minimum *y*-value of the quadratic equation.
The minimum value of the quadratic is -9

and it
occurs when x = −1.



Show the graph to
verify the result.

Find the maximum or minimum value of the quadratic function f(x) = x2 − 8x + 12.

The minimum value of the quadratic function is -4 and it occurs when $x = 4$.

Find the maximum or minimum value of the quadratic function $f(x) = -4x2 + 16x - 11$.
The maximum value of the quadratic function is 5 and it occurs when $x = 2$.

Access these online resources for additional instruction and practice with graphing quadratic functions using properties.

- Quadratic Functions: Axis of Symmetry and Vertex
- Finding x- and y-intercepts of a Quadratic Function
- Graphing Quadratic Functions
- Solve Maxiumum or Minimum Applications
- Quadratic Applications: Minimum and Maximum

Key Concepts

- Parabola Orientation

- For the graph of the quadratic function $f(x)=ax2+bx+c$, if
    - $a > 0$, the parabola opens upward.
    - $a < 0$, the parabola opens downward.

- Axis of Symmetry and Vertex of a Parabola The graph of the function $f(x)=ax2+bx+c$ is a parabola where:
    - the axis of symmetry is the vertical line $x=-b2a$.
    - the vertex is a point on the axis of symmetry, so its *x*-coordinate is $-b2a$.
    - the *y*-coordinate of the vertex is found by substituting $x=-b2a$ into the quadratic equation.

- Find the Intercepts of a Parabola
    - To find the intercepts of a parabola whose function is $f(x)=ax2+bx+c$:
      y-interceptx-interceptsLet$x=0$and solve forf(x).Let$f(x)=0$and solve forx.

- How to graph a quadratic function using properties.

  Determine whether the parabola opens upward or downward. Find the equation of the axis of symmetry. Find the vertex. Find the *y*-intercept. Find the point symmetric to the *y*-intercept

across the axis of symmetry. Find the *x*-intercepts. Find additional points if needed. Graph the parabola.

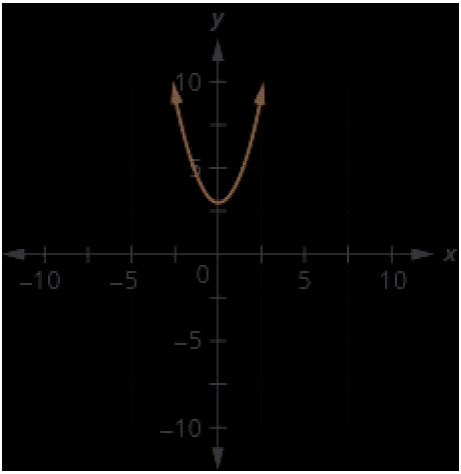
- Minimum or Maximum Values of a Quadratic Equation
    - The *y*-coordinate of the vertex of the graph of a quadratic equation is the
    - *minimum* value of the quadratic equation if the parabola opens *upward*.
    - *maximum* value of the quadratic equation if the parabola opens *downward*.

Practice Makes Perfect

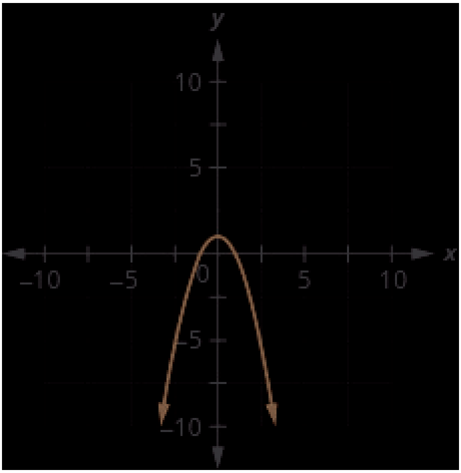
Recognize the Graph of a Quadratic Function

In the following exercises, graph the functions by plotting points.

$f(x) = x2 + 3$



$y = -x2 + 1$



For each of the following exercises, determine if the parabola opens up or down.

ⓐ $f(x) = -2x2 - 6x - 7$
ⓑ $f(x) = 6x2 + 2x + 3$

ⓐ down ⓑ up

ⓐ $f(x) = -3x2 + 5x - 1$
ⓑ $f(x) = 2x2 - 4x + 5$

ⓐ down ⓑ up

Find the Axis of Symmetry and Vertex of a Parabola

In the following functions, find ⓐ the equation of the axis of symmetry and ⓑ the vertex of its graph.

$f(x) = x2 + 8x - 1$

ⓐ $x = -4$; ⓑ $(-4, -17)$

$f(x) = -x2 + 2x + 5$

ⓐ $x = 1$; ⓑ $(1,2)$

Find the Intercepts of a Parabola

In the following exercises, find the intercepts of the parabola whose function is given.

$f(x)=x2+7x+6$

---

y-intercept: (0, 6); x-intercept $(-1, 0),(-6, 0)$

$f(x)=-x2+8x-19$

---

y-intercept: $(0, -19)$; x-intercept: none

$f(x)=4x2-20x+25$

---

y-intercept: $(0, -16)$; x-intercept (52,0)

$f(x)=-x2-6x-9$

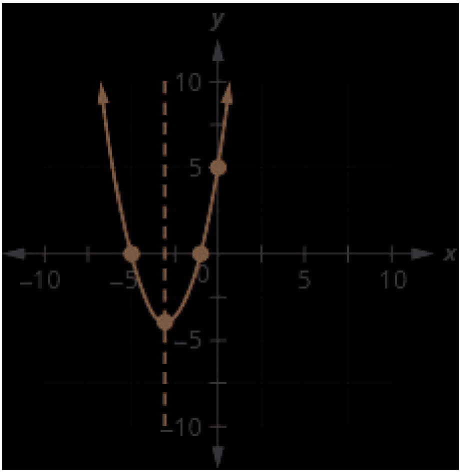
---

y-intercept: (0, 9); x-intercept $(-3, 0)$

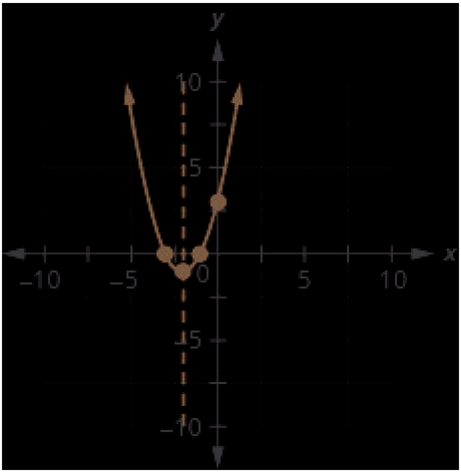
Graph Quadratic Functions Using Properties

In the following exercises, graph the function by using its properties.

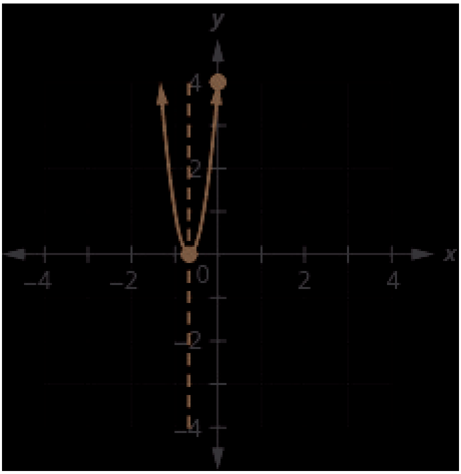
$f(x)=x2+6x+5$



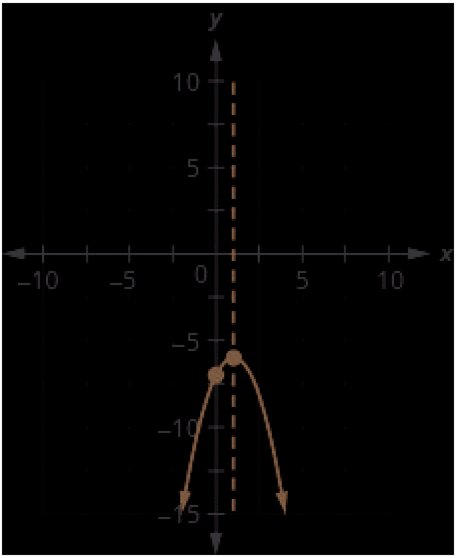
$f(x)=x2+4x+3$



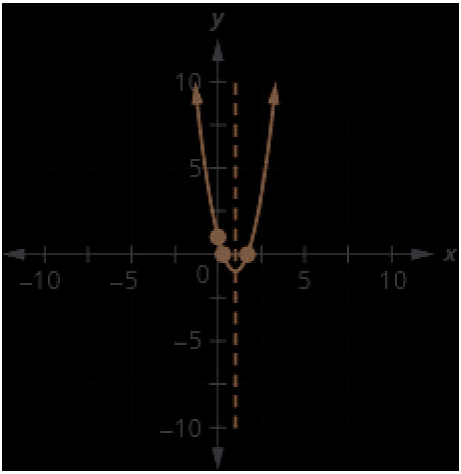
$f(x) = 9x2 + 12x + 4$



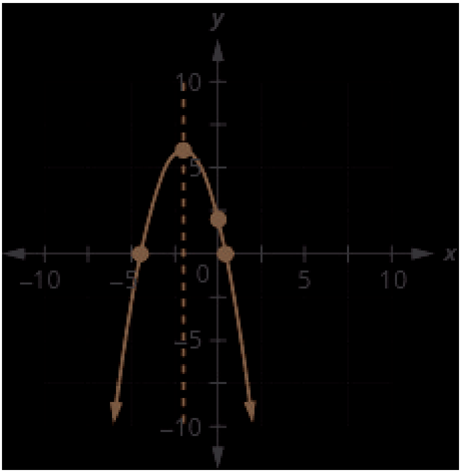
$f(x) = -x2 + 2x - 7$



$f(x) = 2x2 - 4x + 1$



$f(x) = -x^2 - 4x + 2$



Solve Maximum and Minimum Applications

In the following exercises, find the maximum or minimum value of each function.

$f(x) = 2x^2 + x - 1$

The minimum value is -98 when $x = -14$.

$y = x^2 - 6x + 15$

The maximum value is 6 when $x = 3$.

$y = -9x^2 + 16$

The maximum value is 16 when $x = 0$.

An arrow is shot vertically upward from a platform 45 feet high at a rate of 168 ft/sec. Use the quadratic function $h(t) = -16t^2 + 168t + 45$ find how long it will take the arrow to reach its maximum height, and then find the maximum height.

In 5.3 sec the arrow will reach maximum height of 486 ft.

A computer store owner estimates that by charging x dollars each for a certain computer, he can sell $40 - x$ computers each week. The quadratic function $R(x) = -x^2 + 40x$ is used to find the revenue, R, received when the selling price of a computer is x, Find the selling price that will give him the maximum revenue, and then find the amount of the maximum revenue.

---

20 computers will give the maximum of $400 in receipts.

Glossary

quadratic function

A quadratic function, where *a*, *b*, and *c* are real numbers and $a \neq 0$, is a function of the form $f(x) = ax2 + bx + c$.

Polynomial Functions (3.2)
By the end of this section, you will be able to:

- Determine the degree of polynomials
- Add and subtract polynomials
- Evaluate a polynomial function for a given value
- Add and subtract polynomial functions

This Module supports section **3.2 of Mat 1023**.
In case you missed something in class, or just want to review a specific topic covered in this Module, here is a list of topics covered:

1. Polynomial Functions and Leading Coefficient [link]
2. Turning Points, Zeros, and Intercepts of Polynomial Functions [link]
3. Zeros and Multiplicities [link]
4. Graphing Polynomial Functions [link]
5. Intermediate Value Theorem [link]
6. Key Concepts [link]

Polynomial Functions and Leading

Coefficient

Recall earlier we introduced Polynomials. Here is a brief review:

A **polynomial** is a sum of or difference of terms, each consisting of a variable raised to a nonnegative integer power. A number multiplied by a variable raised to an exponent, such as 384π, is known as a **coefficient**. Coefficients can be positive, negative, or zero, and can be whole numbers, decimals, or fractions. Each product $a_i x^i$, such as $384\pi w$, is a **term of a polynomial**. If a term does not contain a variable, it is called a *constant*.

A polynomial containing only one term, such as $5x^4$, is called a **monomial**. A polynomial containing two terms, such as $2x-9$, is called a **binomial**. A polynomial containing three terms, such as $-3x^2+8x-7$, is called a **trinomial**.

Here are some examples of polynomials.

Polynomial	$y+1$	$4a^2-7ab+2b^2$	$4x^4+x^3+8x^2-9x+1$	
Monomial	14	$8y^2$	$-9x^3y^5$	$-13a^3b^2c$
Binomial	$a+7b$	$4x^2-y^2$	y^2-16	$3p^3q$

Trinomial	$x^2-7x+12$	$9m^2+2mn-8n^2$	$6k^4-k^3+8k$	$-9p^2q$	z^4+3z^2-1

Degree of a Polynomial

The **degree of a term** is the sum of the exponents of its variables.

The **degree of a constant** is 0.

The **degree of a polynomial** is the highest degree of all its terms.

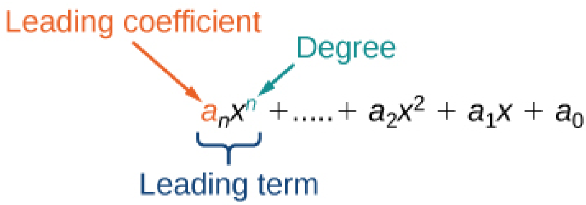
Here are some additional examples.

Monomials	14	$8ab^2$	$-9x^3y^5$	$-13a$
Degree				
Binomial	$h+7$	$7b^2-3b$	x^2y^2-25	$4n^3-8n^2$
Degree of each				
Degree of polynomial	1	2	4	3
Trinomial	$x^2-12x+27$	$9a^2+6ab+b^2$	$6m^4-m^3n^2+8mn^5$	z^4+3z^2-1
Degree of each				
Degree of polynomial	2	2	6	4
Polynomial	$y-1$	$3y^2-2y-5$	$4x^4+x^3+8x^2-9x+1$	
Degree of each				
Degree of polynomial	1	2	4	

Working with polynomials is easier when you list the terms in descending order of degrees. When a polynomial is written this way, it is said to be in

standard form of a polynomial. Get in the habit of writing the term with the highest degree first.

We can find the **degree** of a polynomial by identifying the highest power of the variable that occurs in the polynomial. The term with the highest degree is called the **leading term** because it is usually written first. The coefficient of the leading term is called the **leading coefficient**. When a polynomial is written so that the powers are descending, we say that it is in standard form.



The same process can be applied to polynomial functions.

A **polynomial function** consists of either zero or the sum of a finite number of non-zero terms, each of which is a product of a number, called the coefficient of the term, and a variable raised to a non-negative integer power.

Polynomial Functions

Let n be a non-negative integer. A **polynomial function** is a function that can be written in the

form
$f(x)=a_n x^n+\ldots+a_2 x^2+a_1 x+a_0$
This is called the general form of a polynomial function. Each a_i is a **coefficient** and can be any real number, but a_n cannot $=0$. Each expression $a_i x^i$ is a **term of a polynomial function**. The number a_0 that is not multiplied by a variable is called a **constant**.

Identifying Polynomial Functions

Which of the following are polynomial functions?
$f(x)=2x^3 \cdot 3x+4$ $g(x)=-x(x^2-4)$ $h(x)=5\sqrt{x}+2$

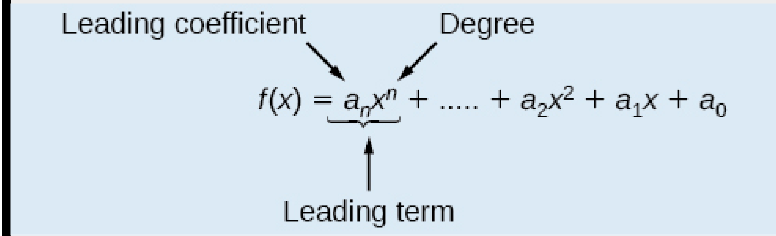
The first two functions are examples of polynomial functions because they can be written in the form $f(x)=a_n x^n+\ldots+a_2 x^2+a_1 x+a_0$, where the powers are non-negative integers and the coefficients are real numbers.

- $f(x)$ can be written as $f(x)=6x^4+4$.
- $g(x)$ can be written as $g(x)=-x^3+4x$.
- $h(x)$ cannot be written in this form and is therefore not a polynomial function.

Remember, like we discussed above, although the order of the terms in the polynomial function is not important for performing operations, we typically arrange the terms in descending order of power, or in general form. The **degree** of the polynomial is the highest power of the variable that occurs in the polynomial; it is the power of the first variable if the function is in general form. The **leading term** is the term containing the highest power of the variable, or the term with the highest degree. The **leading coefficient** is the coefficient of the leading term.

Terminology of Polynomial Functions

We often rearrange polynomials so that the powers are descending.



When a polynomial is written in this way, we say that it is in general form.

Given a polynomial function, identify the degree and leading coefficient.

1. Find the highest power of x to determine the degree function.
2. Identify the term containing the highest power of x to find the leading term.
3. Identify the coefficient of the leading term.

Identifying the Degree and Leading Coefficient of a Polynomial Function

Identify the degree, leading term, and leading coefficient of the following polynomial functions.

$f(x) = 3+2x^2-4x^3$ $g(t) = 5t^5-2t^3+7t$ $h(p) = 6p-p^3-2$

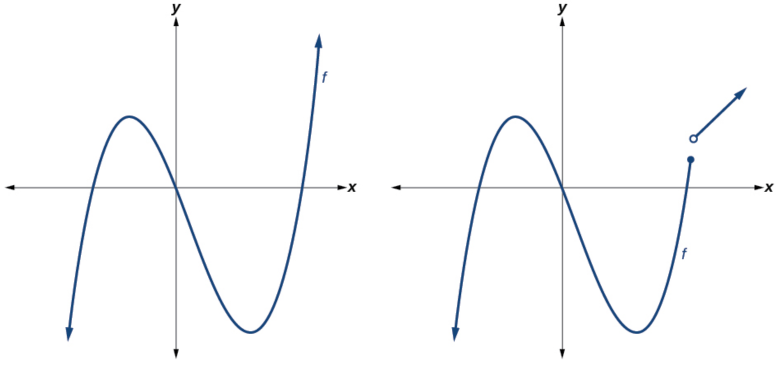
For the function $f(x)$, the highest power of x is 3, so the degree is 3. The leading term is the term containing that degree, $-4x^3$. The leading coefficient is the coefficient of that term, -4.

For the function $g(t)$, the highest power of t is 5, so the degree is 5. The leading term is the term containing that degree, $5t^5$. The leading coefficient is the coefficient of that

term, 5.

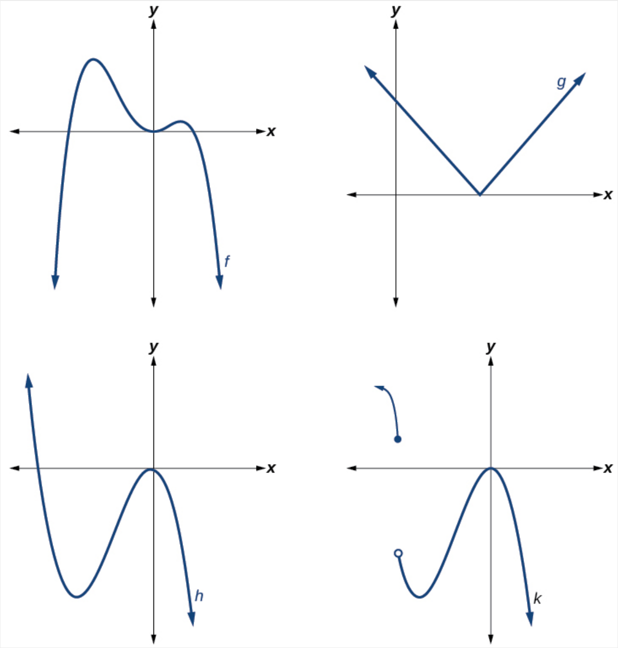
For the function $h(p)$, the highest power of p is 3, so the degree is 3. The leading term is the term containing that degree, $-p^3$. The leading coefficient is the coefficient of that term, -1.

Polynomial functions of degree 2 or more have graphs that do not have sharp corners; these types of graphs are called **smooth** curves. Polynomial functions also display graphs that have no breaks. Curves with no breaks are called **continuous**. [link] shows a graph that represents a polynomial function and a graph that represents a function that is not a polynomial.



Recognizing Polynomial Functions

Which of the graphs in [link] represents a polynomial function?



The graphs of f and h are graphs of polynomial functions. They are smooth and continuous.

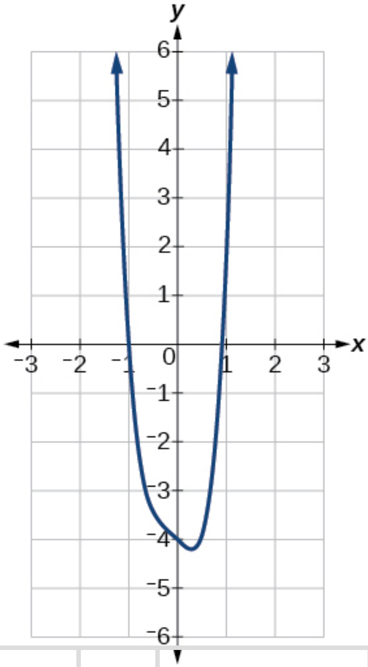
The graphs of g and k are graphs of functions that are not polynomials. The graph of function g has a sharp corner. The graph of function k is not continuous.

The behavior of the graph of a function as the input

values get very small ($x \to -\infty$) and get very large ($x \to \infty$) is referred to as the **end behavior** of the function. We can use words or symbols to describe end behavior.To describe the behavior as numbers become larger and larger, we use the idea of infinity. We use the symbol ∞ for positive infinity and $-\infty$ for negative infinity. When we say that " x approaches infinity," which can be symbolically written as $x \to \infty$, we are describing a behavior; we are saying that x is increasing without bound.

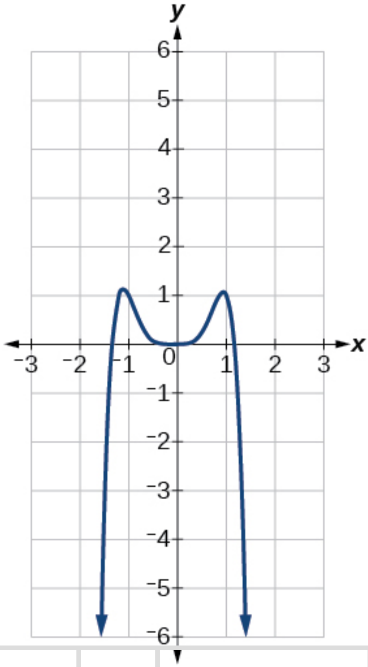
Knowing the degree of a polynomial function is useful in helping us predict its end behavior. To determine its end behavior, look at the leading term of the polynomial function. Because the power of the leading term is the highest, that term will grow significantly faster than the other terms as x gets very large or very small, so its behavior will dominate the graph. For any polynomial, the end behavior of the polynomial will match the end behavior of the leading term.

Polynomial Function	**Leading Term**	**Graph of Polynomial Function**
$f(x)=5x^4+2x^3-x-4$	$5x^4$	

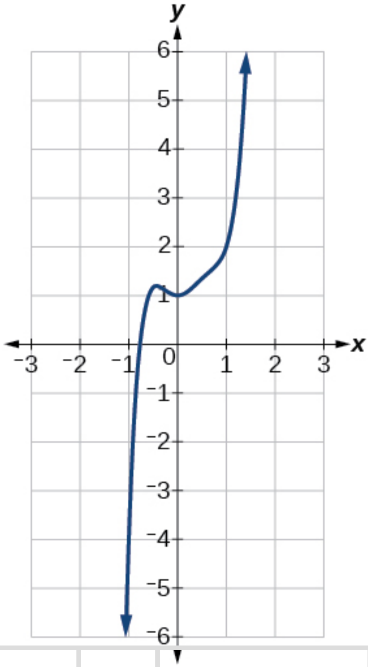


$f(x) = -2x^6 - x^5 + 3x^4 + x^3$

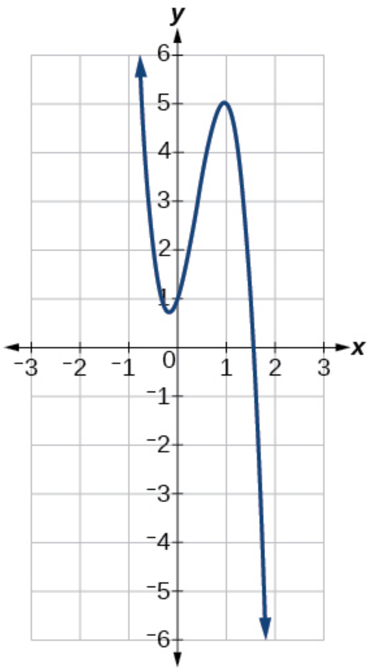
$-2x^6$



f(x) = 3 x 5 − 4 x 3 x 5
4 + 2 x 2 + 1

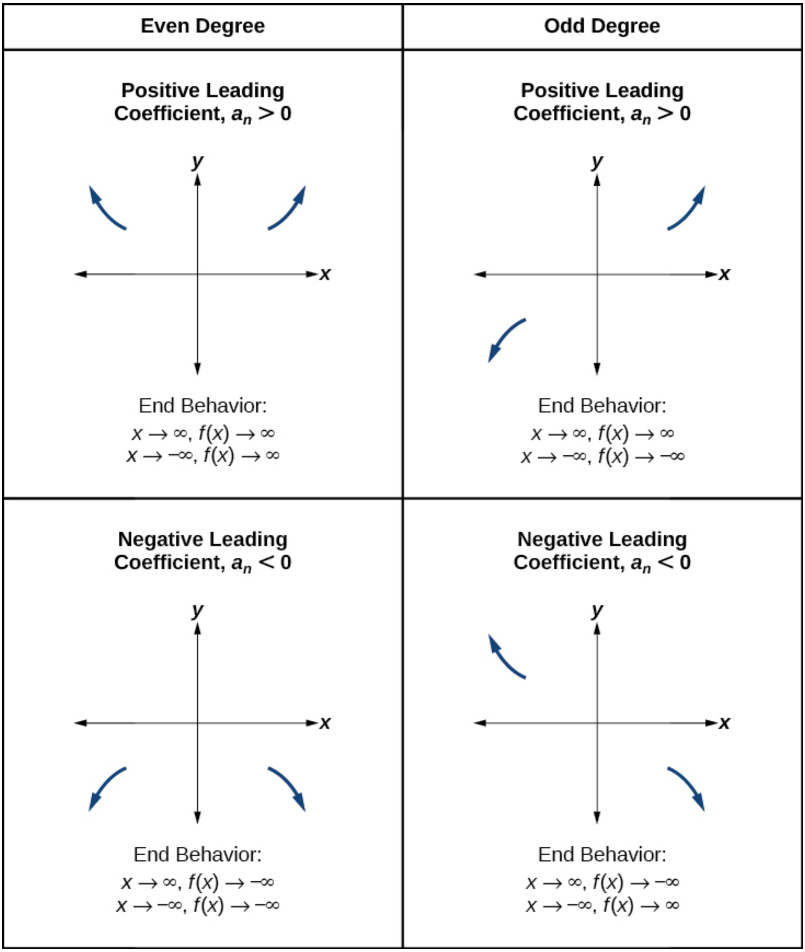


$f(x) = -6\ x\ 3\ + 7 - 6\ x\ 3$
$x\ 2\ + 3x + 1$



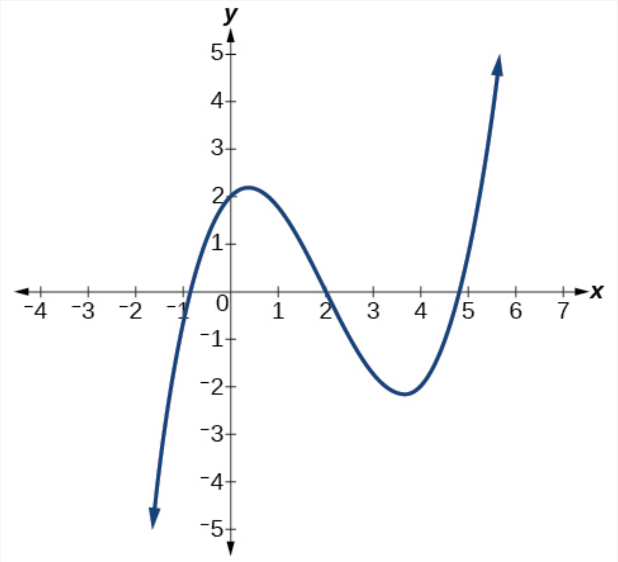
Leading Coeffiient Test

As we pointed out when discussing quadratic equations, when the leading term of a polynomial function, $a_n x^n$, is an even power function, as x increases or decreases without bound, $f(x)$ increases without bound. When the leading term is an odd power function, as x decreases without bound, $f(x)$ also decreases without bound; as x increases without bound, $f(x)$ also increases without bound. If the leading term is negative, it will change the direction of the end behavior. [link] summarizes all four cases.



Identifying End Behavior and Degree of a Polynomial Function

Describe the end behavior and determine a possible degree of the polynomial function in [link].



As the input values x get very large, the output values $f(x)$ increase without bound. As the input values x get very small, the output values $f(x)$ decrease without bound. We can describe the end behavior symbolically by writing

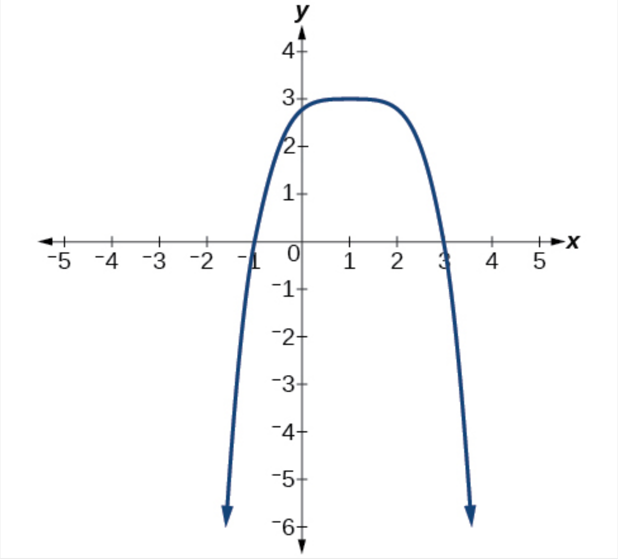
$$\text{as } x \to -\infty,\ f(x) \to -\infty \quad \text{as } x \to \infty,\ f(x) \to \infty$$

In words, we could say that as x values approach infinity, the function values approach infinity, and as x values approach negative infinity, the function values approach negative infinity.

We can tell this graph has the shape of an odd degree power function that has not been reflected, so the degree of the polynomial

creating this graph must be odd and the leading coefficient must be positive.

Describe the end behavior, and determine a possible degree of the polynomial function in [link].



As $x \to \infty$, $f(x) \to -\infty$; as $x \to -\infty$, $f(x) \to -\infty$. It has the shape of an even degree power function with a negative coefficient.

Identifying End Behavior and Degree of a Polynomial Function

Given the function $f(x) = -3x^2(x-1)(x+4)$, express the function as a polynomial in general form, and determine the leading term, degree, and end behavior of the function.

Obtain the general form by expanding the given expression for $f(x)$.

$f(x) = -3x^2(x-1)(x+4) = -3x^2(x^2+3x-4) = -3x^4 - 9x^3 + 12x^2$

The general form is $f(x) = -3x^4 - 9x^3 + 12x^2$. The leading term is $-3x^4$; therefore, the degree of the polynomial is 4. The degree is even (4) and the leading coefficient is negative (–3), so the end behavior is

as $x \to -\infty$, $f(x) \to -\infty$ as $x \to \infty$, $f(x) \to -\infty$

Given the function $f(x) = 0.2(x-2)(x+1)(x-5)$, express the function as a polynomial in general form and determine the leading term, degree, and end behavior of the function.

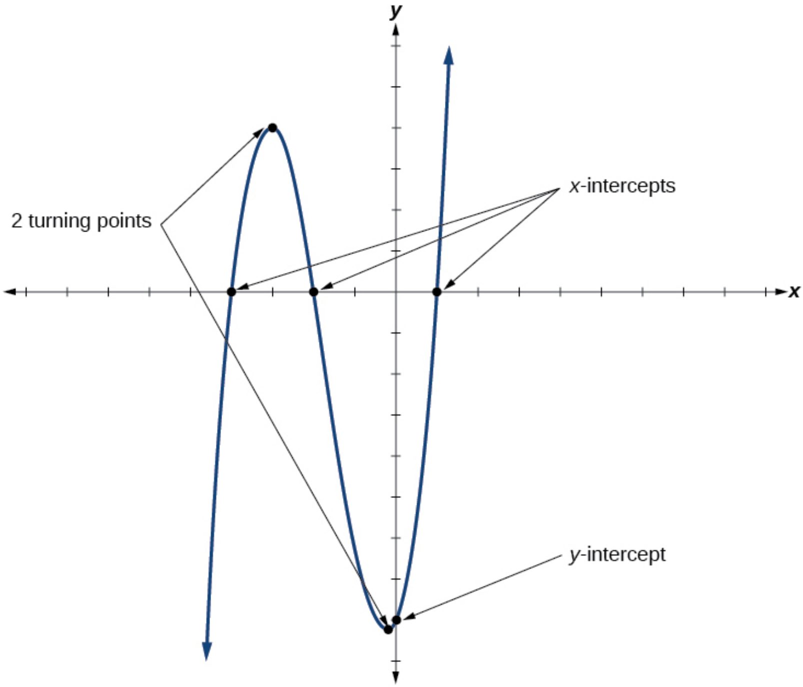
The leading term is $0.2x^3$, so it is a degree 3 polynomial. As x approaches positive infinity, $f(x)$ increases without bound; as x approaches negative infinity, $f(x)$ decreases without bound.

Turning Points, Zeros, and Intercepts of Polynomial Functions

In addition to the end behavior of polynomial functions, we are also interested in what happens in the "middle" of the function. In particular, we are interested in locations where graph behavior changes. A **turning point** is a point at which the function values change from increasing to decreasing or decreasing to increasing.

We are also interested in the intercepts. As with all functions, the *y*-intercept is the point at which the graph intersects the vertical axis. The point corresponds to the coordinate pair in which the input value is zero. Because a polynomial is a function, only one output value corresponds to each input value so there can be only one *y*-intercept $(0, a_0)$. The *x*-intercepts occur at the input values that correspond to an output value of zero. It is possible

to have more than one x-intercept. See [link].

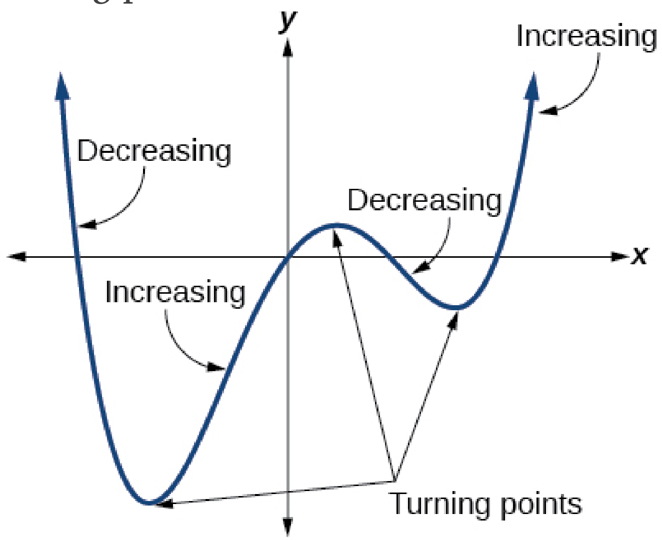


The degree of a polynomial function helps us to determine the number of x-intercepts and the number of turning points. A polynomial function of nth degree is the product of n factors, so it will have at most n roots or zeros, or x-intercepts. The graph of the polynomial function of degree n must have at most $n–1$ turning points. This means the graph has at most one fewer turning point than the degree of the polynomial or one fewer than the number of factors.

Turning Points of Polynomial Functions

A **turning point** of a graph is a point at which the graph changes direction from increasing to decreasing (rising to falling) or decreasing to increasing (falling to rising).
A polynomial of degree n will have at most $n-1$ turning points.

Look at the graph of the polynomial function $f(x)=x^4-x^3-4x^2+4x$ in [link]. The graph has three turning points.



This function f is a 4^{th} degree polynomial function and has 3 turning points. The maximum number of turning points of a polynomial function is always one less than the degree of the function.

A **continuous function** has no breaks in its graph:

the graph can be drawn without lifting the pen from the paper. A **smooth curve** is a graph that has no sharp corners. The turning points of a smooth graph must always occur at rounded curves. The graphs of polynomial functions are both continuous and smooth.

Finding the Maximum Number of Turning Points Using the Degree of a Polynomial Function

Find the maximum number of turning points of each polynomial function.

1. $f(x)=-x^3+4x^5-3x^2+1$
2. $f(x)=-(x-1)^2(1+2x^2)$

---

1. First, rewrite the polynomial function in descending order: $f(x)=4x^5-x^3-3x^2+1$

   Identify the degree of the polynomial function. This polynomial function is of degree 5.

   The maximum number of turning points is $5-1=4$.

2. First, identify the leading term of the polynomial function if the function were expanded.

$$f(x) = -(x-1)^2\,(1+2x^2)$$
$$a_n = -(x^2)\,(2x^2)\ -2x^4$$

Then, identify the degree of the polynomial function. This polynomial function is of degree 4.

The maximum number of turning points is $4-1=3$.

Finding Zeros (x-intercepts)
Recall that if f is a polynomial function, the values of x for which $f(x)=0$ are called zeros of f. If the equation of the polynomial function can be factored, we can set each factor equal to zero and solve for the zeros.

We can use this method to find x- intercepts because at the x- intercepts we find the input values when the output value is zero. For general polynomials, this can be a challenging prospect. While quadratics can be solved using the relatively simple quadratic formula, the corresponding

formulas for cubic and fourth-degree polynomials are not simple enough to remember, and formulas do not exist for general higher-degree polynomials. Consequently, we will limit ourselves to three cases:

1. The polynomial can be factored using known methods: greatest common factor and trinomial factoring.
2. The polynomial is given in factored form.
3. Technology is used to determine the intercepts.

Given a polynomial function f, find the *x*-intercepts by factoring.

1. Set $f(x)=0$.
2. If the polynomial function is not given in factored form:
    1. Factor out any common monomial factors.
    2. Factor any factorable binomials or trinomials.
3. Set each factor equal to zero and solve to find the x- intercepts.

Finding the *x*-Intercepts of a Polynomial

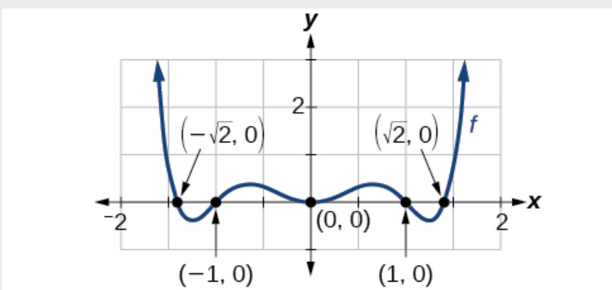
Function by Factoring

Find the *x*-intercepts of $f(x)=x^6-3x^4+2x^2$.

We can attempt to factor this polynomial to find solutions for $f(x)=0$.

$x^6-3x^4+2x^2=0$ Factor out the greatest common factor. $x^2(x^4-3x^2+2)=0$ Factor the trinomial. $x^2(x^2-1)(x^2-2)=0$ Set each factor equal to zero. $(x^2-1)=0$ $(x^2-2)=0$ $x^2=0$ or $x^2=1$ or $x^2=2$ $x=0$ $x=\pm 1$ $x=\pm 2$

This gives us five *x*-intercepts: $(0,0),(1,0),(-1,0),(2,0)$, and $(-2,0)$. See [link]. We can see that this is an even function because it is symmetric about the *y*-axis.



Finding the x-Intercepts of a Polynomial Function by Factoring

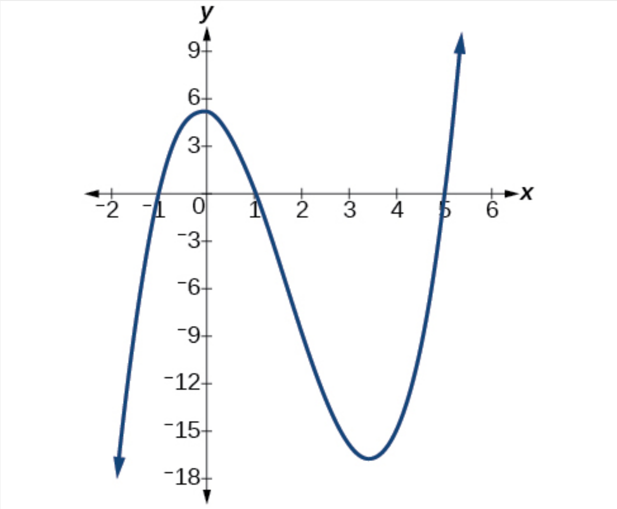
Find the x-intercepts of $f(x) = x^3 - 5x^2 - x + 5$.

Find solutions for $f(x) = 0$ by factoring.

$$\begin{aligned} x^3 - 5x^2 - x + 5 &= 0 && \text{Factor by grouping.} \\ x^2(x-5) - (x-5) &= 0 && \text{Factor out the common factor.} \\ (x^2-1)(x-5) &= 0 && \text{Factor the difference of squares.} \\ (x+1)(x-1)(x-5) &= 0 && \text{Set each factor equal to zero.} \end{aligned}$$

$$\begin{aligned} x+1 &= 0 \text{ or } x-1 = 0 \text{ or } x-5 = 0 \\ x &= -1 \quad x = 1 \quad x = 5 \end{aligned}$$

There are three x-intercepts: $(-1,0)$, $(1,0)$, and $(5,0)$. See [link].



Now that we know how to find the X-intercepts, lets combine finding both x- and y- intercepts.
Remember:

Intercepts of Polynomial Functions
The *y*-intercept is the point at which the function has an input value of zero. The *x*-intercepts are the points at which the output value is zero.
Given a polynomial function, determine the intercepts.

1. Determine the *y*-intercept by setting $x=0$ and finding the corresponding output value.
2. Determine the *x*-intercepts by solving for the input values that yield an output value of zero.

Finding the *y*- and *x*-Intercepts of a Polynomial in Factored Form

Find the *y*- and *x*-intercepts of $g(x)= (x-2) 2 (2x+3)$.

The *y*-intercept can be found by evaluating $g(0)$.
$g(0) = (0-2) 2 (2(0)+3) = 12$

So the *y*-intercept is $(0,12)$.

The *x*-intercepts can be found by solving $g(x)=0$.

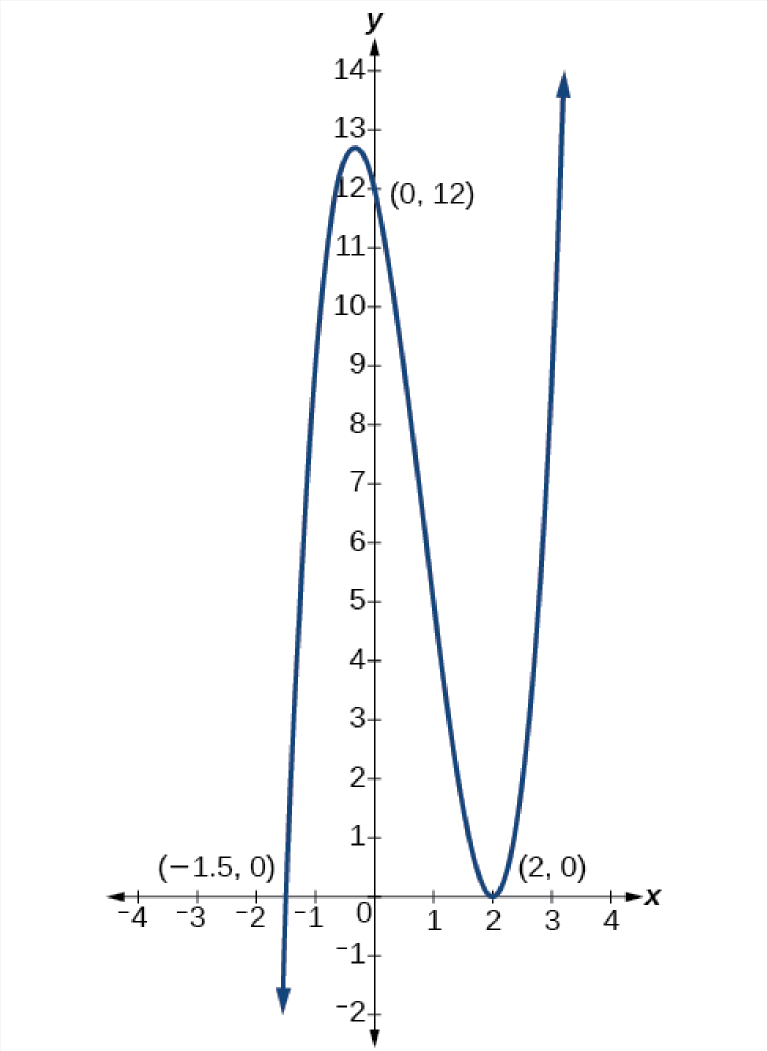
$$(x-2)^2(2x+3)=0$$

$$(x-2)^2 = 0 \quad (2x+3) = 0 \quad x-2 = 0 \text{ or } x = -\frac{3}{2} \quad x = 2$$

So the *x*-intercepts are $(2,0)$ and $\left(-\frac{3}{2},0\right)$.

Analysis

We can always check that our answers are reasonable by using a graphing calculator to graph the polynomial as shown in [link].



Determining the Intercepts of a Polynomial Function

Given the polynomial function $f(x) = (x - 2)(x$

$+1)(x-4),$ written in factored form for your convenience, determine the *y*- and *x*-intercepts.

The *y*-intercept occurs when the input is zero so substitute 0 for x.

$$f(0) = (0)^4 - 4(0)^2 - 45 = -45$$

The *y*-intercept is (0, 8).

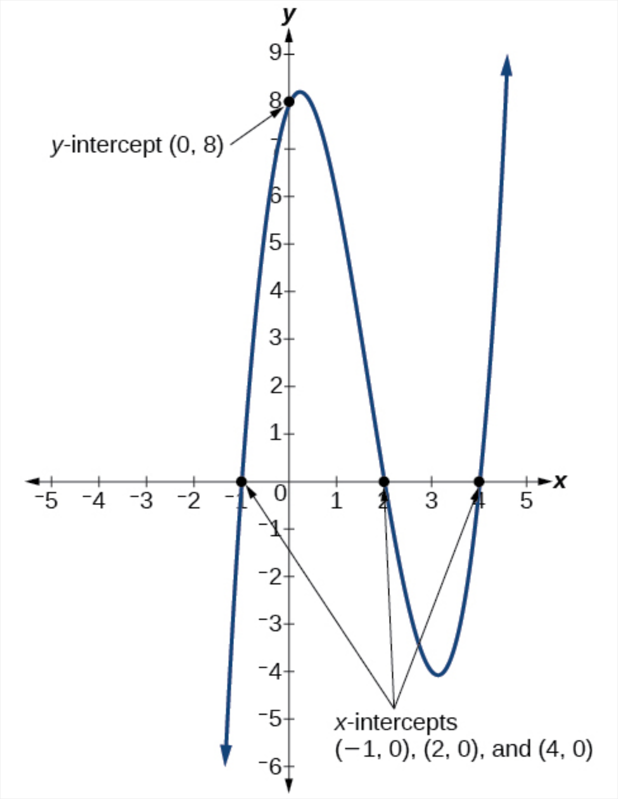
The *x*-intercepts occur when the output is zero.

$$0 = (x-2)(x+1)(x-4)$$

$$x-2 = 0 \text{ or } x+1 = 0 \text{ or } x-4 = 0 \quad x = 2 \text{ or } x = -1 \text{ or } x = 4$$

The *x*-intercepts are $(2,0), (–1,0),$ and $(4,0).$

We can see these intercepts on the graph of the function shown in [link].



Determining the Intercepts of a Polynomial Function with Factoring

Given the polynomial function $f(x) = x^4 - 4x^2 - 45$, determine the *y*- and *x*-intercepts.

The *y*-intercept occurs when the input is zero.

$$f(0) = (0)^4 - 4(0)^2 - 45 = -45$$

The y-intercept is $(0,-45)$.

The x-intercepts occur when the output is zero. To determine when the output is zero, we will need to factor the polynomial.

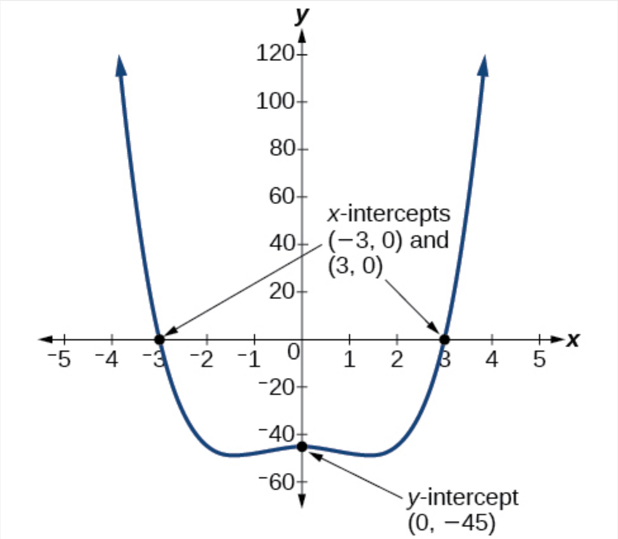
$$f(x) = x^4 - 4x^2 - 45 = (x^2 - 9)(x^2 + 5) = (x-3)(x+3)(x^2+5)$$

$$0 = (x-3)(x+3)(x^2+5)$$

$$x-3 = 0 \text{ or } x+3 = 0 \text{ or } x^2+5=0 \quad x = 3 \text{ or } x = -3 \text{ or (no real solution)}$$

The x-intercepts are $(3,0)$ and $(–3,0)$.

We can see these intercepts on the graph of the function shown in [link]. We can see that the function is even because $f(x) = f(-x)$.



Given the polynomial function $f(x)=2x^3-6x^2-20x$, determine the *y*- and *x*-intercepts.

y-intercept $(0,0)$; *x*-intercepts $(0,0),(–2,0)$, and $(5,0)$

Intercepts and Turning Points of Polynomials

A polynomial of degree n will have, at most, n *x*-intercepts and $n-1$ turning points.

Determining the Number of Intercepts and Turning Points of a Polynomial

Without graphing the function, determine the local behavior of the function by finding the maximum number of *x*-intercepts and turning points for $f(x)=-3x^{10}+4x^7-x^4+2x^3$.

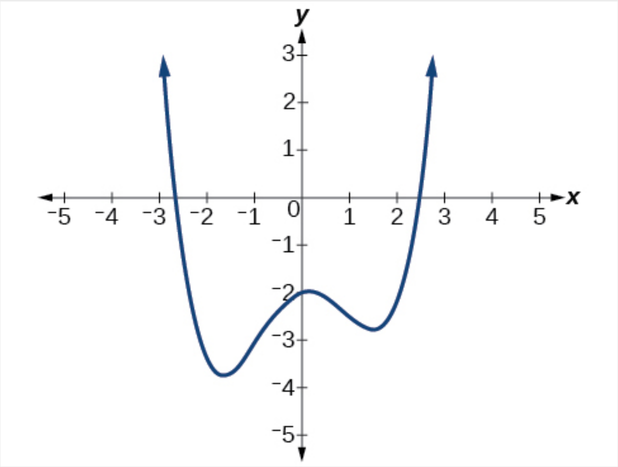
The polynomial has a degree of 10, so there are at most 10 *x*-intercepts and at most 9 turning points.

Without graphing the function, determine the maximum number of *x*-intercepts and turning points for $f(x)=108-13x^9-8x^4+14x^{12}+2x^3$.

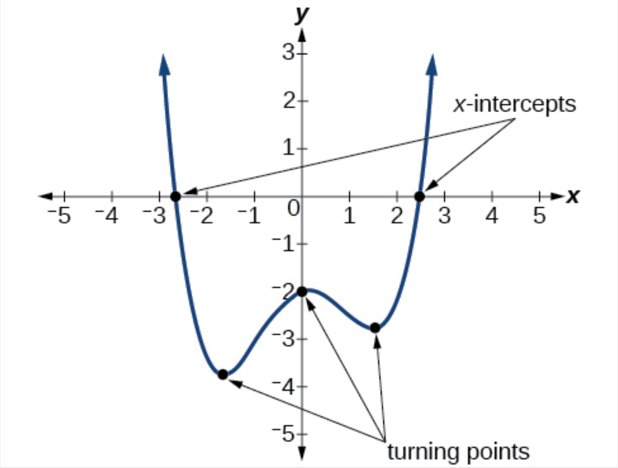
There are at most 12 x- intercepts and at most 11 turning points.

Drawing Conclusions about a Polynomial Function from the Graph

What can we conclude about the polynomial represented by the graph shown in [link] based on its intercepts and turning points?

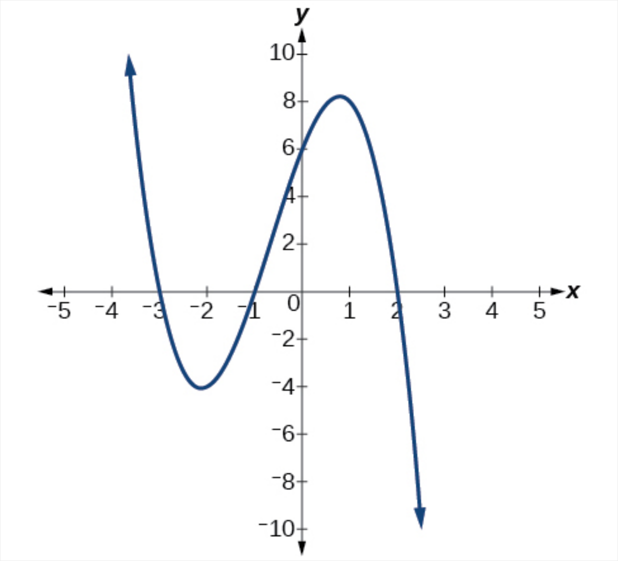


The end behavior of the graph tells us this is the graph of an even-degree polynomial. See [link].



The graph has 2 x-intercepts, suggesting a degree of 2 or greater, and 3 turning points, suggesting a degree of 4 or greater. Based on this, it would be reasonable to conclude that the degree is even and at least 4.

What can we conclude about the polynomial represented by the graph shown in [link] based on its intercepts and turning points?



The end behavior indicates an odd-degree polynomial function; there are 3 x- intercepts and 2 turning points, so the degree is odd and at least 3. Because of the end behavior, we know that the lead coefficient must be negative.

Drawing Conclusions about a Polynomial Function from the Factors

Given the function $f(x) = -4x(x+3)(x-4)$, determine the x- and y-intercepts, and the number of turning points.

The *y*-intercept is found by evaluating $f(0)$.
$f(0) = -4(0)(0+3)(0-4 = 0$

The *y*-intercept is $(0,0)$.

The *x*-intercepts are found by determining the zeros of the function.
$0 = -4x(x+3)(x-4)$
$x = 0$ or $x+3 = 0$ or $x-4 = 0$ $x = 0$ or $x = -3$ or $x = 4$

The *x*-intercepts are $(0,0), (–3,0),$ and $(4,0)$.

The degree is 3 so the graph has at most 2 turning points.

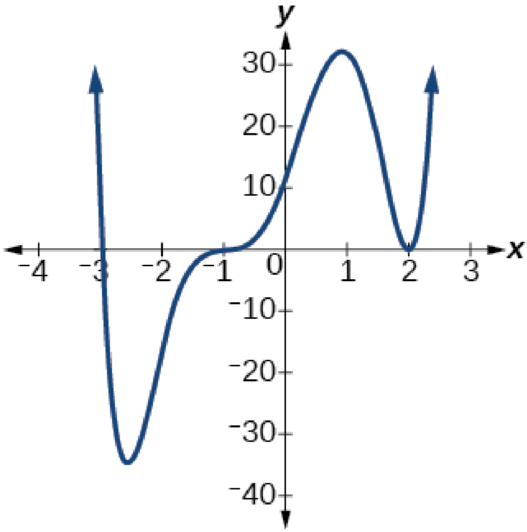
Identifying the behavior of the graph at an *x*-intercept by examining the multiplicity of the zero.

Zeros and Multiplicities

Graphs behave differently at various *x*-intercepts. Sometimes, the graph will cross over the horizontal axis at an intercept. Other times, the graph will touch the horizontal axis and "bounce" off.

Suppose, for example, we graph the function shown.
$f(x) = (x+3)\ (x-2)\ 2\ (x+1)\ 3$

Notice in [link] that the behavior of the function at each of the *x*-intercepts is different.



The *x*-intercept $x=-3$ is the solution of equation $(x+3)=0$. The graph passes directly through the *x*-intercept at $x=-3$. The factor is linear (has a degree of 1), so the behavior near the intercept is like that of a line—it passes directly through the intercept. We call this a single zero because the zero corresponds to a single factor of the function.

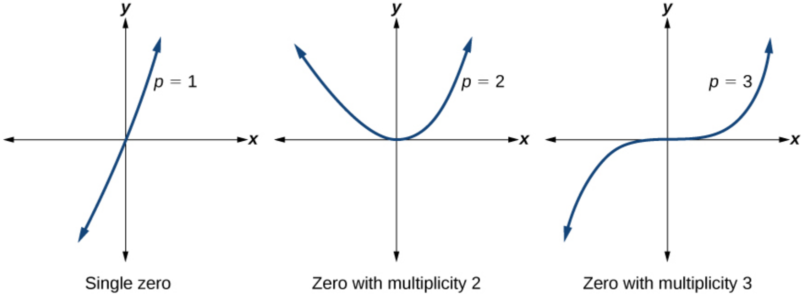
The *x*-intercept $x=2$ is the repeated solution of equation $(x-2)^2=0$. The graph touches the axis at the intercept and changes direction. The factor is quadratic (degree 2), so the behavior near the intercept is like that of a quadratic—it bounces off of the horizontal axis at the intercept.

$$(x-2)^2=(x-2)(x-2)$$

The factor is repeated, that is, the factor $(x-2)$ appears twice. The number of times a given factor appears in the factored form of the equation of a polynomial is called the **multiplicity**. The zero associated with this factor, $x=2,$ has multiplicity 2 because the factor $(x-2)$ occurs twice.

The *x*-intercept $x=-1$ is the repeated solution of factor $(x+1)^3=0.$ The graph passes through the axis at the intercept, but flattens out a bit first. This factor is cubic (degree 3), so the behavior near the intercept is like that of a cubic—with the same S-shape near the intercept as the toolkit function $f(x)=x^3.$ We call this a triple zero, or a zero with multiplicity 3.

For zeros with even multiplicities, the graphs *touch* or are tangent to the *x*-axis. For zeros with odd multiplicities, the graphs *cross* or intersect the *x*-axis. See [link] for examples of graphs of polynomial functions with multiplicity 1, 2, and 3.



For higher even powers, such as 4, 6, and 8, the graph will still touch and bounce off of the horizontal axis but, for each increasing even power,

the graph will appear flatter as it approaches and leaves the x-axis.

For higher odd powers, such as 5, 7, and 9, the graph will still cross through the horizontal axis, but for each increasing odd power, the graph will appear flatter as it approaches and leaves the x-axis.

Graphical Behavior of Polynomials at x-Intercepts

If a polynomial contains a factor of the form $(x-h)^p$, the behavior near the x-intercept h is determined by the power p. We say that $x=h$ is a zero of **multiplicity** p.

The graph of a polynomial function will touch the x-axis at zeros with even multiplicities. The graph will cross the x-axis at zeros with odd multiplicities.

The sum of the multiplicities is the degree of the polynomial function.

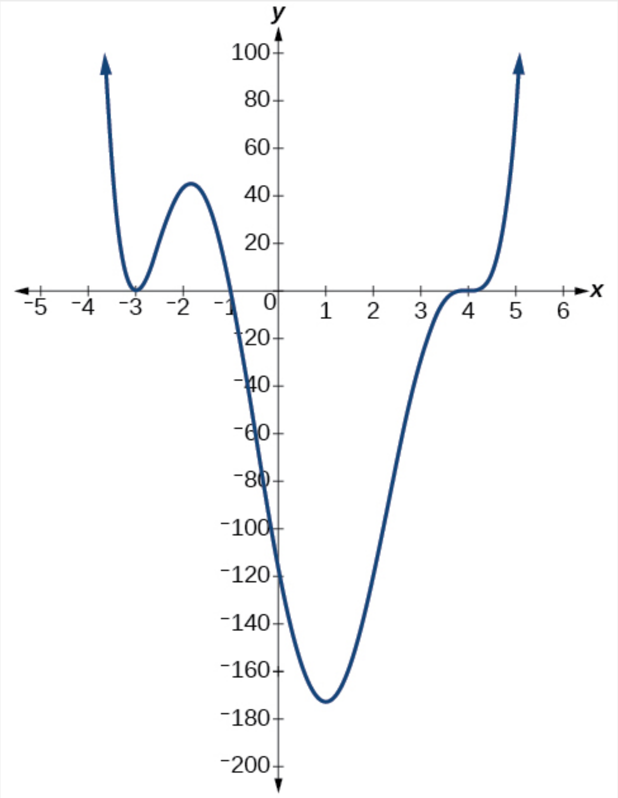
Given a graph of a polynomial function of degree n, identify the zeros and their multiplicities.

1. If the graph crosses the x-axis and appears almost linear at the intercept, it is a single zero.

2. If the graph touches the x-axis and bounces off of the axis, it is a zero with even multiplicity.
3. If the graph crosses the x-axis at a zero, it is a zero with odd multiplicity.
4. The sum of the multiplicities is n.

Identifying Zeros and Their Multiplicities

Use the graph of the function of degree 6 in [link] to identify the zeros of the function and their possible multiplicities.



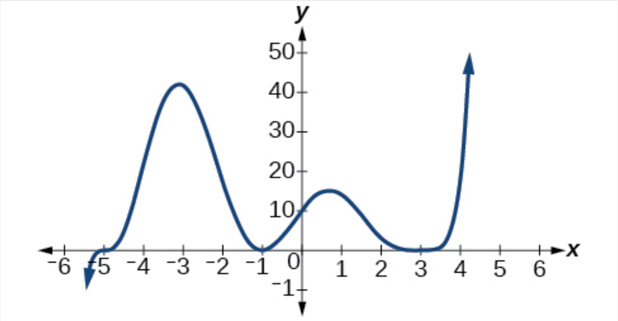
The polynomial function is of degree 6. The sum of the multiplicities must be 6.

Starting from the left, the first zero occurs at $x = -3$. The graph touches the *x*-axis, so the multiplicity of the zero must be even. The zero of -3 most likely has multiplicity 2.

The next zero occurs at $x = -1$. The graph looks almost linear at this point. This is a single zero of multiplicity 1.

The last zero occurs at $x=4$. The graph crosses the *x*-axis, so the multiplicity of the zero must be odd. We know that the multiplicity is likely 3 and that the sum of the multiplicities is 6.

Use the graph of the function of degree 9 in [link] to identify the zeros of the function and their multiplicities.



The graph has a zero of –5 with multiplicity 3, a zero of -1 with multiplicity 2, and a zero of 3 with multiplicity 4.

Graphing Polynomial Functions

We can use what we have learned about multiplicities, end behavior, and turning points to sketch graphs of polynomial functions. Let us put this all together and look at the steps required to graph polynomial functions.

Given a polynomial function, sketch the graph.

1. Find the intercepts.
2. Check for symmetry. If the function is an even function, its graph is symmetrical about the y-axis, that is, $f(-x)=f(x)$. If a function is an odd function, its graph is symmetrical about the origin, that is, $f(-x)=-f(x)$.
3. Use the multiplicities of the zeros to determine the behavior of the polynomial at the x-intercepts.
4. Determine the end behavior by examining the leading term.
5. Use the end behavior and the behavior at the intercepts to sketch a graph.
6. Ensure that the number of turning points does not exceed one less than the degree of the polynomial.
7. Optionally, use technology to check the graph.

Sketching the Graph of a Polynomial Function

Sketch a graph of $f(x) = -2(x+3)^2(x-5)$.

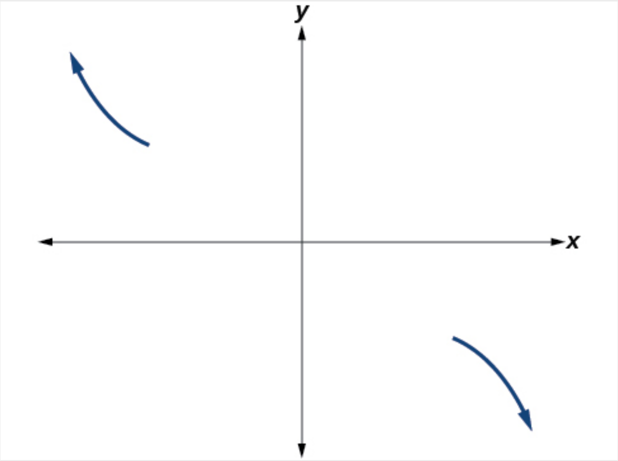
This graph has two *x*-intercepts. At $x = -3$, the factor is squared, indicating a multiplicity of 2. The graph will bounce at this *x*-intercept. At $x = 5$, the function has a multiplicity of one, indicating the graph will cross through the axis at this intercept.

The *y*-intercept is found by evaluating $f(0)$.

$$f(0) = -2(0+3)^2(0-5) = -2 \cdot 9 \cdot (-5) = 90$$

The *y*-intercept is $(0, 90)$.

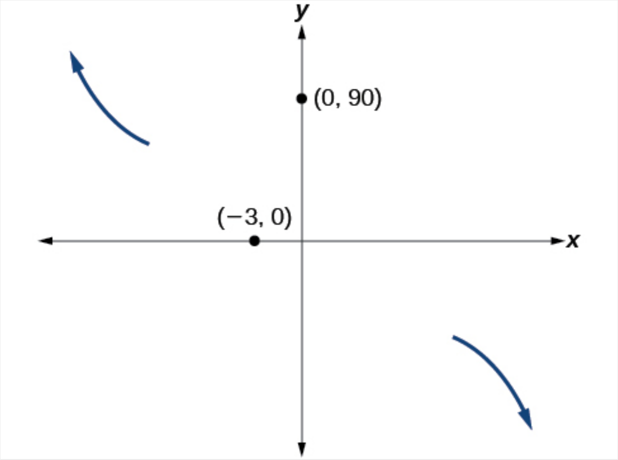
Additionally, we can see the leading term, if this polynomial were multiplied out, would be $-2x^3$, so the end behavior is that of a vertically reflected cubic, with the outputs decreasing as the inputs approach infinity, and the outputs increasing as the inputs approach negative infinity. See [link].



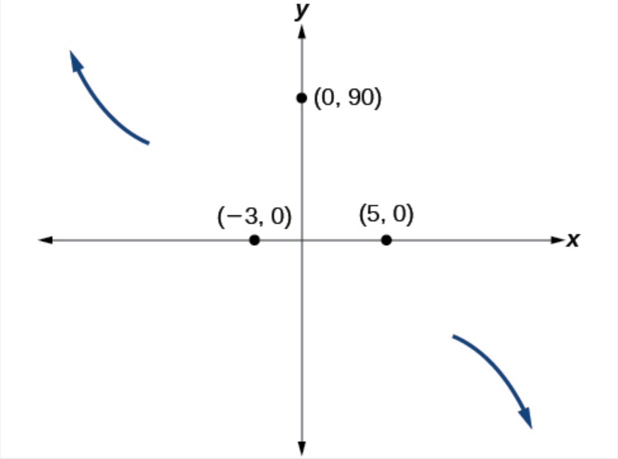
To sketch this, we consider that:

- As $x \to -\infty$ the function $f(x) \to \infty$, so we know the graph starts in the second quadrant and is decreasing toward the x-axis.
- Since $f(-x) = -2(-x+3)^2(-x-5)$ is not equal to $f(x)$, the graph does not display symmetry.
- At $(-3,0)$, the graph bounces off of the *x*-axis, so the function must start increasing.

  At $(0,90)$, the graph crosses the *y*-axis at the *y*-intercept. See [link].



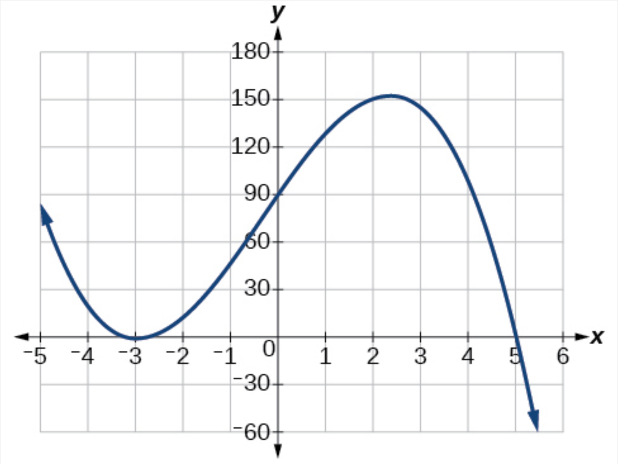
Somewhere after this point, the graph must turn back down or start decreasing toward the horizontal axis because the graph passes through the next intercept at $(5,0)$. See [link].



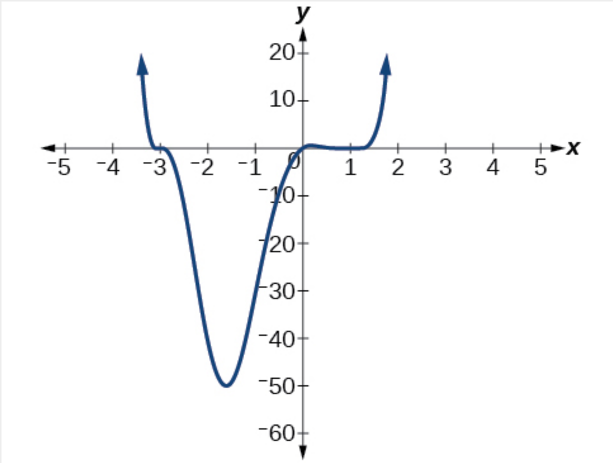
As $x\to\infty$ the function $f(x)\to-\infty$, so we know the graph continues to decrease, and we can stop drawing the graph in the fourth quadrant.

Using technology, we can create the graph for the polynomial function, shown in [link], and verify that the resulting graph looks like our sketch in [link].

The complete graph of the polynomial function $f(x) = -2(x+3)^2(x-5)$



Sketch a graph of $f(x) = \frac{1}{4}x(x-1)^4(x+3)^3$.



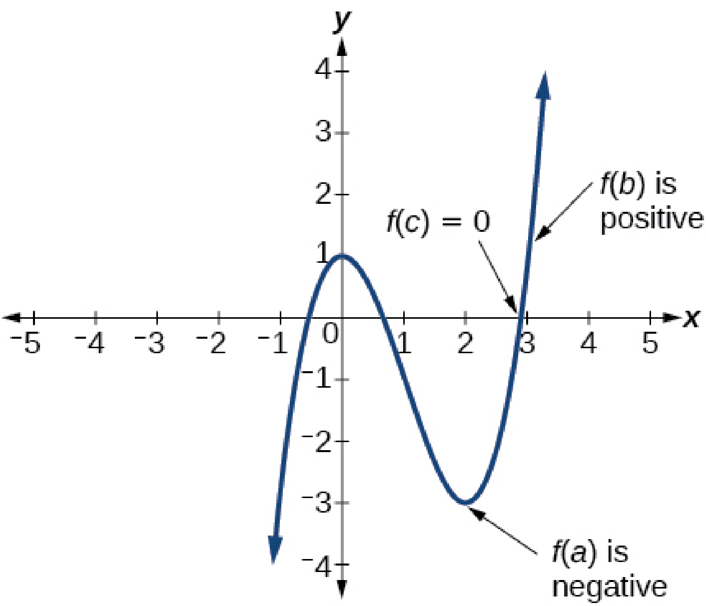
Using the Intermediate Value Theorem to show there exists a zero.

Intermediate Value Theorem

In some situations, we may know two points on a graph but not the zeros. If those two points are on opposite sides of the *x*-axis, we can confirm that there is a zero between them. Consider a polynomial function f whose graph is smooth and continuous. The **Intermediate Value Theorem** states that for two numbers a and b in the domain of f, if $a < b$ and $f(a) \neq f(b)$, then the function f takes on every value between $f(a)$ and $f(b)$. (While the theorem is intuitive, the proof is actually quite complicated and requires higher mathematics.) We can apply

this theorem to a special case that is useful in graphing polynomial functions. If a point on the graph of a continuous function f at $x=a$ lies above the x- axis and another point at $x=b$ lies below the x- axis, there must exist a third point between $x=a$ and $x=b$ where the graph crosses the x- axis. Call this point $(c, f(c))$. This means that we are assured there is a solution c where $f(c)=0$.

In other words, the Intermediate Value Theorem tells us that when a polynomial function changes from a negative value to a positive value, the function must cross the x- axis. [link] shows that there is a zero between a and b.



Intermediate Value Theorem

Let f be a polynomial function. The **Intermediate Value Theorem** states that if $f(a)$ and $f(b)$ have opposite signs, then there exists at least one value c between a and b for which $f(c)=0$.

Using the Intermediate Value Theorem

Show that the function $f(x)=x^3-5x^2+3x+6$ has at least two real zeros between $x=1$ and $x=4$.

As a start, evaluate $f(x)$ at the integer values $x=1,2,3,$ and 4. See [link].

x	1	2	3	4
f(x)	5	0	–3	2

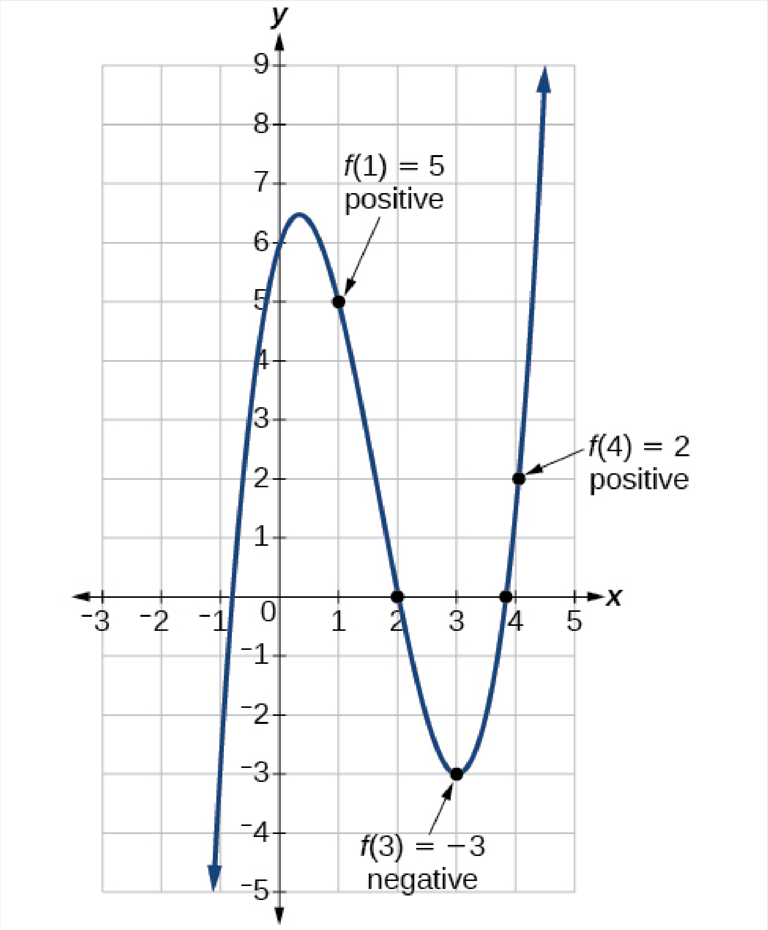
We see that one zero occurs at $x=2$. Also, since $f(3)$ is negative and $f(4)$ is positive, by the Intermediate Value Theorem, there must be at least one real zero between 3 and 4.

We have shown that there are at least two real

zeros between $x=1$ and $x=4$.

Analysis

We can also see on the graph of the function in [link] that there are two real zeros between $x=1$ and $x=4$.



Show that the function $f(x)=7x^5-9x^4-x^2$ has at least one real zero between $x=1$ and $x=2$.

Because f is a polynomial function and since $f(1)$ is negative and $f(2)$ is positive, there is at least one real zero between $x=1$ and $x=2$.

Access these online resources for additional instruction and practice with power and polynomial functions.

- Find Key Information about a Given Polynomial Function
- End Behavior of a Polynomial Function
- Turning Points and x- intercepts of Polynomial Functions
- Least Possible Degree of a Polynomial Function

Key Concepts

- **Polynomials**

- **Polynomial**—A monomial, or two or more algebraic terms combined by addition or subtraction is a polynomial.
- **monomial** —A polynomial with exactly one term is called a monomial.
- **binomial** — A polynomial with exactly two terms is called a binomial.
- **trinomial** —A polynomial with exactly three terms is called a trinomial.

- **Degree of a Polynomial**

    - The **degree of a term** is the sum of the exponents of its variables.
    - The **degree of a constant** is 0.
    - The **degree of a polynomial** is the highest degree of all its terms.

- The behavior of a graph as the input decreases beyond bound and increases beyond bound is called the end behavior.
- The end behavior depends on whether the power is even or odd. See [link] and [link].
- A polynomial function is the sum of terms, each of which consists of a transformed power function with positive whole number power. See [link].
- The degree of a polynomial function is the highest power of the variable that occurs in a polynomial. The term containing the highest power of the variable is called the leading

term. The coefficient of the leading term is called the leading coefficient. See [link].

- The end behavior of a polynomial function is the same as the end behavior of the power function represented by the leading term of the function. See [link] and [link].
- Polynomial functions of degree 2 or more are smooth, continuous functions. See [link].
- To find the zeros of a polynomial function, if it can be factored, factor the function and set each factor equal to zero. See [link], [link], and [link].
- Another way to find the x- intercepts of a polynomial function is to graph the function and identify the points at which the graph crosses the x- axis. See [link].
- The multiplicity of a zero determines how the graph behaves at the x- intercepts. See [link].
- The graph of a polynomial will cross the horizontal axis at a zero with odd multiplicity.
- The graph of a polynomial will touch the horizontal axis at a zero with even multiplicity.
- The end behavior of a polynomial function depends on the leading term.
- The graph of a polynomial function changes direction at its turning points.
- A polynomial function of degree n has at most $n-1$ turning points. See [link].
- To graph polynomial functions, find the zeros and their multiplicities, determine the end behavior, and ensure that the final graph has at

most $n-1$ turning points. See [link] and [link].
- The Intermediate Value Theorem tells us that if $f(a)$ and $f(b)$ have opposite signs, then there exists at least one value c between a and b for which $f(c)=0$. See [link].

Practice Makes Perfect

Determine the Type of Polynomials

In the following exercises, determine if the polynomial is a monomial, binomial, trinomial, or other polynomial.

ⓐ $47x5-17x2y3+y2$
ⓑ $5c3+11c2-c-8$
ⓒ $59ab+13b$
ⓓ 4
ⓔ $4pq+17$

ⓐ trinomial, 5 ⓑ polynomial, 3 ⓒ binomial, 2
ⓓ monomial, 0
ⓔ binomial, 1

ⓐ $8y - 5x$
ⓑ $y2 - 5yz - 6z2$
ⓒ $y3 - 8y2 + 2y - 16$
ⓓ $81ab4 - 24a2b2 + 3b$
ⓔ -18

ⓐ binomial ⓑ trinomial
ⓒ polynomial ⓓ trinomial
ⓔ monomial

For the following exercises, find the degree and leading coefficient for the given polynomial.

$7 - 2 x 2$

Degree $=$ 2, Coefficient $=$ –2

$x(4 - x 2)(2x + 1)$

Degree $= 4$, Coefficient $=$ –2

For the following exercises, determine the end behavior of the functions.

$f(x) = x 4$

As $x \to \infty$, $f(x) \to \infty$, as $x \to -\infty$, $f(x) \to \infty$

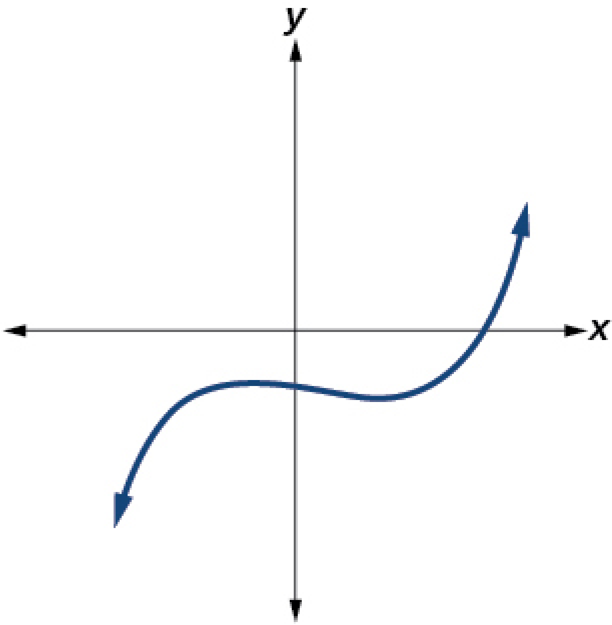
$f(x) = -2x^4 - 3x^2 + x - 1$

As $x \to -\infty$, $f(x) \to -\infty$, as $x \to \infty$, $f(x) \to -\infty$

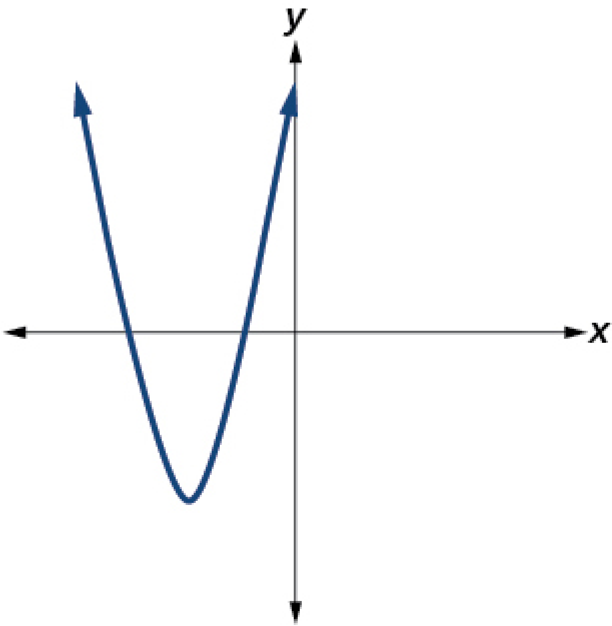
$f(x) = x^2(2x^3 - x + 1)$

As $x \to \infty$, $f(x) \to \infty$, as $x \to -\infty$, $f(x) \to -\infty$

For the following exercises, determine whether the graph of the function provided is a graph of a polynomial function. If so, determine the number of turning points and the least possible degree for the function.



Yes. Number of turning points is 2. Least possible degree is 3.



Yes. Number of turning points is 1. Least possible degree is 2.

For the following exercises, find the x- or *t*-intercepts (zeros) of the polynomial functions.

$C(t)=3(t+2)(t-3)(t+5)$

$(-2,0),(3,0),(-5,0)$

$C(t)=2t(t-3)(t+1)2$

$(3,0),(-1,0),(0,0)$

$f(x)=x^3+x^2-20x$

$(0,0),(-5,0),(4,0)$

$f(x)=2x^3-x^2-8x+4$

$(-2,0),(2,0),(\frac{1}{2},0)$

For the following exercises, find the intercepts of the functions.

$f(t)=2(t-1)(t+2)(t-3)$

---

y-intercept is $(0,12)$, *t*-intercepts are $(1,0);(-2,0);$and $(3,0)$.

$f(x)=x(x^2-2x-8)$

---

y-intercept is $(0,0)$. *x*-intercepts are $(0,0)$, $(4,0)$, and $(-2,0)$.

For the following exercises, find the zeros and give

the multiplicity of each.

$f(x)= x^2 (2x+3)^5 (x-4)^2$

---

0 with multiplicity 2, $-\frac{3}{2}$ with multiplicity 5, 4 with multiplicity 2

$f(x)= x^2 (x^2 +4x+4)$

---

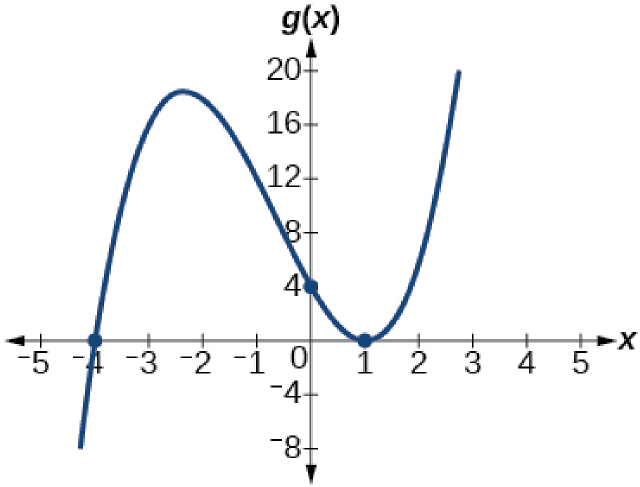
0 with multiplicity 2, –2 with multiplicity 2

For the following exercises, graph the polynomial functions. Note x- and y- intercepts, multiplicity, and end behavior.

$g(x)=(x+4)(x-1)^2$

---

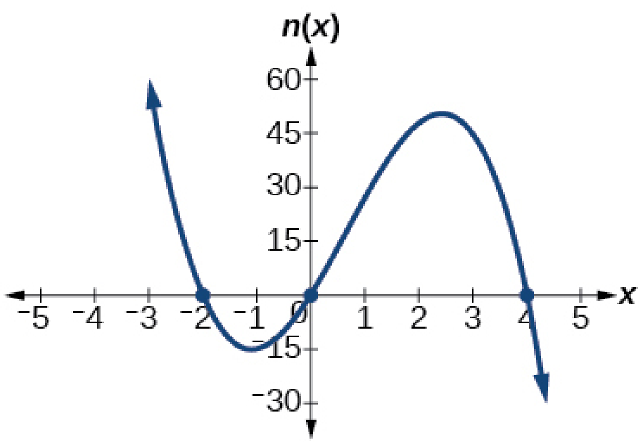
x-intercepts, $(1, 0)$ with multiplicity 2, $(-4, 0)$ with multiplicity 1, y- intercept $(0, 4)$. As $x\to-\infty, f(x)\to-\infty$, as $x\to\infty, f(x)\to\infty$.



$n(x) = -3x(x+2)(x-4)$

---

x-intercepts $(0, 0), (-2, 0), (4,0)$ with multiplicity 1, y- intercept $(0, 0)$. As $x \rightarrow -\infty, f(x) \rightarrow \infty, as x \rightarrow \infty, f(x) \rightarrow -\infty$.



For the following exercises, use the Intermediate Value Theorem to confirm that the given polynomial has at least one zero within the given interval.

$f(x) = x^3 - 9x$, between $x = 2$ and $x = 4$.

$f(2) = -10$ and $f(4) = 28$. Sign change confirms.

$f(x) = x^3 - 100x + 2$, between $x = 0.01$ and $x = 0.1$

$f(0.01) = 1.000001$ and $f(0.1) = -7.999$. Sign change confirms.

For the following exercises, use the given information about the polynomial graph to write the equation.

Degree 3. Zeros at $x = -2$, $x = 1$, and $x = 3$. *y*-intercept at $(0, -4)$.

$f(x) = -\frac{2}{3}(x+2)(x-1)(x-3)$

Degree 5. Double zero at $x=1$, and triple zero at $x=3$. Passes through the point $(2,15)$.

$f(x)=-15(x-1)^2(x-3)^3$

Glossary

binomial
: A binomial is a polynomial with exactly two terms.

degree of a constant
: The degree of any constant is 0.

degree of a polynomial
: The degree of a polynomial is the highest degree of all its terms.

degree of a term
: The degree of a term is the sum of the exponents of its variables.

monomial
: A monomial is an algebraic expression with one term. A monomial in one variable is a term of the form ax^m, where *a* is a constant and *m* is a whole number.

polynomial
: A monomial or two or more monomials

combined by addition or subtraction is a polynomial.

standard form of a polynomial
: A polynomial is in standard form when the terms of a polynomial are written in descending order of degrees.

trinomial
: A trinomial is a polynomial with exactly three terms.

polynomial function
: A polynomial function is a function whose range values are defined by a polynomial.

Dividing Polynomials (3.3)
By the end of this section, you will be able to:

- Dividing a polynomial by a monomial
- Dividing polynomials using long division
- Dividing polynomials using synthetic division
- Use the remainder and factor theorems

This Module supports section **3.3 of Mat 1023**.
In case you missed something in class, or just want to review a specific topic covered in this Module, here is a list of topics covered:

1. Dividing Polynomials Review [link]
2. Divide Using Long Division [link]
3. Synthetic Division [link]
4. Remainder and Factor Theorem [link]
5. Key Concepts [link]

Polynomial Division Review

The method we'll use to divide a polynomial by a monomial is based on the properties of fraction addition. So we'll start with an example to review

fraction addition. The sum $\frac{y}{5}+\frac{2}{5}$ simplifies to $\frac{y+2}{5}$.

Now we will do this in reverse to split a single fraction into separate fractions. For example, $\frac{y+2}{5}$ can be written $\frac{y}{5}+\frac{2}{5}$.

This is the “reverse” of fraction addition and it states that if *a*, *b*, and *c* are numbers where $c \neq 0$, then $\frac{a+b}{c}=\frac{a}{c}+\frac{b}{c}$. We will use this to divide polynomials by monomials.

Division of a Polynomial by a Monomial

To divide a polynomial by a monomial, divide each term of the polynomial by the monomial.

Find the quotient: $(18x^3y - 36xy^2) \div (-3xy)$.

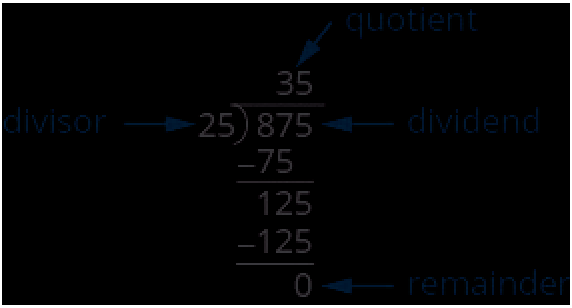
$(18x^3y - 36xy^2) \div (-3xy)$ Rewrite as a fraction. $\frac{18x^3y - 36xy^2}{-3xy}$ Divide each term by the divisor. Be careful with the signs! $\frac{18x^3y}{-3xy} - \frac{36xy^2}{-3xy}$ Simplify. $-6x^2 + 12y$

Find the quotient: $(32a2b-16ab2)\div(-8ab)$.
$-4a+2b$

Find the quotient: $(-48a8b4-36a6b5)\div(-6a3b3)$.
$8a5b+6a3b2$

Divide Polynomials Using Long Division

To divide a polynomial by a binomial, we follow a procedure very similar to long division of numbers. We start by dividing into the digits of the dividend that have the greatest place value. We divide, multiply, subtract, include the digit in the next place value position, and repeat. So let's look carefully the steps we take when we divide a 3-digit number, 875, by a 2-digit number, 25.



We check division by multiplying the quotient by the divisor.

If we did the division correctly, the product of quotient and divisor should equal the dividend.
$35 \cdot 25 875$ ✓

To check your answers, you can look at the form of the solution in a different way.
$\text{dividend} = (\text{divisor} \cdot \text{quotient}) + \text{remainder}$ $875 = (25 \cdot 35) + 0 = 875 + 0 = 875$

This process demonstrates the division algorithm.

Division Algorithm

The **Division Algorithm** states that, given a polynomial dividend $f(x)$ and a non-zero polynomial divisor $d(x)$ where the degree of $d(x)$ is less than or equal to the degree of $f(x)$, there exist unique polynomials $q(x)$ and $r(x)$ such that

$$f(x) = d(x)q(x) + r(x)$$

$q(x)$ is the quotient and $r(x)$ is the remainder. The

remainder is either equal to zero or has degree strictly less than $d(x)$.
If $r(x)=0$, then $d(x)$ divides evenly into $f(x)$. This means that, in this case, both $d(x)$ and $q(x)$ are factors of $f(x)$.

Now we will divide a trinomial by a binomial. As you read through the example, notice how similar the steps are to the numerical example above.

Find the quotient: $(x2+9x+20)\div(x+5)$.

$(x^2+9x+20)\div(x+5)$

Write it as a long division problem. Be sure the dividend is in standard form.

$x+5\overline{)x^2+9x+20}$

Divide x2 by x. It may

he[redacted]
“What do I need to multiply x by to get x2?”

Put the answer, x, in the quotient over the x te[redacted]

Mu[redacted]

Line up the like terms under the dividend.

Subtract x2 + 5x from x2 + 9x.

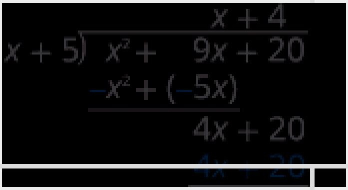
Yo[redacted]
to [redacted]
the[redacted]

Then bring down the last term, 20.

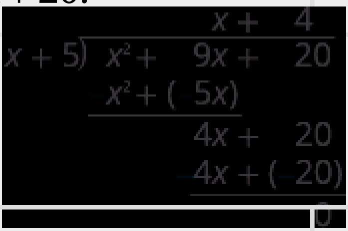
Divide 4x by x. It may he[redacted]
“W[redacted]
ne[redacted]
to get 4x?”

Put the answer, 4, in the quotient over the constant term.

Multiply 4 times x + 5.



Subtract $4x+20$ from $4x+20$.



Check:
Multiply the quotient by the divisor. $(x+4)(x+5)$
You should get the dividend. $x2+9x+20$✓

Find the quotient: $(y2+10y+21)\div(y+3)$.

$y+7$

When we divided 875 by 25, we had no remainder. But sometimes division of numbers does leave a remainder. The same is true when we divide polynomials. In the next example, we'll have a division that leaves a remainder. We write the remainder as a fraction with the divisor as the denominator.

Look back at the dividends in previous examples. The terms were written in descending order of degrees, and there were no missing degrees. The dividend in this example will be $x4-x2+5x-6$. It is missing an x3 term. We will add in 0x3 as a placeholder.

Given a polynomial and a binomial, use long division to divide the polynomial by the binomial.

1. Arrange the terms of in descending order of degrees,for both the dividend and divisor. Remember to add in any missing degrees as a placeholder for the dividend
2. Set up the division problem.
3. Determine the first term of the quotient by dividing the leading term of the dividend by the leading term of the divisor.
4. Multiply the answer by the divisor and write it below the like terms of the dividend.

5. Subtract the bottom binomial from the top binomial.
6. Bring down the next term of the dividend.
7. Repeat steps 2–5 until reaching the last term of the dividend.
8. If the remainder is non-zero, express as a fraction using the divisor as the denominator.

Find the quotient: $(x4 - x2 + 5x - 6) \div (x + 2)$.

Notice that there is no x3 term in the dividend. We will add 0x3 as a placeholder.

[redacted]

Write it as a long division problem. Be [redacted] standard form with placeholders for missing terms.

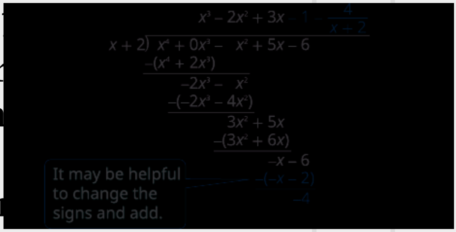
Divide x4 by x.
Put the answer, x3, in
the [redacted]
x3 [redacted]
Multiply x3 times x + 2.
Line up the like terms.
Subtract and then bring down the next term.

Divide − 2x3 by x.
Put the answer, − 2x2, in [redacted]
the [redacted]
Mu [redacted] − 2x2 times x + 1. Line up the like terms
Subtract and bring down the next term.

Divide 3x2 by x.
Put the answer, 3x, in
the [redacted]
ter [redacted]
Mu [redacted]
Li [redacted]
Subtract and bring down the next term.

Divide − x by x.
Put the answer, − 1, in the quotient over the constant term.
Multiply − 1 times x

+
ter
Ch



Wr

a fraction with the divisor as the denominator.

To check, multiply $(x+2)(x^3-2x^2+3x-1-\frac{4}{x+2})$.

The result should be x^4-x^2+5x-6.

Find the quotient: $(x^4-7x^2+7x+6)\div(x+3)$.

$x^3-3x^2+2x+1+\frac{3}{x+3}$

In the next example, we will divide by $2a+3$. As we divide, we will have to consider the constants as well as the variables.

Find the quotient: $(8a^3+27)\div(2a+3)$.

This time we will show the division all in one step. We need to add two placeholders in order to divide.

[redacted]

[redacted]

To check, multiply $(2a+3)(4a^2-6a+9)$.

The result should be $8a^3+27$.

Find the quotient: $(125x3-8)\div(5x-2)$.

$25x2+10x+4$

In the following example, we use division algorithm form to write the solution. Notice the answer is not just the quotient.

Using Long Division to Divide a Second-Degree Polynomial

Divide $5\ x\ 2\ +3x-2$ by $x+1$.

$x+1\overline{)5x^2+3x-2}$	Set up the division problem.
$\begin{array}{r} 5x \\ x+1\overline{)5x^2+3x-2} \end{array}$	$5x^2$ divided by x is $5x$.
$\begin{array}{r} 5x \\ x+1\overline{)5x^2+3x-2} \\ \underline{-(5x^2+5x)} \\ -2x-2 \end{array}$	Multiply $x+1$ by $5x$. Subtract. Bring down the next term.
$\begin{array}{r} 5x-2 \\ x+1\overline{)5x^2+3x-2} \\ \underline{-(5x^2+5x)} \\ -2x-2 \\ \underline{-(-2x-2)} \\ 0 \end{array}$	$-2x$ divided by x is -2. Multiply $x+1$ by -2. Subtract.

The quotient is $5x-2$. The remainder is 0. We write the result as

$5\ x\ 2\ +3x-2\ x+1\ =5x-2$

or

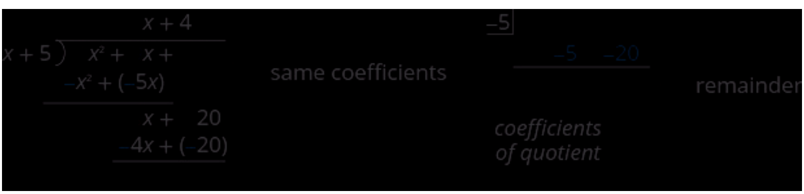
$$5x^2+3x-2=(x+1)(5x-2)$$

Analysis

This division problem had a remainder of 0. This tells us that the dividend is divided evenly by the divisor, and that the divisor is a factor of the dividend.

Divide Polynomials using Synthetic Division

As we have mentioned before, mathematicians like to find patterns to make their work easier. Since long division can be tedious, let’s look back at the long division we did in [link] and look for some patterns. We will use this as a basis for what is called synthetic division. The same problem in the synthetic division format is shown next.



Synthetic division basically just removes

unnecessary repeated variables and numbers. Here all the x and x2 are removed. as well as the $-x2$ and $-4x$ as they are opposite the term above.

The first row of the synthetic division is the coefficients of the dividend. The -5 is the opposite of the 5 in the divisor.

The second row of the synthetic division are the numbers shown in red in the division problem.

The third row of the synthetic division are the numbers shown in blue in the division problem.

Notice the quotient and remainder are shown in the third row.
Synthetic division only works when the divisor is of the form$x-c$.

Synthetic Division
Synthetic division is a shortcut that can be used when the divisor is a binomial in the form $x-c$ where c is a real number. In **synthetic division**, only the coefficients are used in the division process.

The following example will explain the process.

Notice that the divisor is x+2. In this case, it is not a minus sign, but a plus. All you do is write the x+2 as x-(-2). Making c=-2.

Use synthetic division to find the quotient and remainder when 2x3+3x2+x+8 is divided by x+2.

Write the dividend with decreasing powers of x.	$2x^3+3x^2+x+8$
Write the coefficients of the terms as the first row of synthetic division.	2 3 1 8
Write the divisor as x − c and place *c* in synthetic division in the divisor box.	−2 | 2 3 1 8
Bring down the first coefficient to the third	

ro[redacted]

Multiply that coefficient by the di[redacted] coefficient.

Add the second column, putting the res[redacted]

Multiply that result by the divisor and place the[redacted] coefficient.

Add the third column, putting the result in the[redacted]

Multiply that result by the divisor and place the[redacted] coefficient.

Add the final column, putting the result in the

-2	2	3	1	8
		-4	2	-6
	2	-1	3	$\boxed{2}$

The quotient is $2x2 - 1x + 3$ and the remainder is 2.

The division is complete. The numbers in the third row give us the result. The $2 - 13$ are the coefficients of the quotient. The quotient is $2x2 - 1x + 3$. The 2 in the box in the third row is the remainder.

Check:

(quotient)
$(\text{divisor}) + \text{remainder} = \text{dividend}(2x2 - 1x + 3)$
$(x + 2) + 2 = ?2x3 + 3x2 + x + 8$ $2x3 - x2 + 3x$
$+ 4x2 - 2x + 6 + 2 = ?2x3 + 3x2 + x + 8$
$2x3 + 3x2 + x + 8 = 2x3 + 3x2 + x + 8$✓

How To Divide a Polynomial by $x - c$

1. Arrange the terms of in descending order of degrees,for both the dividend and divisor, adding a 0 coefficient for any missing degrees.

2. Write c for the divisor.
3. Write the coefficients of the dividend to the right.
4. Bring the lead coefficient down to the bottom row.
5. Multiply the lead coefficient by c. Write the product in the next column on the second row.
6. Add the terms of the second column and write the sum on the bottom row.
7. Repeat Multiplication/Addition steps until all columns are filled: Multiply the result by c. Write the product in the next column.
8. Repeat steps 5 and 6 for the remaining columns.
9. Use the bottom numbers to write the quotient. The number in the last column is the remainder and has degree 0, the next number from the right has degree 1, the next number from the right has degree 2, and so on. You may need to write the remainder over the divisor, rather than just state the remainder separately. This format is used in [link]

Use synthetic division to find the quotient and remainder when $3x3+10x2+6x-2$ is divided by $x+2$.

$3x^2+4x-2;2$

In the next example, we will do all the steps together.

Use synthetic division to find the quotient and remainder when $x^4-16x^2+3x+12$ is divided by $x+4$.

The polynomial $x^4-16x^2+3x+12$ has its term in order with descending degree but we notice there is no x^3 term. We will add a 0 as a placeholder for the x^3 term. In $x-c$ form, the divisor is $x-(-4)$.

$\boxed{-4}$	1	0	−16	3	12
		−4	16	0	−12
	1	−4	0	3	$\boxed{0}$

We divided a 4th degree polynomial by a 1st degree polynomial so the quotient will be a 3rd degree polynomial.

Reading from the third row, the quotient has

the coefficients $1 - 403$, which is $x3 - 4x2 + 3$. The remainder
is 0.

Use synthetic division to find the quotient and remainder when $x4 - 16x2 + 5x + 20$ is divided by $x + 4$.

$x3 - 4x2 + 5;0$

Using Synthetic Division to Divide a Fourth-Degree Polynomial

Use synthetic division to divide $-9 x 4 + 10 x 3 + 7 x 2 - 6$ by $x - 1$.

Notice there is no *x*-term. We will use a zero as the coefficient for that term.

1	−9	10	7	0	−6
		−9	1	8	8
	−9	1	8	8	2

The result is $-9x^3 + x^2 + 8x + 8 + \frac{2}{x-1}$.

$(4x^3 - 5x^2 + 13) \div (x+4)$

$4x^2 - 21x + 84 - \frac{323}{x+4}$

Use the Remainder and Factor Theorem

Let’s look at the division problems we have just worked that ended up with a remainder. They are summarized in the chart below. If we take the dividend from each division problem and use it to define a function, we get the functions shown in the chart. When the divisor is written as $x-c$, the value of the function at $c, f(c)$, is the same as the remainder from the division problem.

Dividend	Divisor $x-c$	Remainder	Function	$f(c)$
x^4-x^2+5x-6	$x-(-2)$	-4	$f(x)=x^4-x^2+5x-6$	-4
$3x^3-2x^2-10x+8$	$x-2$	4	$f(x)=3x^3-2x^2-10x+8$	4
$x^4-16x^2+3x+15$	$x-(-4)$	3	$f(x)=x^4-16x^2+3x+15$	3

To see this more generally, we realize we can check a division problem by multiplying the quotient times the divisor and add the remainder. This should look familiar as it uses the Division Algorithm [link] we showed you earlier. $f(x)=d(x)q(x)+r(x)$. In function notation we could say, to get the dividend f(x), we multiply the quotient, q(x) times the divisor, $x-c$, and add the remainder, *r*.

[redacted]

If we evaluate this at c, we get:

[redacted]

[redacted]

[redacted]

This leads us to the Remainder Theorem.

Remainder Theorem
If the polynomial function $f(x)$ is divided by $x-c$, then the remainder is $f(c)$.

Use the Remainder Theorem to find the remainder when $f(x)=x3+3x+19$ is divided by $x+2$.

To use the Remainder Theorem, we must use the divisor in the $x-c$ form. We can write the divisor $x+2$ as $x-(-2)$. So, our c is -2.

To find the remainder, we evaluate $f(c)$ which is $f(-2)$.

[redacted]

To evaluate f(− 2), substitute x = − 2.

[redacted]

Simplify.

[redacted]

The remainder is 5 when f(x) = x3 + 3x + 19 is divided by x + 2.

Check:
Use synthetic division to check.

$\underline{-2}\rvert$	1	0	3	19
		−2	4	−14
	1	−2	7	$\boxed{5}$

The remainder is 5.

Use the Remainder Theorem to find the

remainder when $f(x)=x3-7x+12$ is divided by $x+3$.

6

In the next example, we evaluate the function by using synthetic division first.

Using the Remainder Theorem to Evaluate a Polynomial

Use the Remainder Theorem to evaluate $f(x)=6\ x\ 4\ -\ x\ 3\ -15\ x\ 2\ +2x-7$ at $x=2$.

To find the remainder using the Remainder Theorem, use synthetic division to divide the polynomial by $x-2$.

$$\begin{array}{r|rrrrr} 2 & 6 & -1 & -15 & 2 & -7 \\ & & 12 & 22 & 14 & 32 \\ \hline & 6 & 11 & 7 & 16 & 25 \end{array}$$

The remainder is 25. Therefore, $f(2)=25$.

Analysis

We can check our answer by evaluating $f(2)$.
$f(x) = 6x^4 - x^3 - 15x^2 + 2x - 7$ $f(2) = 6(2)^4 - (2)^3 - 15(2)^2 + 2(2) - 7 = 25$

Factor Theorem

The **Factor Theorem** is another theorem that helps us analyze polynomial equations. It tells us how the zeros of a polynomial are related to the factors. Recall the Division Algorithm. $f(x) = (x - c)q(x) + r$

When we divided $8a^3 + 27$ by $2a + 3$ in [link] the result was $4a^2 - 6a + 9$. To check our work, we multiply $4a^2 - 6a + 9$ by $2a + 3$ to get $8a^3 + 27$.
$(4a^2 - 6a + 9)(2a + 3) = 8a^3 + 27$

Written this way, we can see that $4a^2 - 6a + 9$ and $2a + 3$ are factors of $8a^3 + 27$. When we did the division, the remainder was zero.

Whenever a divisor, $x - c$, divides a polynomial function, $f(x)$, and resulting in a remainder of zero, we say $x - c$ is a factor of $f(x)$.

The reverse is also true. If $x - c$ is a factor of $f(x)$ then $x - c$ will divide the polynomial function resulting in a remainder of zero.

We will state this in the Factor Theorem.

Factor Theorem

For any polynomial function $f(x)$,

- if $x-c$ is a factor of $f(x)$, then $f(c)=0$
- if $f(c)=0$, then $x-c$ is a factor of $f(x)$

Use the Remainder Theorem to determine if $x-4$ is a factor of $f(x)=x3-64$.

The Factor Theorem tells us that $x-4$ is a factor of $f(x)=x3-64$ if $f(4)=0$.

$f(x)=x3-64$To evaluatef(4)substitute$x=4$.$f(4)=43-64$Simplify.$f(4)=6$

Since $f(4)=0$, $x-4$ is a factor of $f(x)=x3-64$.

Use the Factor Theorem to determine if $x-5$ is a factor of $f(x)=x3-125$.

yes

Using the Factor Theorem to Find the Zeros of a Polynomial Expression

Show that $(x+2)$ is a factor of x^3-6x^2-x+30. Find the remaining factors. Use the factors to determine the zeros of the polynomial.

We can use synthetic division to show that $(x+2)$ is a factor of the polynomial.

$$\begin{array}{r|rrrr} -2 & 1 & -6 & -1 & 30 \\ & & -2 & 16 & -30 \\ \hline & 1 & -8 & 15 & 0 \end{array}$$

The remainder is zero, so $(x+2)$ is a factor of the polynomial. We can use the Division Algorithm to write the polynomial as the product of the divisor and the quotient:

$(x+2)(x^2-8x+15)$

We can factor the quadratic factor to write the polynomial as

$(x+2)(x-3)(x-5)$

By the Factor Theorem, the zeros of $x^3 - 6x^2 - x + 30$ are –2, 3, and 5.

Use the Factor Theorem to find the zeros of $f(x) = x^3 + 4x^2 - 4x - 16$ given that $(x - 2)$ is a factor of the polynomial.

The zeros are 2, –2, and –4.

Access these online resources for additional instruction and practice with dividing polynomials.

- Dividing a Polynomial by a Binomial
- Synthetic Division & Remainder Theorem
- Dividing a Trinomial by a Binomial Using Long Division
- Dividing a Polynomial by a Binomial Using Long Division
- Ex 2: Dividing a Polynomial by a Binomial Using Synthetic Division
- Ex 4: Dividing a Polynomial by a Binomial Using Synthetic Division

Key Concepts

- **Division of a Polynomial by a Monomial**
  - To divide a polynomial by a monomial, divide each term of the polynomial by the monomial.
- Polynomial long division can be used to divide a polynomial by any polynomial with equal or lower degree.
- The Division Algorithm tells us that a polynomial dividend can be written as the product of the divisor and the quotient added to the remainder.
- Synthetic division is a shortcut that can be used to divide a polynomial by a binomial in the form $x-k$.
- **Remainder Theorem**
  - If the polynomial function $f(x)$ is divided by $x-c$, then the remainder is $f(c)$.
- **Factor Theorem:** For any polynomial function $f(x)$,
  - if $x-c$ is a factor of $f(x)$, then $f(c)=0$
  - if $f(c)=0$, then $x-c$ is a factor of $f(x)$

Section Exercises

Practice Makes Perfect

Divide a Polynomial by a Monomial

In the following exercises, divide each polynomial by the monomial.

$(8x3+6x2)\div 2x$

$4x2+3x$

$(48y4-24y3)\div(-8y2)$

$-6y2+3y$

$72r5s2+132r4s3-96r3s512r2s2$

$6r3+11r2s-8rs3$

$20y2+12y-1-4y$

$-5y-3+14y$

Divide Polynomials using Long Division

In the following exercises, divide each polynomial by the binomial.

$(a2-2a-35)\div(a+5)$

$a-7$

$(4x2-17x-15)\div(x-5)$

$4x+3$

$(2n3-10n+28)\div(n+3)$

$2n2-6n+8+4n+3$

$(125y3-64)\div(5y-4)$

$25y2+20x+16$

For the following exercises, use long division to

divide. Specify the quotient and the remainder.

$(x^2+5x-1)\div(x-1)$

$x+6+\frac{5}{x-1}$, quotient: $x+6$, remainder: 5

$(2x^2-3x+2)\div(x+2)$

$2x-7+\frac{16}{x+2}$, quotient: $2x-7$, remainder: 16

Divide Polynomials using Synthetic Division

In the following exercises, use synthetic Division to find the quotient and remainder.

$x^3-3x^2-4x+12$ is divided by $x+2$

$x^2-5x+6;0$

$2x^3-11x^2+16x-12$ is divided by $x-4$

$2x^2-3x+4;4$

$x^4 + x^2 + 6x - 10$ is divided by $x + 2$

$x^3 - 2x^2 + 5x - 4; -2$

$3x^4 - 11x^3 + 2x^2 + 10x + 6$ is divided by $x - 3$

$3x^3 - 2x^2 - 4x - 2; 0$

For the following exercises, use synthetic division to find the quotient. Ensure the equation is in the form required by synthetic division. (Hint: divide the dividend and divisor by the coefficient of the linear term in the divisor.)

$(2x^3 - 6x^2 - 7x + 6) \div (x - 4)$

$2x^2 + 2x + 1 + \frac{10}{x - 4}$

$(4x^3 - 12x^2 - 5x - 1) \div (2x + 1)$

$2x^2 - 7x + 1 - \frac{2}{2x + 1}$

$(9x^3 - x + 2) \div (3x - 1)$

$3 x 2 + x + 2 3x - 1$

$(3 x 3 - 2 x 2 + x - 4) \div (x + 3)$

$3 x 2 - 11x + 34 - 106 x + 3$

Use the Remainder and Factor Theorem

In the following exercises, use the Remainder Theorem to find the remainder.

$f(x) = x3 - 4x - 9$ is divided by $x + 2$

-9

$f(x) = 7x2 - 5x - 8$ divided by $x - 1$

-6

$(3 x 3 - 2 x 2 + x - 4) \div (x + 3)$

-106

In the following exercises, use the Factor Theorem

to determine if $x - c$ is a factor of the polynomial function.

Determine whether $x + 4$ a factor of $x^3 + x^2 - 14x + 8$

---

no

Determine whether $x - 3$ a factor of $x^3 - 7x^2 + 11x + 3$

---

yes

For the following exercises, use the Factor Theorem to find all real zeros for the given polynomial function and one factor.

$f(x) = 2x^3 + x^2 - 5x + 2;\ x + 2$

$-2, 1, \frac{1}{2}$

$f(x) = 2x^3 + 3x^2 + x + 6;\ \ x + 2$

-2

$2x^3 + 5x^2 - 12x - 30, 2x + 5$

$-5\ 2, 6, -6$

Glossary

Division Algorithm
: given a polynomial dividend $f(x)$ and a non-zero polynomial divisor $d(x)$ where the degree of $d(x)$ is less than or equal to the degree of $f(x)$, there exist unique polynomials $q(x)$ and $r(x)$ such that $f(x)=d(x)q(x)+r(x)$ where $q(x)$ is the quotient and $r(x)$ is the remainder. The remainder is either equal to zero or has degree strictly less than $d(x)$.

Factor Theorem
: c is a zero of polynomial function $f(x)$ if and only if $(x-c)$ is a factor of $f(x)$

Remainder Theorem
: if a polynomial $f(x)$ is divided by $x-k$, then the remainder is equal to the value $f(k)$

synthetic division
: a shortcut method that can be used to divide a polynomial by a binomial of the form $x-k$

Exponential Functions (4.1)
By the end of this section, you will be able to:

- Graph exponential functions
- Use exponential models in applications

This Module supports section **4.1 of Mat 1023**.
In case you missed something in class, or just want to review a specific topic covered in this Module, here is a list of topics covered:

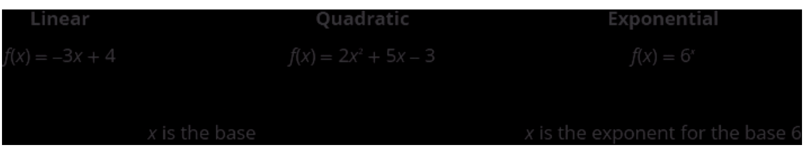
Define and Evaluate Exponential Functions

The functions we have studied so far do not give us

a model for many naturally occurring phenomena. From the growth of populations and the spread of viruses to radioactive decay and compounding interest, the models are very different from what we have studied so far. These models involve exponential functions.

An **exponential function** is a function of the form $f(x)=ax$ where $a>0$ and $a\neq 1$.

> Exponential Function
>
> An exponential function, where $a>0$ and $a\neq 1$, is a function of the form
>
> $f(x)=ax$
>
> Notice that in this function, **the variable is the exponent** . In our functions so far, the variables were the base.



Our definition says $a\neq 1$. If we let $a=1$, then $f(x)=ax$ becomes $f(x)=1x$. Since $1x=1$ for all real numbers, $f(x)=1$. This is the constant function.

Recall that the base of an exponential function must

be a positive real number other than 1. Why do we limit the base a to positive values? To ensure that the outputs will be real numbers. Observe what happens if the base is not positive:

$f(x)=(-4)x$ $f(12)=(-4)12$ $f(12)=-4$not a real number

In fact, $f(x)=(-4)x$ would not be a real number any time x is a fraction with an even denominator. So our definition requires $a>0$.

Identifying Exponential Functions

Which of the following equations are *not* exponential functions?

- $f(x)=4\ 3(x-2)$
- $g(x)=x\ 3$
- $h(x)=(1\ 3)\ x$
- $j(x)=(-2)\ x$

By definition, an exponential function has a constant as a base and an independent variable as an exponent. Thus, $g(x)=x\ 3$ does not represent an exponential function because the base is an independent variable. In fact, $g(x)=x\ 3$ is a power function.

Recall that the base a of an exponential function is always a positive constant, and $b \neq 1$. Thus, $j(x) = (-2)^x$ does not represent an exponential function because the base, -2, is less than 0.

Which of the following equations represent exponential functions?

- $f(x) = 2x^2 - 3x + 1$
- $g(x) = 0.875^x$
- $h(x) = 1.75x + 2$
- $j(x) = 1095.6^{-2x}$

$g(x) = 0.875^x$ and $j(x) = 1095.6^{-2x}$ represent exponential functions.

Why do we limit the base to positive values other than 1? Because base 1 results in the constant function. Observe what happens if the base is 1:

- Let $a = 1$. Then $f(x) = 1^x = 1$ for any value of x.

To evaluate an exponential function with the form $f(x) = a^x$, we simply substitute x with the given value, and calculate the resulting power. For example:

Let $f(x) = 2^x$. What is $f(3)$?

$$\begin{aligned} f(x) &= 2^x \\ f(3) &= 2^3 && \text{Substitute } x=3. \\ &= 8 && \text{Evaluate the power.} \end{aligned}$$

To evaluate an exponential function with a form other than the basic form, it is important to follow the order of operations. For example:

Let $f(x) = 30(2)^x$. What is $f(3)$?

$$\begin{aligned} f(x) &= 30(2)^x \\ f(3) &= 30(2)^3 && \text{Substitute } x=3. \\ &= 30(8) && \text{Simplify the power first.} \\ &= 240 && \text{Multiply.} \end{aligned}$$

Note that if the order of operations were not followed, the result would be incorrect:

$$f(3) = 30(2)^3 \neq 60^3 = 216{,}000$$

Evaluating Exponential Functions

Let $f(x) = 5(3)^{x+1}$. Evaluate $f(2)$ without using a calculator.

Follow the order of operations. Be sure to pay attention to the parentheses.

$f(x) = 5(3)^{x+1}$ $f(2) = 5(3)^{2+1}$ Substitute $x = 2$. $= 5(3)^3$ Add the exponents. $= 5(27)$ Simplify the power. $= 135$ Multiply.

Let $f(x) = 8(1.2)^{x-5}$. Evaluate $f(3)$ using a calculator. Round to four decimal places.

5.5556

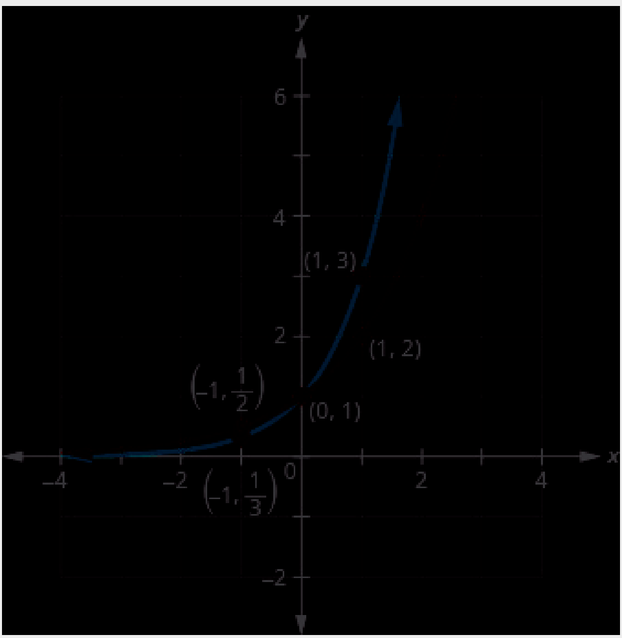
Graph Exponential Functions

By graphing a few exponential functions, we will be able to see their unique properties.

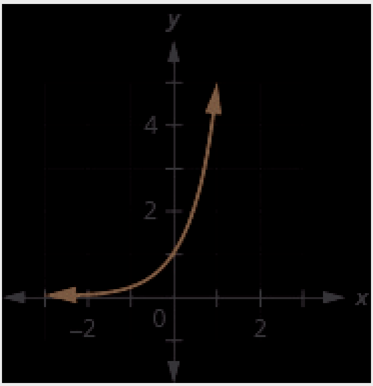
On the same coordinate system graph $f(x) = 2^x$ and $g(x) = 3^x$.

We will use point plotting to graph the functions.

x		$(x, f(x))$		$g(x) = 3^x$	$(x, g(x))$
-2	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$	$\left(-2, \frac{1}{4}\right)$		$3^{-2} = \frac{1}{3^2} = \frac{1}{9}$	$\left(-2, \frac{1}{9}\right)$
-1	$2^{-1} = \frac{1}{2^1} = \frac{1}{2}$	$\left(-1, \frac{1}{2}\right)$		$3^{-1} = \frac{1}{3^1} = \frac{1}{3}$	$\left(-1, \frac{1}{3}\right)$
0	$2^0 = 1$	(0, 1)		$3^0 = 1$	(0, 1)
1	$2^1 = 2$	(1, 2)		$3^1 = 3$	(1, 3)
2	$2^2 = 4$	(2, 4)		$3^2 = 9$	(2, 9)
3	$2^3 = 8$	(3, 8)		$3^3 = 27$	(3, 27)



Graph: $g(x)=5x$.



If we look at the graphs from the previous Example and Try Its, we can identify some of the properties of exponential functions.

The graphs of $f(x)=2x$ and $g(x)=3x$, as well as the graphs of $f(x)=4x$ and $g(x)=5x$, all have the same basic shape. This is the shape we expect from an exponential function where $a>1$.

We notice, that for each function, the graph contains the point $(0,1)$. This make sense because

$a0=1$ for any a.

The graph of each function, $f(x)=ax$ also contains the point $(1,a)$. The graph of $f(x)=2x$ contained $(1,2)$ and the graph of $g(x)=3x$ contained $(1,3)$. This makes sense as $a1=a$.

Notice too, the graph of each function $f(x)=ax$ also contains the point $(-1,1a)$. The graph of $f(x)=2x$ contained $(-1,12)$ and the graph of $g(x)=3x$ contained $(-1,13)$. This makes sense as $a-1=1a$.

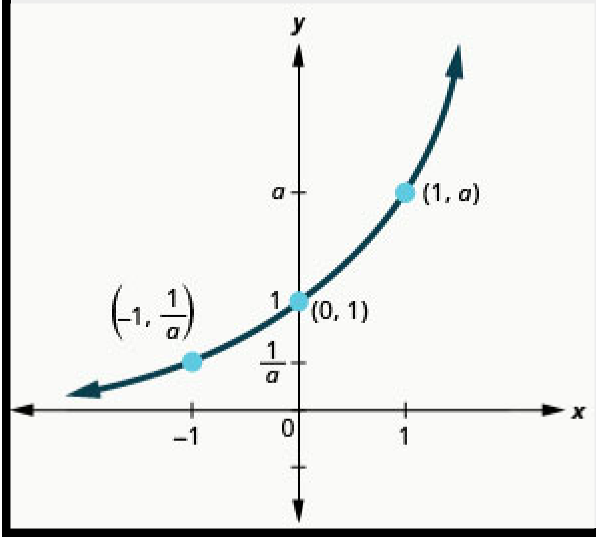
What is the domain for each function? From the graphs we can see that the domain is the set of all real numbers. There is no restriction on the domain. We write the domain in interval notation as $(-\infty,\infty)$.

Look at each graph. What is the range of the function? The graph never hits the x-axis. The range is all positive numbers. We write the range in interval notation as $(0,\infty)$.

Whenever a graph of a function approaches a line but never touches it, we call that line an **asymptote**. For the exponential functions we are looking at, the graph approaches the x-axis very closely but will never cross it, we call the line $y=0$, the x-axis, a horizontal asymptote.

Properties of the Graph of $f(x)=ax$ when $a>1$

Domain	$(-\infty,\infty)$
Range	$(0,\infty)$
x intercept	None
y intercept	$(0,1)$
Contains	$(1,a),(-1,1a)$
Asymptote	x-axis, the line $y=0$



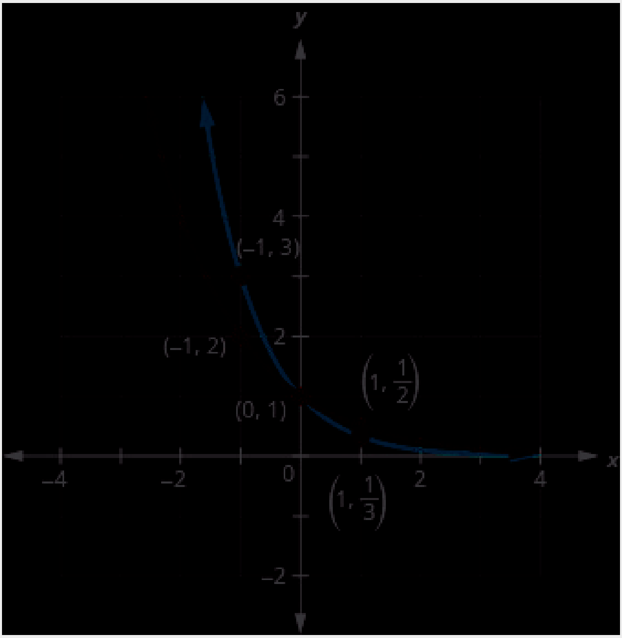
Our definition of an exponential function $f(x)=ax$

says a > 0, but the examples and discussion so far has been about functions where a > 1. What happens when 0 < a < 1? The next example will explore this possibility.

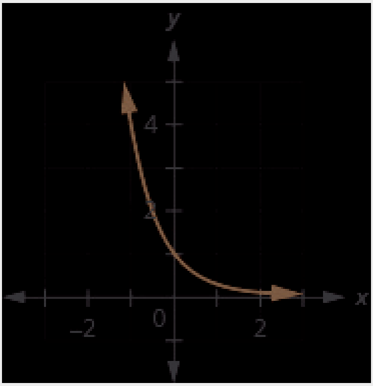
On the same coordinate system, graph f(x) = (12)x and g(x) = (13)x.

We will use point plotting to graph the functions.

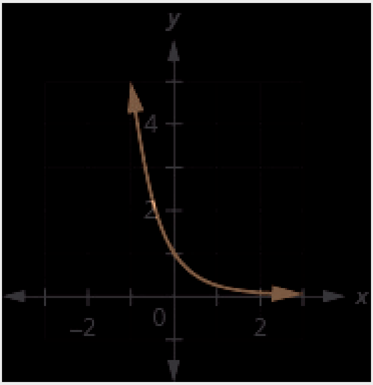
x		$(x, f(x))$		$g(x) = \left(\frac{1}{3}\right)^x$	$(x, g(x))$
-2	$\left(\frac{1}{2}\right)^{-2} = 2^2 = 4$	$(-2, 4)$		$\left(\frac{1}{3}\right)^{-2} = 3^2 = 9$	$(-2, 9)$
-1	$\left(\frac{1}{2}\right)^{-1} = 2^1 = 2$	$(-1, 2)$		$\left(\frac{1}{3}\right)^{-1} = 3^1 = 3$	$(-1, 3)$
0	$\left(\frac{1}{2}\right)^{0} = 1$	$(0, 1)$		$\left(\frac{1}{3}\right)^{0} = 1$	$(0, 1)$
1	$\left(\frac{1}{2}\right)^{1} = \frac{1}{2}$	$\left(1, \frac{1}{2}\right)$		$\left(\frac{1}{3}\right)^{1} = \frac{1}{3}$	$\left(1, \frac{1}{3}\right)$
2	$\left(\frac{1}{2}\right)^{2} = \frac{1}{4}$	$\left(2, \frac{1}{4}\right)$		$\left(\frac{1}{3}\right)^{2} = \frac{1}{9}$	$\left(2, \frac{1}{9}\right)$
3	$\left(\frac{1}{2}\right)^{3} = \frac{1}{8}$	$\left(3, \frac{1}{8}\right)$		$\left(\frac{1}{3}\right)^{3} = \frac{1}{27}$	$\left(3, \frac{1}{27}\right)$



Graph: f(x) = (14)x.



Graph: g(x) = (15)x.



Now let's look at the graphs from the previous Example and Try Its so we can now identify some of the properties of exponential functions where $0<a<1$.

The graphs of $f(x)=(12)x$ and $g(x)=(13)x$ as well as the graphs of $f(x)=(14)x$ and $g(x)=(15)x$ all have the same basic shape. While this is the shape we expect from an exponential function where $0<a<1$, the graphs go down from left to right while the previous graphs, when $a>1$, went from up from left to right.

We notice that for each function, the graph still contains the point (0, 1). This make sense because $a0=1$ for any *a*.

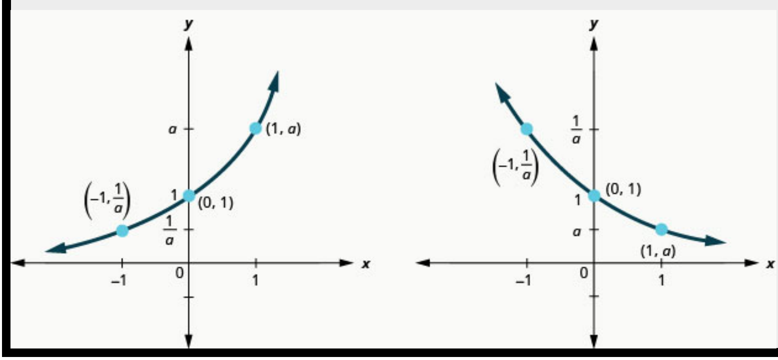
As before, the graph of each function, $f(x)=ax$, also contains the point $(1,a)$. The graph of $f(x)=(12)x$ contained $(1,12)$ and the graph of $g(x)=(13)x$ contained $(1,13)$. This makes sense as $a1=a$.

Notice too that the graph of each function, $f(x)=ax$, also contains the point $(-1,1a)$. The graph of $f(x)=(12)x$ contained $(-1,2)$ and the graph of $g(x)=(13)x$ contained $(-1,3)$. This makes sense as $a-1=1a$.

We will summarize these properties in the chart below. Which also include when $a>1$.

Properties of the Graph of f(x) = ax

when a > 1		when 0 < a < 1	
Domain	$(-\infty,\infty)$	Domain	$(-\infty,\infty)$
Range	$(0,\infty)$	Range	$(0,\infty)$
x intercept	none	x intercept	none
y intercept	(0,1)	y intercept	(0,1)
Contains	(1,a), (−1,1a)	Contains	(1,a), (−1,1a)
Asymptote	x-axis, the line y = 0	Asymptote	x-axis, the line y = 0
Basic shape	increasing	Basic shape	decreasing



It is important for us to notice that both of these graphs are **one-to-one**, as they both pass the horizontal line test. This means the exponential function will have an inverse. We will look at this later.

(a) $g(x)=3(2)^x$ stretches the graph of $f(x)=2^x$ vertically by a factor of 3. (b) $h(x)=\frac{1}{3}(2)^x$ compresses the graph of $f(x)=2^x$ vertically by a factor of $\frac{1}{3}$.(a) $g(x)=-2^x$ reflects the graph of $f(x)=2^x$ about the x-axis. (b) $g(x)=2^{-x}$ reflects the graph of $f(x)=2^x$ about the *y*-axis.

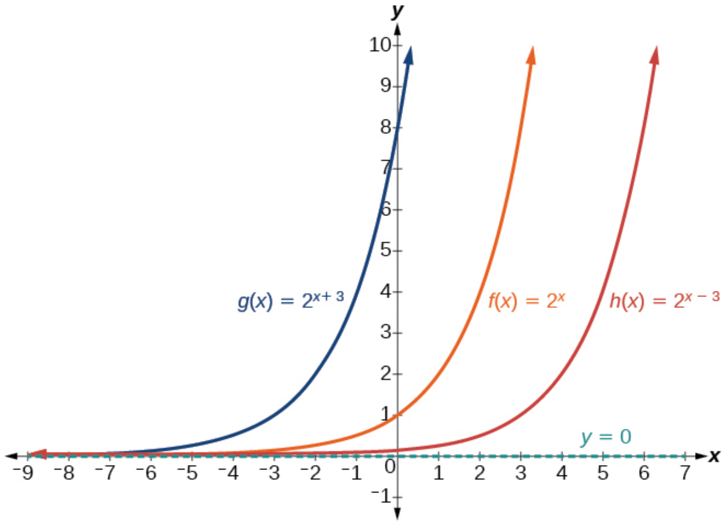
Transformations

When we graphed quadratic functions, we were able to graph using translation rather than just plotting points. Will that work in graphing exponential functions?

Transformations of exponential graphs behave similarly to those of other functions. Just as with other parent functions, we can apply the four types of transformations—shifts, reflections, stretches, and compressions—to the parent function $f(x)=a^x$ without loss of shape. For instance, just as the quadratic function maintains its parabolic shape when shifted, reflected, stretched, or compressed, the exponential function also maintains its general shape regardless of the transformations applied.

The first transformation occurs when we add a

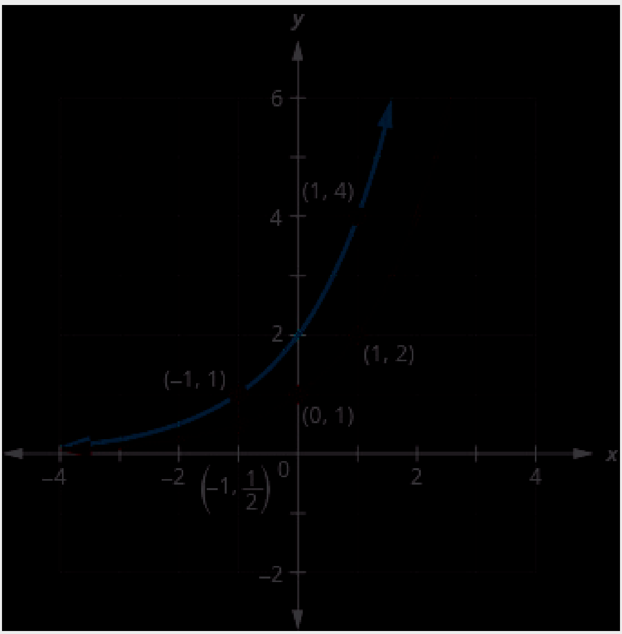
constant c to the input of the parent function $f(x) = a^x$, giving us a horizontal shift c units in the *opposite* direction of the sign. For example, if we begin by graphing the parent function $f(x) = 2^x$, we can then graph two horizontal shifts alongside it, using $c = 3$: the shift left, $g(x) = 2^{x+3}$, and the shift right, $h(x) = 2^{x-3}$. Both horizontal shifts are shown in [link].



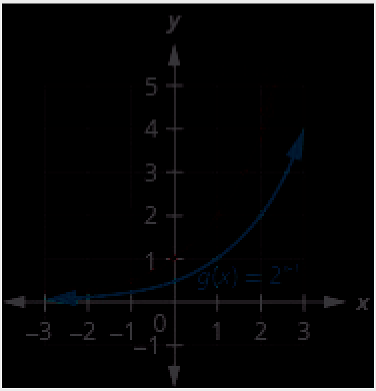
On the same coordinate system graph $f(x) = 2^x$ and $g(x) = 2^{x+1}$.

We will use point plotting to graph the functions.

x		$(x, f(x))$		$g(x) = 2^{x+1}$	$(x, g(x))$
-2	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$	$\left(-2, \frac{1}{4}\right)$		$2^{-2+1} = \frac{1}{2^1} = \frac{1}{2}$	$\left(-2, \frac{1}{2}\right)$
-1	$2^{-1} = \frac{1}{2^1} = \frac{1}{2}$	$\left(-1, \frac{1}{2}\right)$		$2^{-1+1} = 2^0 = 1$	$(-1, 1)$
0	$2^0 = 1$	$(0, 1)$		$2^{0+1} = 2^1 = 2$	$(0, 2)$
1	$2^1 = 2$	$(1, 2)$		$2^{1+1} = 2^2 = 4$	$(1, 4)$
2	$2^2 = 4$	$(2, 4)$		$2^{2+1} = 2^3 = 8$	$(2, 8)$
3	$2^3 = 8$	$(3, 8)$		$2^{3+1} = 2^4 = 16$	$(3, 16)$



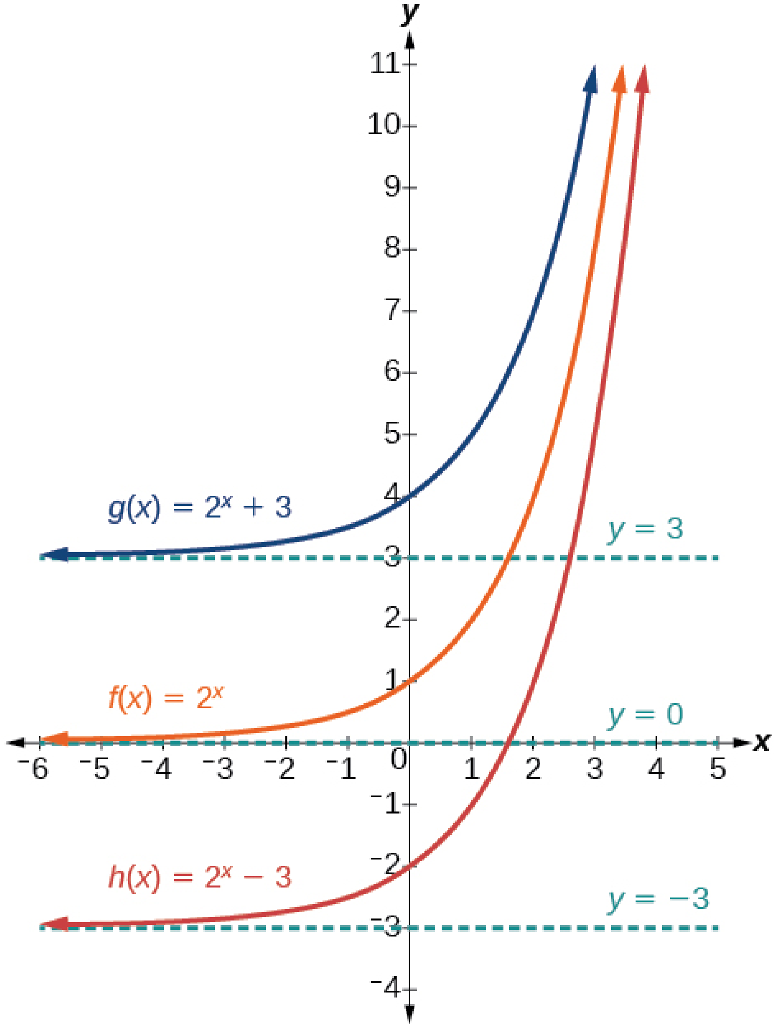
On the same coordinate system, graph:
$f(x) = 2x$ and $g(x) = 2x - 1$.



Looking at the graphs of the functions $f(x) = 2x$ and $g(x) = 2x + 1$ in the last example, we see that adding one in the exponent caused a horizontal shift of one unit to the left. Recognizing this pattern allows us to graph other functions with the same pattern by translation.

The next transformation occurs when we add a constant d to the parent function $f(x) = b x$, giving us a vertical shift d units in the same direction as the sign. For example, if we begin by graphing a parent function, $f(x) = 2 x$, we can then graph two

vertical shifts alongside it, using $d=3$: the upward shift, $g(x)= 2 x +3$ and the downward shift, $h(x)= 2 x -3$. Both vertical shifts are shown in [link].



- The domain, $(-\infty, \infty)$ remains unchanged.
- When the function is shifted up 3 units to $g(x)= 2 x +3$:

    - The *y*-intercept shifts up 3 units to $(0,4)$.
    - The asymptote shifts up 3 units to $y=3$.
    - The range becomes $(3,\infty)$.

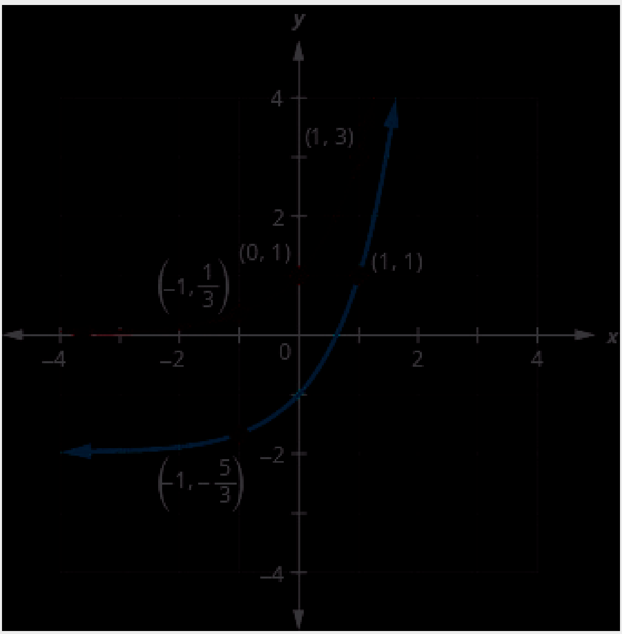
- When the function is shifted down 3 units to $h(x)=2\ x\ -3$:

    - The *y*-intercept shifts down 3 units to $(0,-2)$.
    - The asymptote also shifts down 3 units to $y=-3$.
    - The range becomes $(-3,\infty)$.

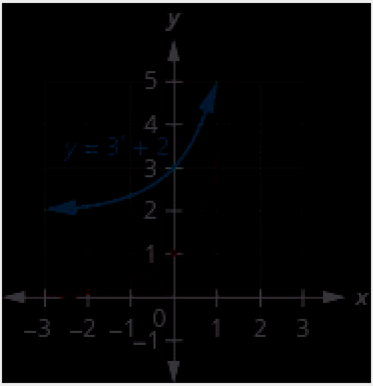
Let’s now consider another situation that might be graphed more easily by **translation**, once we recognize the pattern.

On the same coordinate system graph $f(x)=3x$ and $g(x)=3x-2$.
We will use point plotting to graph the functions.

x		$(x, g(x))$		$g(x) = 3^x - 2$	$(x, g(x))$
-2	$3^{-2} = \frac{1}{9}$	$\left(-2, \frac{1}{9}\right)$		$3^{-2} - 2 = \frac{1}{9} - 2 = -\frac{17}{9}$	$\left(-2, -\frac{17}{9}\right)$
-1	$3^{-1} = \frac{1}{3}$	$\left(-1, \frac{1}{3}\right)$		$3^{-1} - 2 = \frac{1}{3} - 2 = -\frac{5}{3}$	$\left(-1, -\frac{5}{3}\right)$
0	$3^0 = 1$	(0, 1)		$3^0 - 2 = 1 - 2 = -1$	(0, −1)
1	$3^1 = 3$	(1, 3)		$3^1 - 2 = 3 - 2 = 1$	(1, 1)
2	$3^2 = 9$	(2, 9)		$3^2 - 2 = 9 - 2 = 7$	(2, 8)



On the same coordinate system, graph:
$f(x) = 3x$ and $g(x) = 3x + 2$.



Looking at the graphs of the functions f(x) = 3x and g(x) = 3x − 2 in the last example, we see that subtracting 2 caused a vertical shift of down two units. Notice that the horizontal asymptote also shifted down 2 units. Recognizing this pattern allows us to graph other functions with the same pattern by translation.

For any constants c and d, the function f(x) = a x +c + d shifts the parent function f(x) = a x

- vertically d units, in the *same* direction of the sign of d.

- horizontally c units, in the *opposite* direction of the sign of c.
- The *y*-intercept becomes $(0, a c + d)$.
- The horizontal asymptote becomes $y = d$.
- The range becomes (d, ∞).
- The domain, $(-\infty, \infty)$, remains unchanged.

Graphing a Shift of an Exponential Function

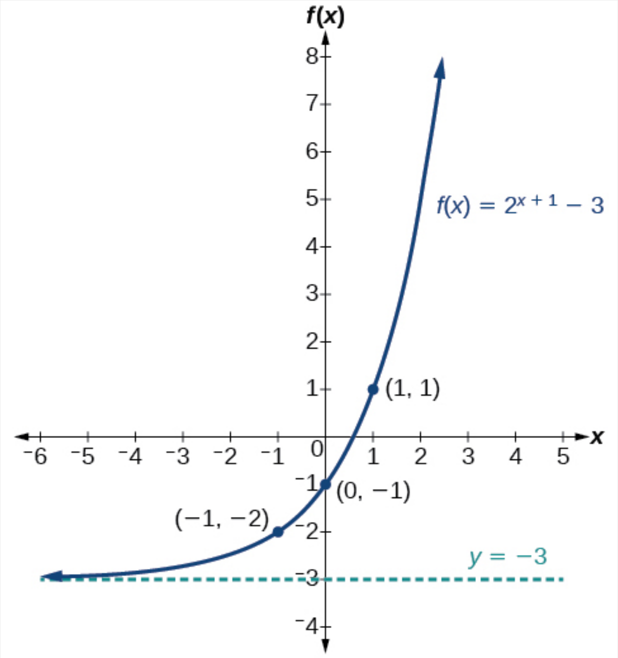
Graph $f(x) = 2 x+1 - 3$. State the domain, range, and asymptote.

We have an exponential equation of the form $f(x) = b x+c + d$, with $b = 2$, $c = 1$, and $d = -3$.

Draw the horizontal asymptote $y = d$, so draw $y = -3$.

Identify the shift as $(-c, d)$, so the shift is $(-1, -3)$.

Shift the graph of $f(x) = b x$ left 1 units and down 3 units.

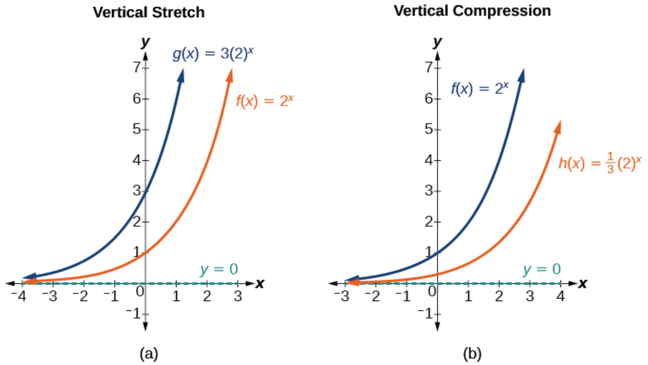


The domain is $(-\infty,\infty)$; the range is $(-3,\infty)$; the horizontal asymptote is $y=-3$.

Stretch or Compress

While horizontal and vertical shifts involve adding constants to the input or to the function itself, a stretch or compression occurs when we multiply the parent function $f(x)=a^x$ by a constant $|c|>0$. For example, if we begin by graphing the parent function $f(x)=2^x$, we can then graph the stretch, using $c=3$, to get $g(x)=3(2)^x$ as shown on the left in [link], and the compression, using $c=\frac{1}{3}$, to get $h(x)=\frac{1}{3}(2)^x$ as shown on the right in

[link].



For any factor $a>0$, the function $f(x)=c(a)^x$

- is stretched vertically by a factor of c if $|c|>1$.
- is compressed vertically by a factor of c if $|c|<1$.
- has a *y*-intercept of $(0,c)$.
- has a horizontal asymptote at $y=0$, a range of $(0,\infty)$, and a domain of $(-\infty,\infty)$, which are unchanged from the parent function.

Graphing the Stretch of an Exponential Function

Sketch a graph of $f(x)=4\left(\frac{1}{2}\right)^x$. State the domain, range, and asymptote.

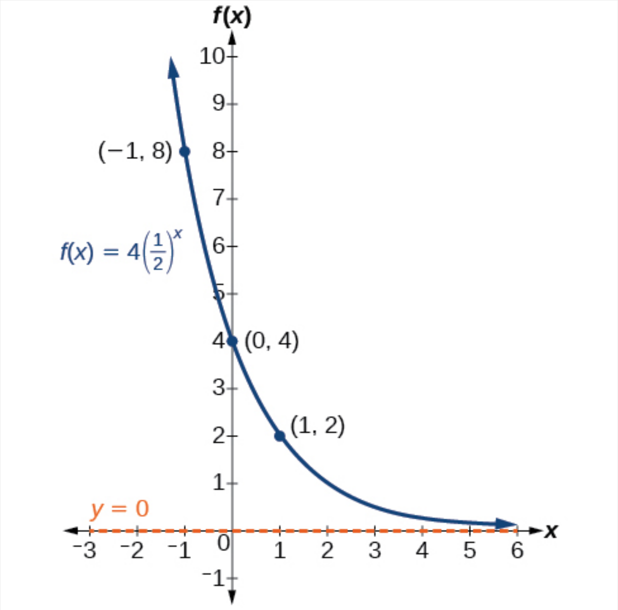
Before graphing, identify the behavior and key points on the graph.

- Since $b = \frac{1}{2}$ is between zero and one, the left tail of the graph will increase without bound as x decreases, and the right tail will approach the *x*-axis as x increases.
- Since $a = 4$, the graph of $f(x) = \left(\frac{1}{2}\right)^x$ will be stretched by a factor of 4.
- Create a table of points as shown in [link].

x	-3	-2	-1	0	1	2	3
$f(x) = 4\left(\frac{1}{2}\right)^x$	32	16	8	4	2	1	0.5

- Plot the *y*-intercept, $(0,4)$, along with two other points. We can use $(-1,8)$ and $(1,2)$.

Draw a smooth curve connecting the points, as shown in [link].

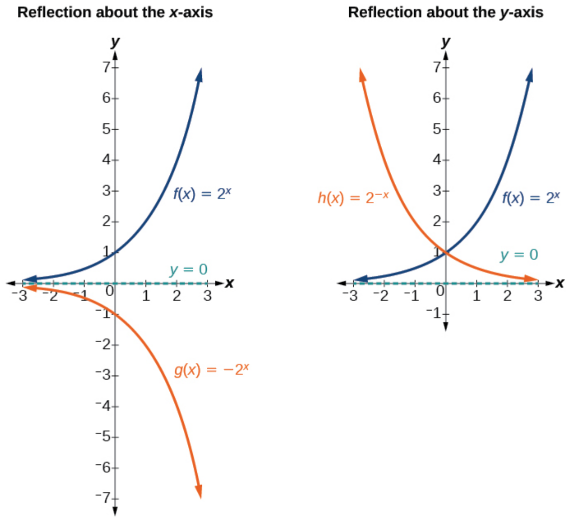


The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.

Reflections

In addition to shifting, compressing, and stretching a graph, we can also reflect it about the *x*-axis or the *y*-axis. When we multiply the parent function $f(x) = b^x$ by -1, we get a reflection about the *x*-axis. When we multiply the input by -1, we get a reflection about the *y*-axis. For example, if we begin by graphing the parent function $f(x) = 2^x$, we can then graph the two reflections alongside it. The reflection about the *x*-axis, $g(x) = -2^x$, is shown on the left side of [link], and the reflection about

the *y*-axis $h(x)=2^{-x}$, is shown on the right side of [link].



The function $f(x)=-a^x$

- reflects the parent function $f(x)=a^x$ about the *x*-axis.
- has a *y*-intercept of $(0,-1)$.
- has a range of $(-\infty,0)$.
- has a horizontal asymptote at $y=0$ and domain of $(-\infty,\infty)$, which are unchanged from the parent function.

The function $f(x)=a^{-x}$

- reflects the parent function $f(x)=a^x$ about the *y*-axis.
- has a *y*-intercept of $(0,1)$, a horizontal

asymptote at $y=0$, a range of $(0,\infty)$, and a domain of $(-\infty,\infty)$, which are unchanged from the parent function.

Writing and Graphing the Reflection of an Exponential Function

Find and graph the equation for a function, $g(x)$, that reflects $f(x)=\left(\frac{1}{4}\right)^x$ about the *x*-axis. State its domain, range, and asymptote.

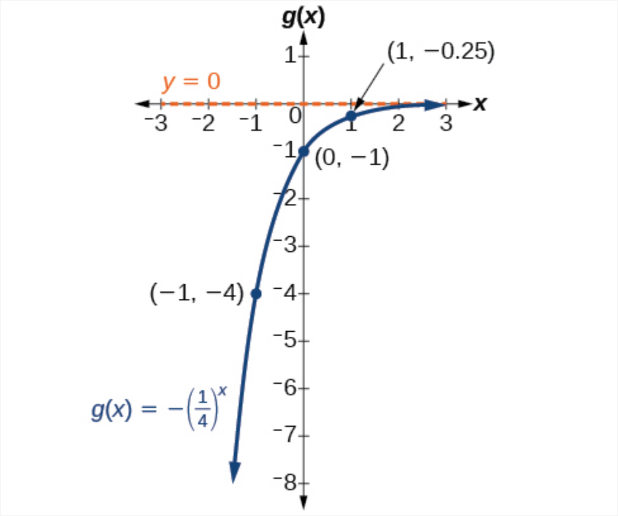
Since we want to reflect the parent function $f(x)=\left(\frac{1}{4}\right)^x$ about the *x*-axis, we multiply $f(x)$ by -1 to get, $g(x)=-\left(\frac{1}{4}\right)^x$. Next we create a table of points as in [link].

x	-3	-2	-1	0	1	2	3
$g(x)=-\left(\frac{1}{4}\right)^x$	-64	-16	-4	-1	-0.25	-0.0625	-0.0156

Plot the *y*-intercept, $(0,-1)$, along with two other points. We can use $(-1,-4)$ and $(1,$

-0.25).

Draw a smooth curve connecting the points:



The domain is $(-\infty,\infty)$; the range is $(-\infty,0)$; the horizontal asymptote is $y=0$.

Translations of the Parent Function $f(x)=b^x$	
Translation	**Form**
Shift • Horizontally c units to the left	$f(x)=b^{x+c}+d$

• Vertically d units up	
Stretch and Compress • Stretch if $\lvert a \rvert > 1$ • Compression if $0 < \lvert a \rvert < 1$	$f(x)=ab^x$
Reflect about the *x* axis	$f(x)=-b^x$
Reflect about the *y* axis	$f(x)=b^{-x}=\left(\frac{1}{b}\right)^x$
General equation for all translations	$f(x)=ab^{x+c}+d$

Translations of Exponential Functions

A translation of an exponential function has the form

$$f(x)=ab^{x+c}+d$$

Where the parent function, $y=b^x$, $b>1$, is

- shifted horizontally c units to the left.
- stretched vertically by a factor of $\lvert a \rvert$ if $\lvert a \rvert > 0$.
- compressed vertically by a factor of $\lvert a \rvert$ if $0 < \lvert a \rvert < 1$.
- shifted vertically d units.
- reflected about the *x*-axis when $a<0$.

Note the order of the shifts, transformations, and reflections follow the order of operations.

Natural Base e

All of our exponential functions have had either an integer or a rational number as the base. We will now look at an exponential function with an irrational number as the base.

Before we can look at this exponential function, we need to define the irrational number, *e*. This number is used as a base in many applications in the sciences and business that are modeled by exponential functions. The number is defined as the value of $(1+1n)n$ as *n* gets larger and larger. We say, as *n* approaches infinity, or increases without bound. The table shows the value of $(1+1n)n$ for several values of n.

n	$(1+1n)n$
1	2
2	2.25
5	2.48832
10	2.59374246
100	2.704813829…

1,000	2.716923932…
10,000	2.718145927…
100,000	2.718268237…
1,000,000	2.718280469…
1,000,000,000	2.718281827…

$e \approx 2.718281827$

The number *e* is like the number π in that we use a symbol to represent it because its decimal representation never stops or repeats. The irrational number *e* is called the **natural base**.

Natural Base e

The number *e* is defined as the value of $(1+\frac{1}{n})^n$, as *n* increases without bound. We say, as *n* approaches infinity,

$e \approx 2.718281827\ldots$

The exponential function whose base is e, $f(x)=e^x$ is called the **natural exponential function**.

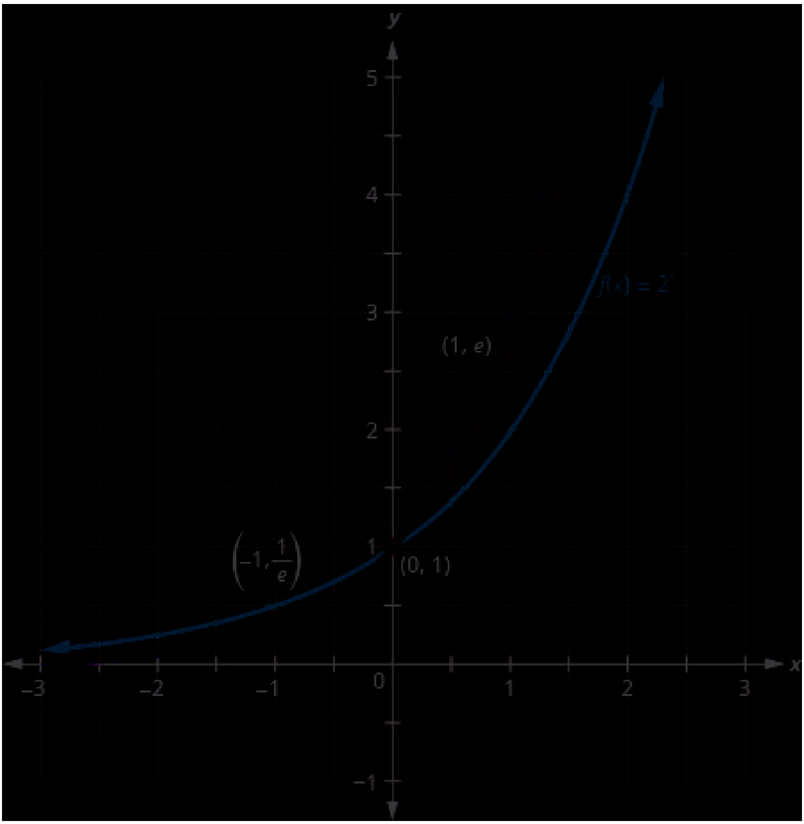
Natural Exponential Function

The natural exponential function is an exponential function whose base is e

$f(x) = ex$

The domain is $(-\infty, \infty)$ and the range is $(0, \infty)$.

Let’s graph the function $f(x) = ex$ on the same coordinate system as $g(x) = 2x$ and $h(x) = 3x$.



Notice that the graph of $f(x) = ex$ is “between” the graphs of $g(x) = 2x$ and $h(x) = 3x$. Does this make sense as $2 < e < 3$?

Use Exponential Models in Applications

Exponential functions model many situations. If you own a bank account, you have experienced the use of an exponential function. There are two formulas that are used to determine the balance in the account when interest is earned. If a principal, *P*, is invested at an interest rate, *r*, for *t* years, the new balance, *A*, will depend on how often the interest is compounded. If the interest is compounded *n* times a year we use the formula $A=P(1+rn)nt$. If the interest is compounded continuously, we use the formula $A=Pert$. These are the formulas for **compound interest**.

Compound Interest
For a principal, *P*, invested at an interest rate, *r*, for *t* years, the new balance, *A*, is:
$A=P(1+rn)nt$when compoundedntimes a year.
$A=Pert$when compounded continuously.

As you work with the Interest formulas, it is often helpful to identify the values of the variables first and then substitute them into the formula.

A total of \$10,000 was invested in a college fund for a new grandchild. If the interest rate is 5%, how much will be in the account in 18 years by each method of compounding?

ⓐ compound quarterly

ⓑ compound monthly

ⓒ compound continuously

Identify the values of each variable in the formulas.Remember to express the percent as a decimal.$A=?P=\$10,000r=0.05t=18$years

ⓐ

For quarterly compounding,$n=4$.There are 4quarters in a year.$A=P(1+rn)nt$ Substitute the values in the formula.$A=10,000(1+0.054)4\cdot18$ Compute the amount. Be careful to consider theorder of operations as you enter the expression intoyour calculator.$A=\$24,459.20$

ⓑ

For monthly compounding,$n=12$.There are 12months in a year.$A=P(1+rn)nt$ Substitute the values in the formula.$A=10,000(1+0.0512)12\cdot18$ Compute the amount.$A=\$24,550.08$

ⓒ
For compounding continuously, $A=Pe^{rt}$ Substitute the values in the formula. $A=10{,}000e^{0.05\cdot 18}$ Compute the amount. $A=\$24{,}596.03$

Angela invested \$15,000 in a savings account. If the interest rate is 4%, how much will be in the account in 10 years by each method of compounding?

ⓐ compound quarterly

ⓑ compound monthly

ⓒ compound continuously

ⓐ \$22,332.96
ⓑ \$22,362.49 ⓒ \$22,377.37

Other topics that are modeled by exponential functions involve growth and decay. Both also use the formula $A=Pe^{rt}$ we used for the growth of money. For growth and decay, generally we use A_0,

as the original amount instead of calling it P, the principal. We see that **exponential growth** has a positive rate of growth and **exponential decay** has a negative rate of growth.

Exponential Growth and Decay

For an original amount, A_0, that grows or decays at a rate, *r*, for a certain time, *t*, the final amount, *A*, is:

$A = A_0 e^{rt}$

Exponential growth is typically seen in the growth of populations of humans or animals or bacteria. Our next example looks at the growth of a virus.

Chris is a researcher at the Center for Disease Control and Prevention and he is trying to understand the behavior of a new and dangerous virus. He starts his experiment with 100 of the virus that grows at a rate of 25% per hour. He will check on the virus in 24 hours. How many viruses will he find?

Identify the values of each variable in the formulas.Be sure to put the percent in decimal form.Be sure the units match—the rate is per hour andthe time is in hours.A = ? $A0 = 100r = 0.25/hourt = 24$hours Substitute the values in the formula:$A = A0ert.A = 100e0.25·24$ Compute the amount.A = 40,342.88 Round to the nearest whole virus.A = 40,343 The researcher will find 40,343viruses.

Maria, a biologist is observing the growth pattern of a virus. She starts with 100 of the virus that grows at a rate of 10% per hour. She will check on the virus in 24 hours. How many viruses will she find?

She will find 1,102 viruses.

Access these online resources for additional instruction and practice with evaluating and graphing exponential functions.

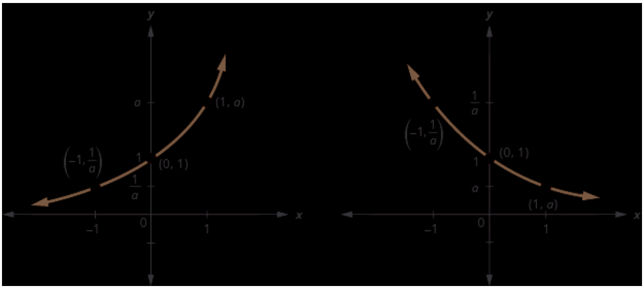
- Graphing Exponential Functions

- Solving Exponential Equations
- Applications of Exponential Functions
- Continuously Compound Interest
- Radioactive Decay and Exponential Growth

Key Concepts

- **Properties of the Graph of** $f(x) = ax$:

when $a > 1$		**when** $0 < a < 1$	
Domain	$(-\infty, \infty)$	Domain	$(-\infty, \infty)$
Range	$(0, \infty)$	Range	$(0, \infty)$
x intercept	none	x intercept	none
y intercept	$(0,1)$	y intercept	$(0,1)$
Contains	$(1,a)$, $(-1,1a)$	Contains	$(1,a)$, $(-1,1a)$
Asymptote	x-axis, the line $y = 0$	Asymptote	x-axis, the line $y = 0$
Basic shape	increasing	Basic shape	decreasing



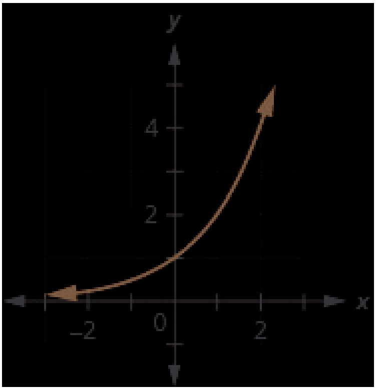
- **One-to-One Property of Exponential Equations:**
  For $a > 0$ and $a \neq 1$,
  $A = A0ert$
- **Compound Interest:** For a principal, P, invested at an interest rate, r, for t years, the new balance, A, is
  $A = P(1 + rn)ntwhen$ compoundedntimes a year.
  $A = Pertwhen$ compounded continuously.
- **Exponential Growth and Decay:** For an original amount, A0 that grows or decays at a rate, r, for a certain time t, the final amount,A, is $A = A0ert$.

Practice Makes Perfect

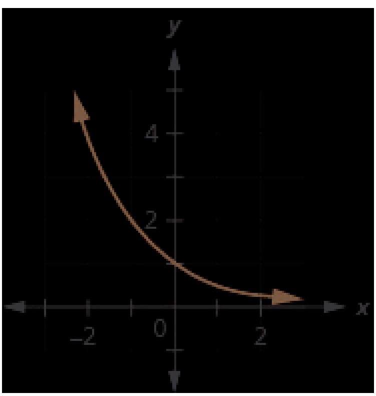
Graph Exponential Functions

In the following exercises, graph each exponential function.

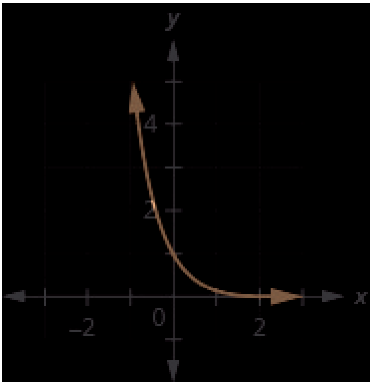
$f(x) = 2x$



$f(x) = (12)x$

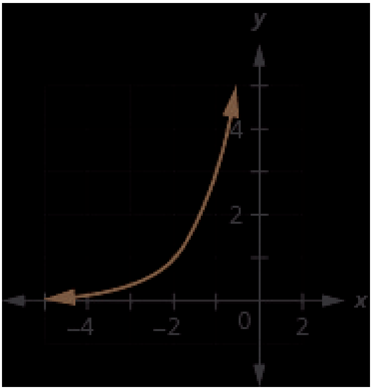


$f(x) = (16)x$

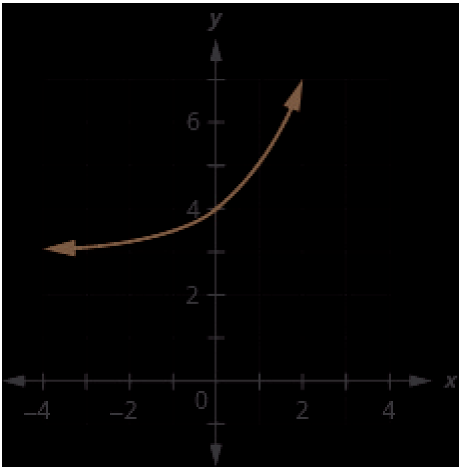


In the following exercises, graph each exponential function.

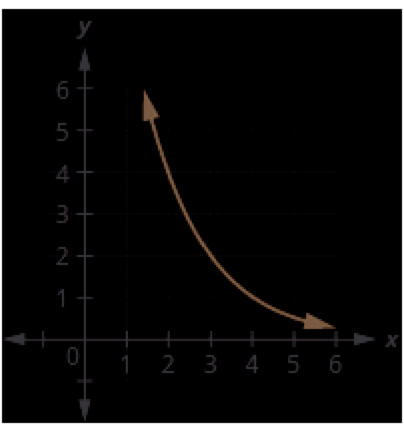
$f(x) = 3x + 2$



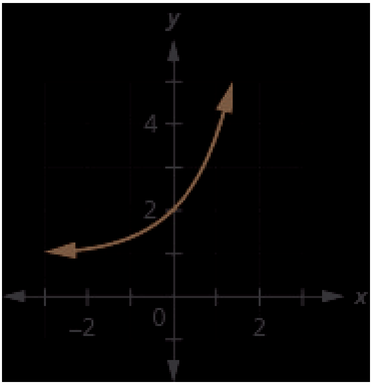
$f(x) = 2x + 3$



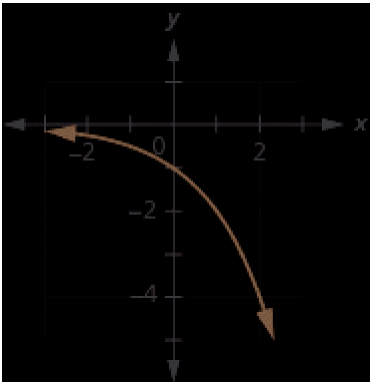
$f(x) = (12)x - 4$



$f(x) = ex + 1$

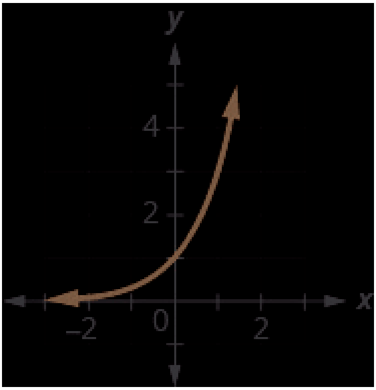


$f(x) = -2x$

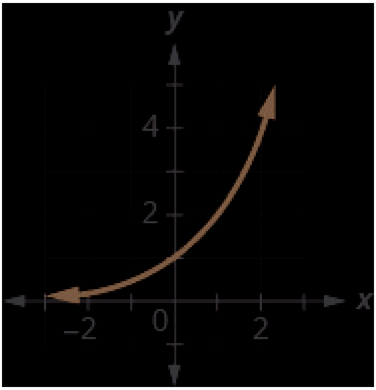


In the following exercises, match the graphs to one of the following functions: ⓐ $2x$ ⓑ $2x+1$ ⓒ $2x-1$

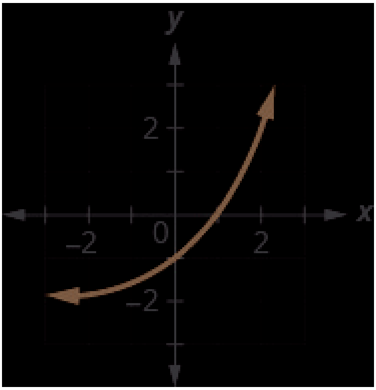
ⓓ 2^{x+2} ⓔ 2^{x-2} ⓕ 3^{x}



ⓕ



ⓐ



ⓔ

Use exponential models in applications

In the following exercises, use an exponential model to solve.

Edgar accumulated $5,000 in credit card debt. If the interest rate is 20% per year, and he does not make any payments for 2 years, how much will he owe on this debt in 2 years by each method of compounding?
ⓐ compound quarterly
ⓑ compound monthly
ⓒ compound continuously

ⓐ \$7,387.28 ⓑ \$7,434.57 ⓒ \$7,459.12

Rochelle deposits \$5,000 in an IRA. What will be the value of her investment in 25 years if the investment is earning 8% per year and is compounded continuously?

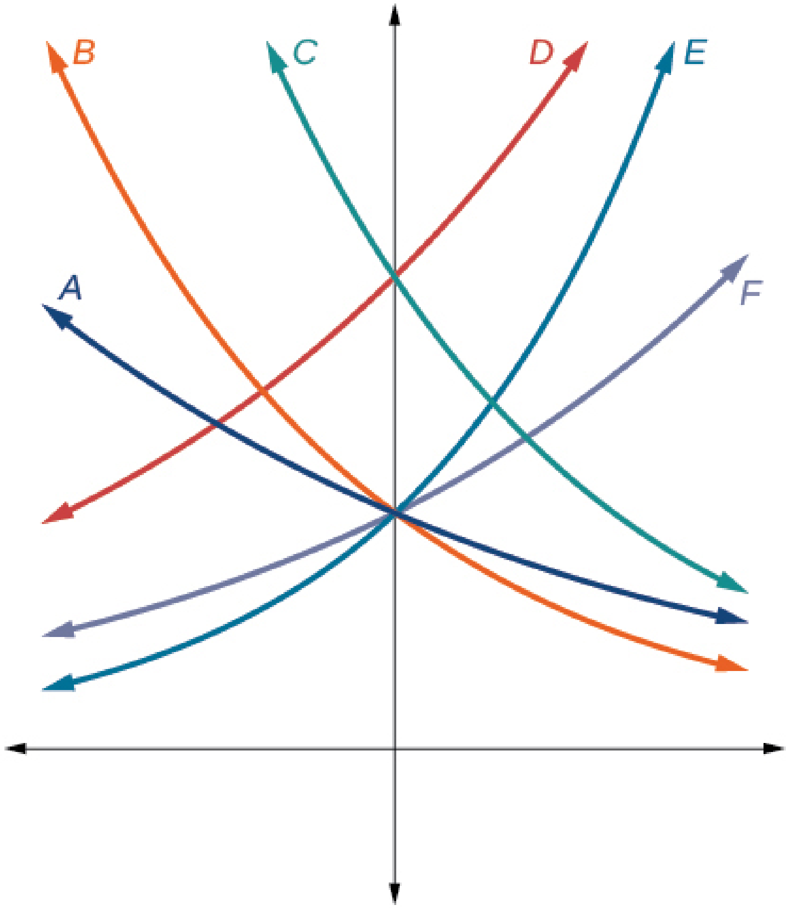
\$36,945.28

A researcher at the Center for Disease Control and Prevention is studying the growth of a bacteria. He starts his experiment with 100 of the bacteria that grows at a rate of 6% per hour. He will check on the bacteria every 8 hours. How many bacteria will he find in 8 hours?

---

159 bacteria

For the following exercises, match each function with one of the graphs in [link].



$f(x) = 2(0.69)^x$

B

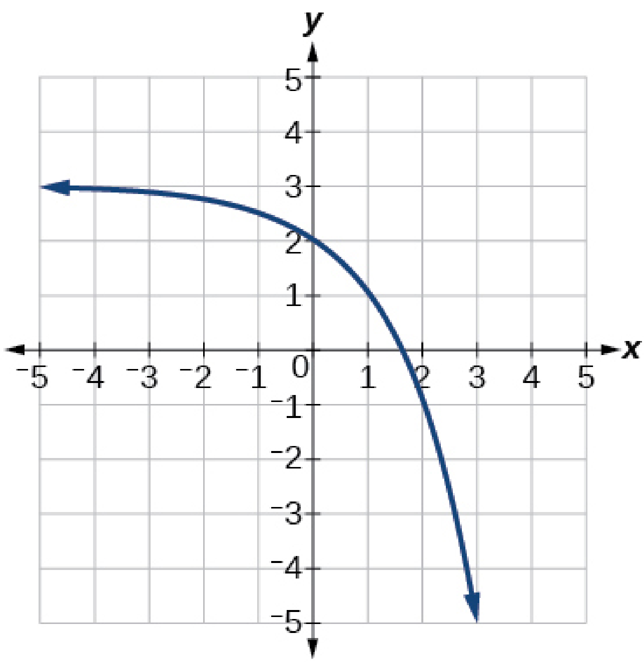
$f(x) = 2(0.81)^x$

A

$f(x)=2(1.59)x$

E

For the following exercises, each graph is a transformation of $y=2x$. Write an equation describing the transformation.



$y=-2x+3$

Glossary

asymptote
: A line which a graph of a function approaches closely but never touches.

exponential function
: An exponential function, where $a > 0$ and $a \neq 1$, is a function of the form $f(x) = ax$.

natural base
: The number *e* is defined as the value of $(1 + 1n)n$, as *n* gets larger and larger. We say, as *n* increases without bound, $e \approx 2.718281827...$

natural exponential function
: The natural exponential function is an exponential function whose base is *e*: $f(x) = ex$. The domain is $(-\infty, \infty)$ and the range is $(0, \infty)$.

Logarithmic Functions (4.2)

By the end of this section, you will be able to:

- Convert between exponential and logarithmic form
- Evaluate logarithmic functions
- Graph Logarithmic functions
- Solve logarithmic equations
- Use logarithmic models in applications

This Module supports section **4.2 of Mat 1023**.

In case you missed something in class, or just want to review a specific topic covered in this Module, here is a list of topics covered:

1. Concert Exponential to Logarithmic Functions [link]
2. Evaluate Logarithmic Functions [link]
3. Basic Log Properties [link]
4. Graphing Logarithmic Functions [link]
5. Domain of Logarithmic Functions [link]
6. Transformations of Logarithmic Functions [link]
7. Summary of Transformations of Logarithms [link]
8. Natural and Common Logarithms [link]
9. Applications with Logarithms [link]
10. Key Concepts [link]

Function and Inverse Recap

- **Horizontal Line Test:** If every horizontal line, intersects the graph of a function in at most one point, it is a one-to-one function.
- **Inverse of a Function Defined by Ordered Pairs:** If f(x) is a one-to-one function whose ordered pairs are of the form (x,y), then its inverse function $f-1(x)$ is the set of ordered pairs (y,x).
- **Inverse Functions:** For every x in the domain of one-to-one function f and $f-1$, $f-1(f(x))=xf(f-1(x))=x$
- **How to Find the Inverse of a One-to-One Function:**

  Substitute *y* for f(x). Interchange the variables *x* and *y*. Solve for *y*. Substitute $f-1(x)$ for y.

- The inverse function 'undoes' what the original function did to a value in its domain in order to get back to the original *x*-value. This holds for all x in the domain of f. Informally, this means that inverse functions "undo" each other. However, only one-to-one functions have inverses that are functions.

We have spent some time finding the inverse of many functions. It works well to 'undo' an operation with another operation. Subtracting 'undoes' addition, multiplication 'undoes' division, taking the

square root ‘undoes’ squaring.

As we studied the exponential function, we saw that it is one-to-one as its graphs pass the horizontal line test. This means an exponential function does have an inverse. If we try our algebraic method for finding an inverse, we run into a problem.

Rewrite with$y=f(x)$.Interchange the variables$x$and$y$.$f(x)=a^x$ $y=a^x$ $x=a^y$Solve for$y$.Oops! We have no way to solve for$y$!

To deal with this we define the logarithm function with base *a* to be the inverse of the exponential function $f(x)=a^x$. We use the notation $f^{-1}(x)=\log_a x$ and say the inverse function of the exponential function is the logarithmic function.

Logarithmic Function
The function $f(x)=\log_a x$ is the **logarithmic function** with base a, where $a>0$, $x>0$, and $a\neq 1$.
$y=\log_a x$is equivalent to$x=a^y$

Convert Between Exponential and Logarithmic Form

Since the equations $y=\log_a x$ and $x=a^y$ are equivalent, we can go back and forth between them. This will often be the method to solve some exponential and logarithmic equations. To help with converting back and forth let's take a close look at the equations. See [link]. Notice the positions of the exponent and base.

If we realize the logarithm is the exponent it makes the conversion easier. You may want to repeat, "base to the exponent give us the number."

Can we take the logarithm of a negative number?

No. Because the base of an exponential function is always positive, no power of that base can ever be negative. We can never take the logarithm of a negative number. Also, we cannot take the logarithm of zero. Calculators may output a log of a negative number when in complex mode, but the log of a negative number is not a real number.

Convert to logarithmic form: ⓐ $23=8$, ⓑ $512=5$, and ⓒ $(12)x=116$.

[redacted]

Convert to logarithmic form: ⓐ $32=9$ ⓑ $712=7$ ⓒ $(13)x=127$

ⓐ $\log 39=2$
ⓑ $\log 77=12$ ⓒ $\log 13127=x$

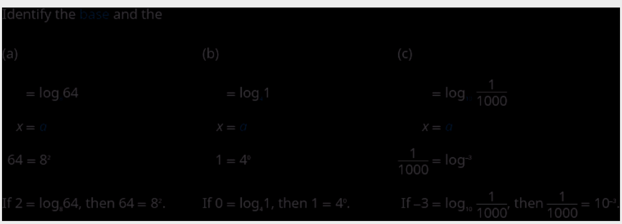
Convert to logarithmic form: ⓐ $43=64$ ⓑ $413=43$ ⓒ $(12)x=132$

ⓐ $\log 464=3$

ⓑ log443 = 13 ⓒ log12132 = x

In the next example we do the reverse—convert logarithmic form to exponential form.

Convert to exponential form: ⓐ 2 = log864, ⓑ 0 = log41, and ⓒ −3 = log1011000.



Convert to exponential form: ⓐ 3 = log464 ⓑ 0 = logx1 ⓒ −2 = log101100

ⓐ 64 = 43

ⓑ $1 = x0$ ⓒ $1100 = 10 - 2$

Evaluate Logarithmic Functions

We can solve and evaluate logarithmic equations by using the technique of converting the equation to its equivalent exponential equation.

Find the value of x: ⓐ $\log x36 = 2$, ⓑ $\log 4x = 3$, and ⓒ $\log 1218 = x$.

ⓐ

$\log x36 = 2$ Convert to exponential form. $x2 = 36$ Solve the quadratic. $x = 6, x = -6$ The base of a logarithmic function must be positive, so we eliminate $x = -6$. $x = 6$ Therefore, $\log 636 = 2$.

ⓑ

$\log 4x = 3$ Convert to exponential form. $43 = x$ Simplify. $x = 64$ Therefore, $\log 464 = 3$.

ⓒ

log1218 = x Convert to exponential form. (12)x = 18 Rewrite18as(12)3.(12)x = (12)3 With the same base, the exponents must be equal.x = 3Therefore,log1218 = 3

Find the value of x: ⓐ logx64 = 2 ⓑ log5x = 3 ⓒ log1214 = x

ⓐ x = 8 ⓑ x = 125 ⓒ x = 2

When see an expression such as log327, we can find its exact value two ways. By inspection we realize it means “3 to what power will be 27”? Since 33 = 27, we know log327 = 3. An alternate way is to set the expression equal to x and then convert it into an exponential equation.

Now consider solving $\log_7 49$ and $\log_3 27$ mentally.

- We ask, “To what exponent must 7 be raised in order to get 49?” We know $7^2 = 49$. Therefore, $\log_7 49 = 2$

- We ask, “To what exponent must 3 be raised in order to get 27?” We know $3^3=27$. Therefore, $\log_3 27=3$

Even some seemingly more complicated logarithms can be evaluated without a calculator. For example, let’s evaluate $\log_{\frac{2}{3}} \frac{4}{9}$ mentally.

- We ask, “To what exponent must $\frac{2}{3}$ be raised in order to get $\frac{4}{9}$? ” We know $2^2=4$ and $3^2=9$, so $\left(\frac{2}{3}\right)^2=\frac{4}{9}$. Therefore, $\log_{\frac{2}{3}}\left(\frac{4}{9}\right)=2$.

Find the exact value of each logarithm without using a calculator:
ⓐ $\log_5 25$,
ⓑ $\log_9 3$, and ⓒ $\log_2 \frac{1}{16}$.

ⓐ
$\log_5 25$ 5 to what power will be25? $\log_5 25=2$
Or Set the expression equal to x. $\log_5 25=x$
Change to exponential form. $5^x=25$ Rewrite 25 as 5^2. $5^x=5^2$ With the same base the exponents must be equal. $x=2$ Therefore, $\log_5 25=2$.

ⓑ

log93Set the expression equal tox.log93 = x Change to exponential form.9x = 3 Rewrite 9 as32.(32)x = 31 Simplify the exponents.32x = 31 With the same base the exponents must be equal.2x = 1Solve the equation.x = 12Therefore,log93 = 12.

ⓒ
log2116 Set the expression equal tox.log2116 = x Change to exponential form.2x = 116 Rewrite 16 as24.2x = 124
2x = 2 − 4 With the same base the exponents must be equal.x = − 4Therefore,log2116 = − 4.

Find the exact value of each logarithm without using a calculator:
ⓐ log12144
ⓑ log42
ⓒ log2132

ⓐ
2 ⓑ 12 ⓒ − 5

Basic Log Properties

Now that we have learned about exponential and logarithmic functions, we can introduce some of the properties of logarithms. These will be very helpful as we continue to solve both exponential and logarithmic equations.

The first two properties derive from the definition of logarithms. Since $a^0=1$, we can convert this to logarithmic form and get $\log_a 1=0$. Also, since $a^1=a$, we get $\log_a a=1$.

Properties of Logarithms
$\log_a 1=0 \quad \log_a a=1$

In the next example we could evaluate the logarithm by converting to exponential form, as we have done previously, but recognizing and then applying the properties saves time.

Evaluate using the properties of logarithms: ⓐ log81 and ⓑ log66.

ⓐ

log81Use the property,loga1 = 0.0log81 = 0

ⓑ

log66Use the property,logaa = 1.1log66 = 1

Evaluate using the properties of logarithms: ⓐ log131 ⓑ log99.

ⓐ 0 ⓑ 1

The next two properties can also be verified by converting them from exponential form to logarithmic form, or the reverse.

The exponential equation alogax = x converts to the logarithmic equation logax = logax, which is a true statement for positive values for *x* only.

The logarithmic equation logaax = x converts to the

exponential equation ax=ax, which is also a true statement.

These two properties are called inverse properties because, when we have the same base, raising to a power “undoes” the log and taking the log “undoes” raising to a power. These two properties show the composition of functions. Both ended up with the identity function which shows again that the exponential and logarithmic functions are inverse functions.

Inverse Properties of Logarithms
For $a>0, x>0$ and $a\neq 1$,
alogax=xlogaax=x

In the next example, apply the inverse properties of logarithms.

Evaluate using the properties of logarithms: ⓐ 4log49 and ⓑ log335.
ⓐ

4log49 Use the property,alogax=x.94log49=9

ⓑ
log335 Use the property,alogax=x.5log335=5

Evaluate using the properties of logarithms: ⓐ 5log515 ⓑ log774.

ⓐ 15 ⓑ 4

Notice that the graphs of $f(x)=2^x$ and $g(x)=\log_2(x)$ are reflections about the line $y=x$.

Graph Logarithmic Functions

To graph a logarithmic function y=logax, it is easiest to convert the equation to its exponential form, x=ay. Generally, when we look for ordered pairs for the graph of a function, we usually choose an *x*-value and then determine its corresponding *y*-value. In this case you may find it easier to choose *y*-values and then determine its corresponding *x*-value.

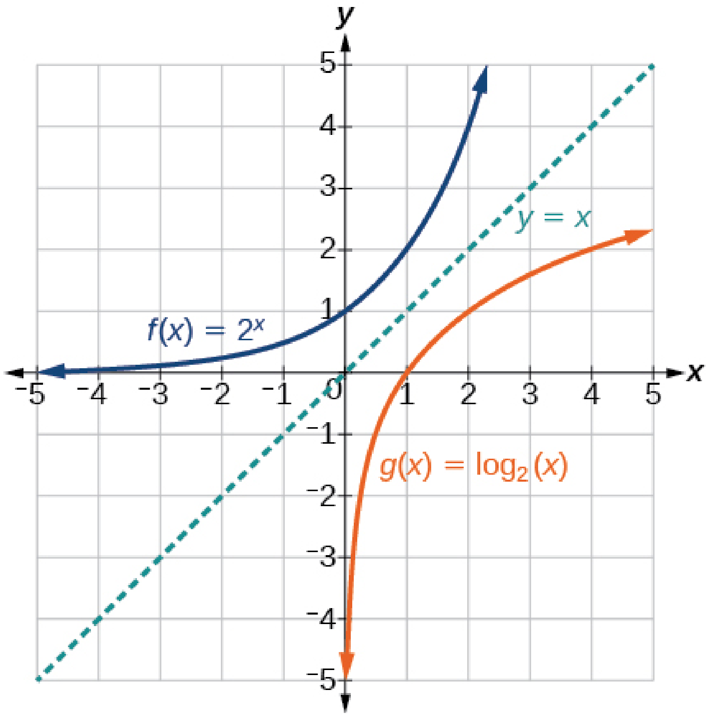
We begin with the parent function $y = \log_a(x)$. Because every logarithmic function of this form is the inverse of an exponential function with the form $y = a^x$, their graphs will be reflections of each other across the line $y = x$. To illustrate this, we can observe the relationship between the input and output values of $y = 2^x$ and its equivalent $x = \log_2(y)$ in [link].

x	-3	-2	-1	0	1	2	3
$2^x = y$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$\log_2(y) = x$	-3	-2	-1	0	1	2	3

Using the inputs and outputs from [link], we can build another table to observe the relationship between points on the graphs of the inverse functions $f(x) = 2^x$ and $g(x) = \log_2(x)$. See [link].

$f(x)=2^x$	$\left(-3,\frac{1}{8}\right)$	$\left(-2,\frac{1}{4}\right)$	$\left(-1,\frac{1}{2}\right)$	$(0,1)$	$(1,2)$	$(2,4)$	$(3,8)$
$g(x)=\log_2(x)$	$\left(\frac{1}{8},-3\right)$	$\left(\frac{1}{4},-2\right)$	$\left(\frac{1}{2},-1\right)$	$(1,0)$	$(2,1)$	$(4,2)$	$(8,3)$

As we'd expect, the *x*- and *y*-coordinates are reversed for the inverse functions. [link] shows the graph of f and g.

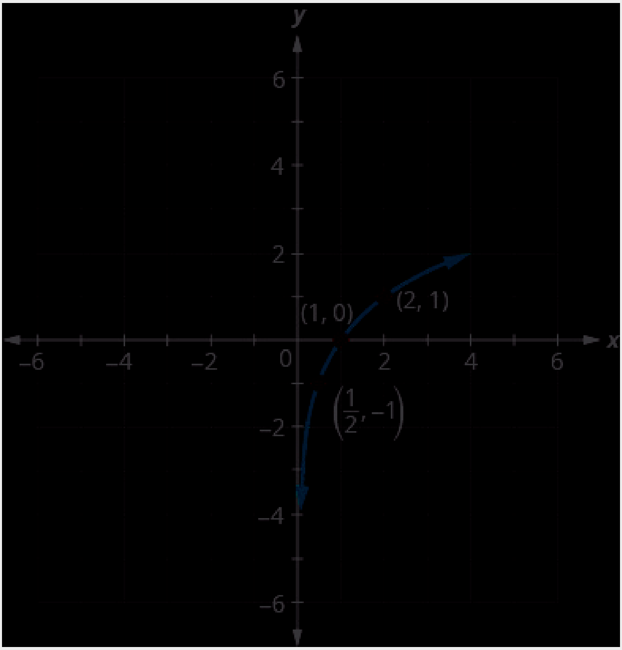


Graph $y=\log_2 x$.

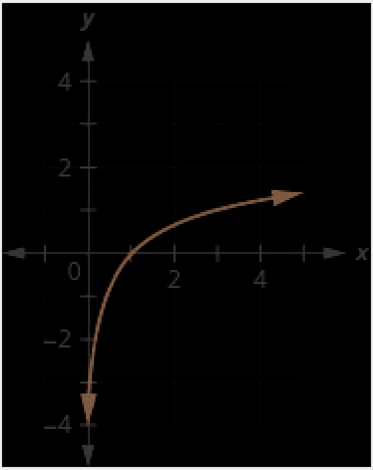
To graph the function, we will first rewrite the logarithmic equation, $y = \log2x$, in exponential form, $2y = x$.

We will use point plotting to graph the function. It will be easier to start with values of *y* and then get *x*.

y	**$2y = x$**	**(x,y)**
-2	$2-2=122=14$	$(14,2)$
-1	$2-1=121=12$	$(12,-1)$
0	$20=1$	$(1,0)$
1	$21=2$	$(2,1)$
2	$22=4$	$(4,2)$
3	$23=8$	$(8,3)$



Graph: $y = \log3x$.



The graphs of $y = log2x$, and $y = log3x$, are the shape we expect from a logarithmic function where $a > 1$.

We notice that for each function the graph contains the point (1,0). This make sense because $0 = loga1$ means $a0 = 1$ which is true for any *a*.

The graph of each function, also contains the point (a,1). This makes sense as $1 = logaa$ means $a1 = a$. which is true for any *a*.

Notice too, the graph of each function $y = logax$ also contains the point $(1a, -1)$. This makes sense as $-1 = loga1a$ means $a-1 = 1a$, which is true for any *a*.

Look at each graph again. Now we will see that

many characteristics of the logarithm function are simply 'mirror images' of the characteristics of the corresponding exponential function.

What is the domain of the function? The graph never hits the *y*-axis. The domain is all positive numbers. We write the domain in interval notation as $(0, \infty)$.

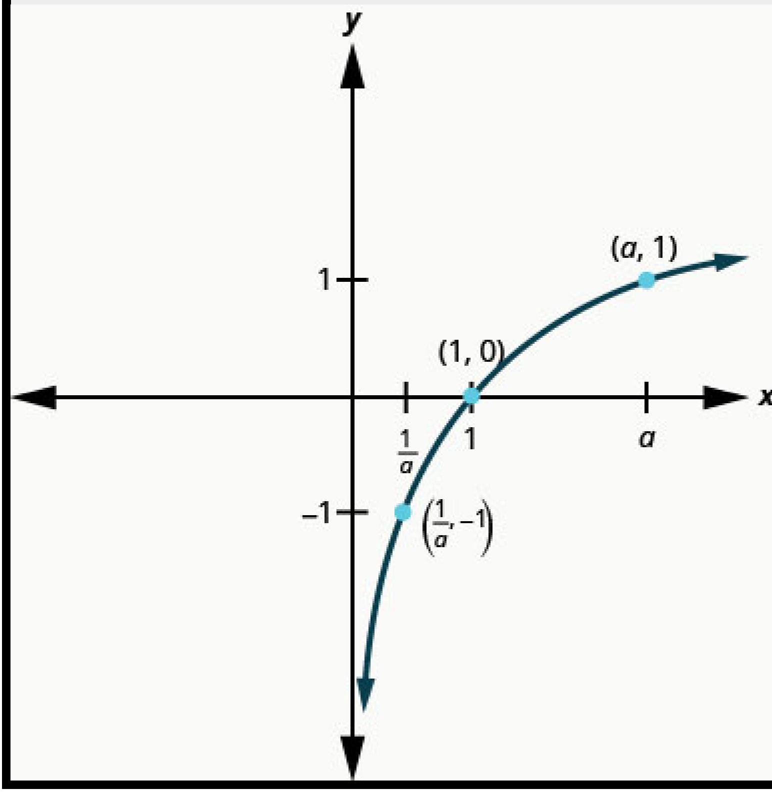
What is the range for each function? From the graphs we can see that the range is the set of all real numbers. There is no restriction on the range. We write the range in interval notation as $(-\infty, \infty)$.

When the graph approaches the *y*-axis so very closely but will never cross it, we call the line $x=0$, the *y*-axis, a vertical asymptote.

Properties of the Graph of $y=\log_a x$ when $a>1$

Domain	$(0,\infty)$
Range	$(-\infty,\infty)$
x intercept	$(1,0)$
y intercept	None
Contains	$(a,1)$,$(\frac{1}{a},-1)$

Asymptote	y-axis



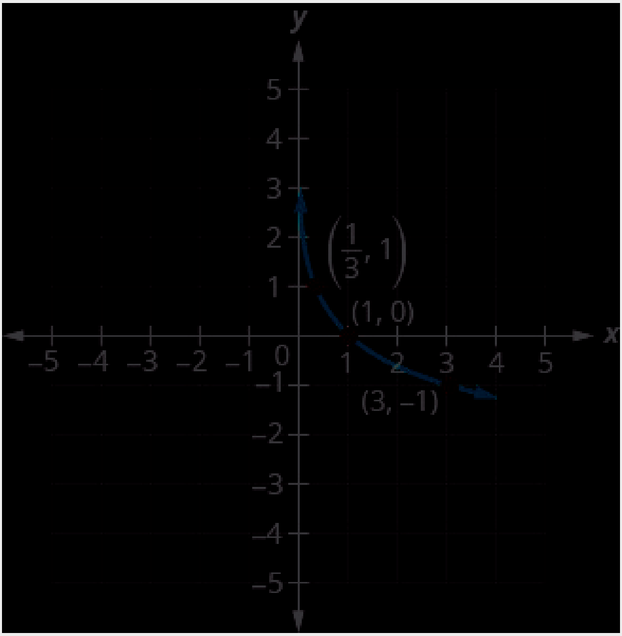
Our next example looks at the graph of $y = \log_a x$ when $0 < a < 1$.

Graph $y = \log 13x$.

To graph the function, we will first rewrite the logarithmic equation, $y = \log 13x$, in exponential form, $(13)y = x$.

We will use point plotting to graph the function. It will be easier to start with values of *y* and then get *x*.

y	**(13)y = x**	**(x,y)**
−2	(13)−2 = 32 = 9	(9, −2)
−1	(13)−1 = 31 = 3	(3, −1)
0	(13)0 = 1	(1,0)
1	(13)1 = 13	(13,1)
2	(13)2 = 19	(19,2)
3	(13)3 = 127	(127,3)



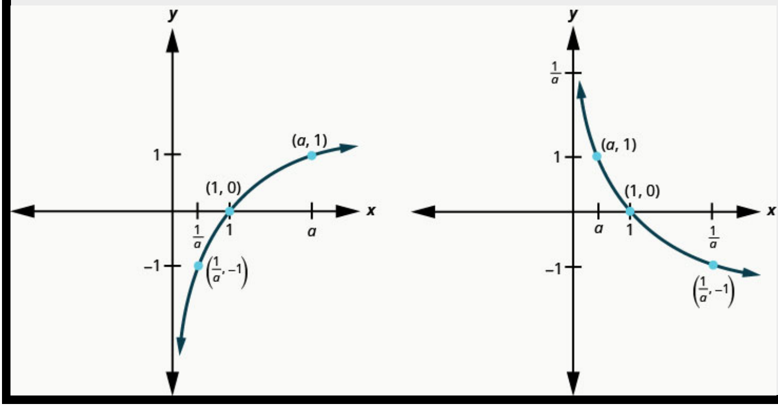
The graphs of all have the same basic shape. While this is the shape we expect from a logarithmic function where $0<a<1$.

We notice, that for each function again, the graph contains the points,$(1,0)$,$(a,1)$,$(1a,-1)$. This make sense for the same reasons we argued above.

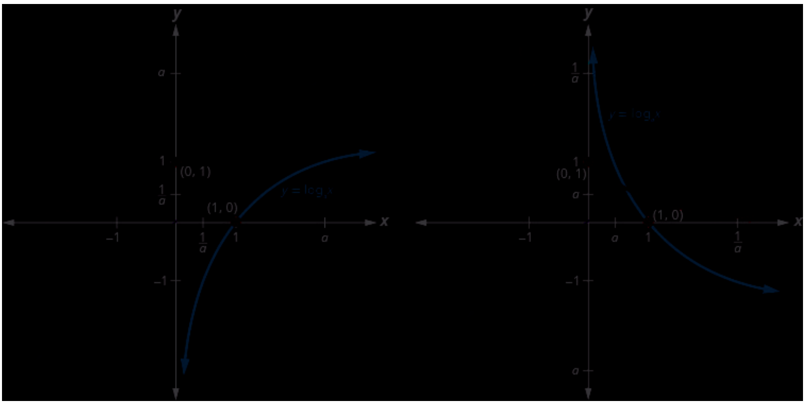
We notice the domain and range are also the same—the domain is $(0,\infty)$ and the range is $(-\infty,\infty)$. The y-axis is again the vertical asymptote.

Properties of the Graph of $y = \log_a x$

when $a > 1$		**when $0 < a < 1$**	
Domain	$(0, \infty)$	Domain	$(0, \infty)$
Range	$(-\infty, \infty)$	Range	$(-\infty, \infty)$
x intercept	$(1, 0)$	x intercept	$(1, 0)$
y intercept	none	y intercept	None
Contains	$(a,1),(\frac{1}{a},-1)$	Contains	$(a,1),(\frac{1}{a},-1)$
Asymptote	y axis	Asymptote	y axis
Basic shape	increasing	Basic shape	Decreasing



We talked earlier about how the logarithmic function $f^{-1}(x) = \log_a x$ is the inverse of the exponential function $f(x) = a^x$. The graphs in [link] show both the exponential (blue) and logarithmic (red) functions on the same graph for both $a > 1$ and $0 < a < 1$.



Notice how the graphs are reflections of each other through the line $y = x$. We know this is true of inverse functions. Keeping a visual in your mind of these graphs will help you remember the domain and range of each function. Notice the *x*-axis is the horizontal asymptote for the exponential functions and the *y*-axis is the vertical asymptote for the logarithmic functions.

Domain of Logarithmic Functions

Recall that the exponential function is defined as $y = b^x$ for any real number x and constant $b > 0$, $b \neq 1$, where

- The domain of y is $(-\infty, \infty)$.
- The range of y is $(0, \infty)$.

In the last section we learned that the logarithmic function $y = \log_b(x)$ is the inverse of the exponential function $y = b^x$. So, as inverse functions:

- The domain of $y = \log_b(x)$ is the range of $y = b^x$: $(0, \infty)$.
- The range of $y = \log_b(x)$ is the domain of $y = b^x$: $(-\infty, \infty)$.

Transformations of the parent function $y = \log_b(x)$ behave similarly to those of other functions. Just as with other parent functions, we can apply the four types of transformations—shifts, stretches, compressions, and reflections—to the parent function without loss of shape.

In the previous section we saw that certain transformations can change the *range* of $y = b^x$. Similarly, applying transformations to the parent function $y = \log_b(x)$ can change the *domain*. When finding the domain of a logarithmic function, therefore, it is important to remember that the domain consists *only of positive real numbers*. That is, the argument of the logarithmic function must be greater than zero.

For example, consider $f(x) = \log_4(2x - 3)$. This function is defined for any values of x such that the

argument, in this case $2x-3$, is greater than zero. To find the domain, we set up an inequality and solve for x:

$2x-3>0$ Show the argument greater than zero.
$2x>3$ Add 3. $x>1.5$ Divide by 2.

In interval notation, the domain of $f(x)=\log_4(2x-3)$ is $(1.5,\infty)$.

Given a logarithmic function, identify the domain.

1. Set up an inequality showing the argument greater than zero.
2. Solve for x.
3. Write the domain in interval notation.

Identifying the Domain of a Logarithmic Shift

What is the domain of $f(x)=\log_2(x+3)$?

The logarithmic function is defined only when the input is positive, so this function is defined when $x+3>0$. Solving this inequality,

$x+3>0$ The input must be positive. $x>

-3 Subtract 3.

The domain of $f(x) = \log_2(x+3)$ is $(-3, \infty)$.

Identifying the Domain of a Logarithmic Shift and Reflection

What is the domain of $f(x) = \log(5-2x)$?

The logarithmic function is defined only when the input is positive, so this function is defined when $5–2x > 0$. Solving this inequality,

$$\begin{aligned} 5-2x &> 0 && \text{The input must be positive.} \\ -2x &> -5 && \text{Subtract 5.} \\ x &< \frac{5}{2} && \text{Divide by } -2 \text{ and switch the inequality.} \end{aligned}$$

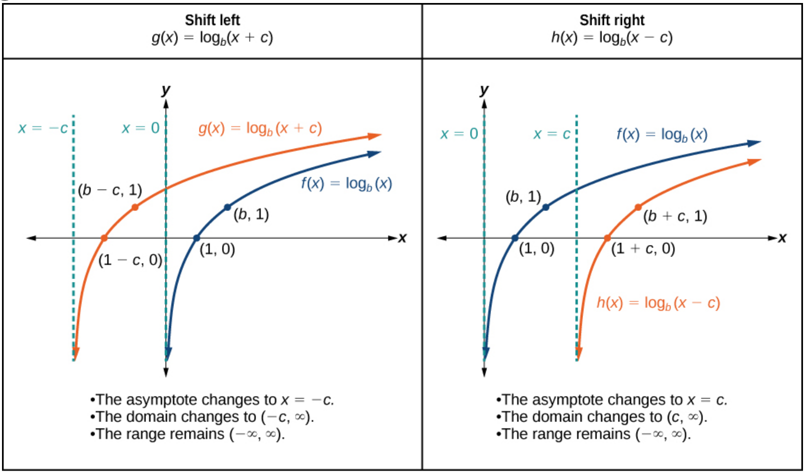
The domain of $f(x) = \log(5-2x)$ is $\left(-\infty, \frac{5}{2}\right)$.

Graphing Transformations of Logarithmic Functions

As we mentioned in the beginning of the section, transformations of logarithmic graphs behave similarly to those of other parent functions. We can shift, stretch, compress, and reflect the parent function $y=\log_a(x)$ without loss of shape.

Graphing a Horizontal Shift of $f(x) = \log_a(x)$

When a constant c is added to the input of the parent function $f(x)=\log_a(x)$, the result is a horizontal shift c units in the *opposite* direction of the sign on c. To visualize horizontal shifts, we can observe the general graph of the parent function $f(x)=\log_a(x)$ and for $c>0$ alongside the shift left, $g(x)=\log_a(x+c)$, and the shift right, $h(x)=\log_a(x-c)$. See [link].



Horizontal Shifts of the Parent Function $f(x)=\log$

$a(x)$
For any constant c, the function $f(x) = \log a(x+c)$

- shifts the parent function $y = \log a(x)$ left c units if $c > 0$.
- shifts the parent function $y = \log a(x)$ right c units if $c < 0$.
- has the vertical asymptote $x = -c$.
- has domain $(-c, \infty)$.
- has range $(-\infty, \infty)$.

Given a logarithmic function with the form $f(x) = \log a(x+c)$, graph the translation.

1. Identify the horizontal shift:
   1. If $c > 0$, shift the graph of $f(x) = \log a(x)$ left c units.
   2. If $c < 0$, shift the graph of $f(x) = \log a(x)$ right c units.
2. Draw the vertical asymptote $x = -c$.
3. Identify three key points from the parent function. Find new coordinates for the shifted functions by subtracting c from the x coordinate.
4. Label the three points.
5. The Domain is $(-c, \infty)$, the range is $($

$-\infty,\infty$), and the vertical asymptote is $x=-c$.

Graphing a Horizontal Shift of the Parent Function $y = \log_a(x)$

Sketch the horizontal shift $f(x)=\log_3(x-2)$ alongside its parent function. Include the key points and asymptotes on the graph. State the domain, range, and asymptote.

Since the function is $f(x)=\log_3(x-2)$, we notice $x+(-2)=x–2$.

Thus $c=-2$, so $c<0$. This means we will shift the function $f(x)=\log_3(x)$ right 2 units.

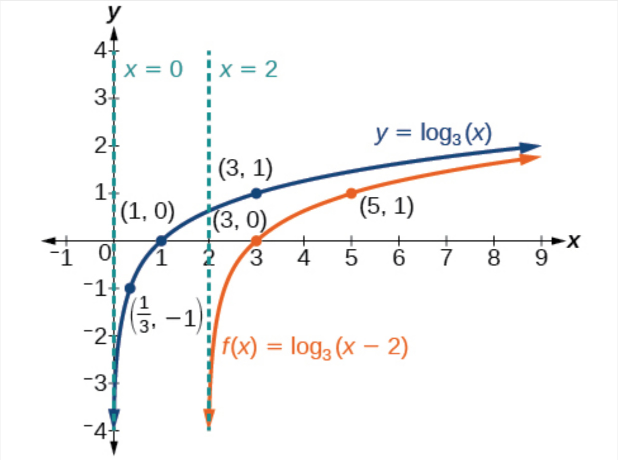
The vertical asymptote is $x=-(-2)$ or $x=2$.

Consider the three key points from the parent function, $\left(\frac{1}{3},-1\right)$, $(1,0)$, and $(3,1)$.

The new coordinates are found by adding 2 to the x coordinates.

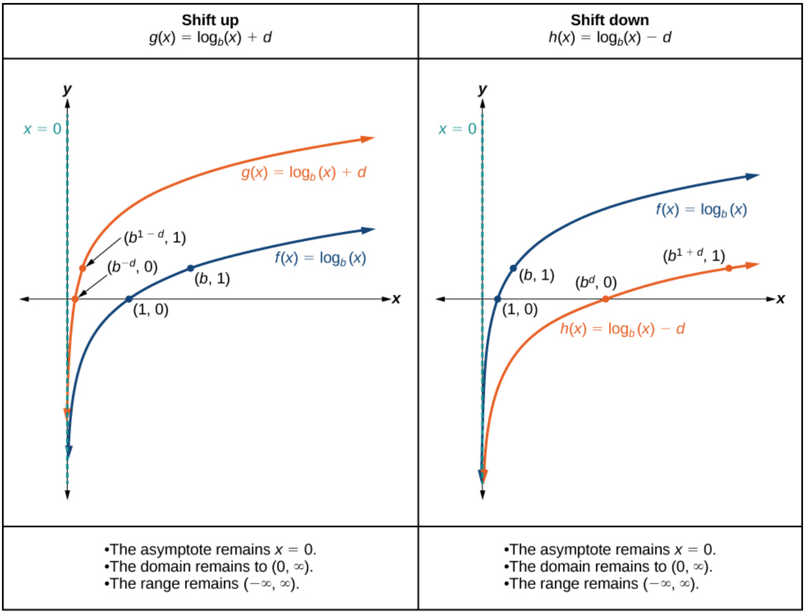
Label the points $\left(\frac{7}{3},-1\right)$, $(3,0)$, and $(5,1)$.

The domain is $(2, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 2$.



Graphing a Vertical Shift of $y = \log_a(x)$

When a constant d is added to the parent function $f(x) = \log_a(x)$, the result is a vertical shift d units in the direction of the sign on d. To visualize vertical shifts, we can observe the general graph of the parent function $f(x) = \log_a(x)$ alongside the shift up, $g(x) = \log_a(x) + d$ and the shift down, $h(x) = \log_a(x) - d$. See [link].



Vertical Shifts of the Parent Function $y = \log a(x)$

For any constant d, the function $f(x) = \log a(x) + d$

- shifts the parent function $y = \log a(x)$ up d units if $d > 0$.
- shifts the parent function $y = \log a(x)$ down d units if $d < 0$.
- has the vertical asymptote $x = 0$.
- has domain $(0, \infty)$.
- has range $(-\infty, \infty)$.

Given a logarithmic function with the form $f(x) = \log_a(x) + d$, graph the translation.

1. Identify the vertical shift:

   - If $d > 0$, shift the graph of $f(x) = \log_a(x)$ up d units.
   - If $d < 0$, shift the graph of $f(x) = \log_a(x)$ down d units.

4. Draw the vertical asymptote $x = 0$.
5. Identify three key points from the parent function. Find new coordinates for the shifted functions by adding d to the y coordinate.
6. Label the three points.
7. The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

Graphing a Vertical Shift of the Parent Function $y = \log_a(x)$

Sketch a graph of $f(x) = \log_3(x) - 2$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Since the function is $f(x) = \log_3(x) - 2$, we will notice $d = -2$. Thus $d < 0$.

This means we will shift the function $f(x) = \log_3(x)$ down 2 units.

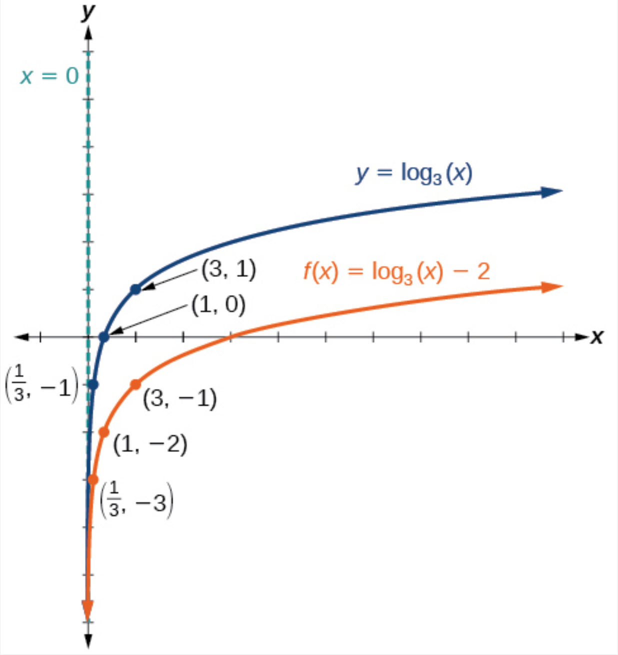
The vertical asymptote is $x = 0$.

Consider the three key points from the parent function, $\left(\frac{1}{3}, -1\right)$, $(1, 0)$, and $(3, 1)$.

The new coordinates are found by subtracting 2 from the *y* coordinates.

Label the points $\left(\frac{1}{3}, -3\right)$, $(1, -2)$, and $(3, -1)$.

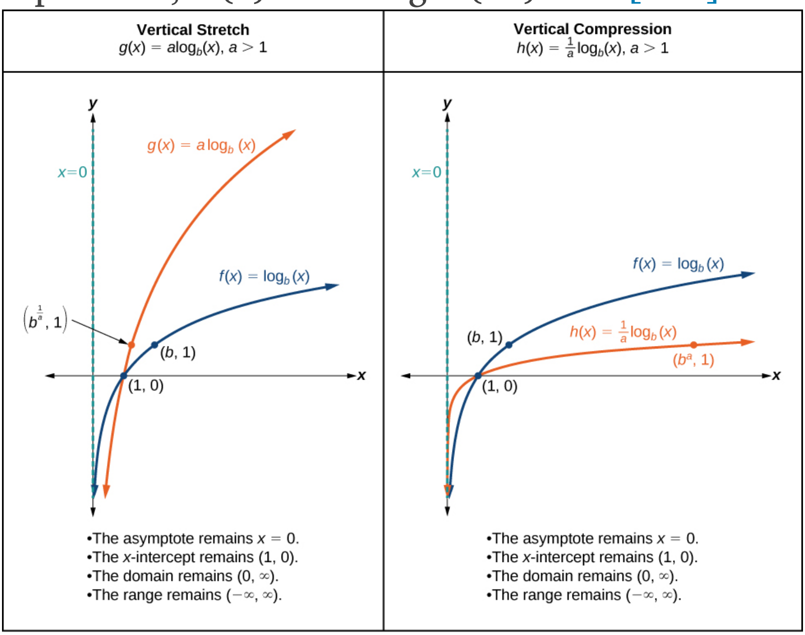
The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.



The domain is $(0,\infty)$, the range is $(-\infty,\infty)$, and the vertical asymptote is $x=0$.

Graphing Stretches and Compressions of $y=\log_a(x)$

When the parent function $f(x)=\log_a(x)$ is multiplied by a constant $a>0$, the result is a vertical stretch or compression of the original graph. To visualize stretches and compressions, we set $a>1$ and observe the general graph of the parent function $f(x)=\log_a(x)$ alongside the vertical stretch, $g(x)=a\log_a(x)$ and the vertical compression, $h(x)=\frac{1}{a}\log_a(x)$. See [link].



Vertical Stretches and Compressions of the Parent Function $y = \log_a(x)$

For any constant $a > 1$, the function $f(x) = a \log_a(x)$

- stretches the parent function $y = \log_a(x)$ vertically by a factor of a if $a > 1$.
- compresses the parent function $y = \log_a(x)$ vertically by a factor of a if $0 < a < 1$.
- has the vertical asymptote $x = 0$.
- has the x-intercept $(1, 0)$.
- has domain $(0, \infty)$.
- has range $(-\infty, \infty)$.

Given a logarithmic function with the form $f(x) = a \log_a(x)$, $a > 0$, graph the translation.

1. Identify the vertical stretch or compressions:
   - If $|a| > 1$, the graph of $f(x) = \log_a(x)$ is stretched by a factor of a units.
   - If $|a| < 1$, the graph of $f(x) = \log_a(x)$ is compressed by a factor of a units.

4. Draw the vertical asymptote $x = 0$.
5. Identify three key points from the parent function. Find new coordinates for the shifted functions by multiplying the y coordinates by a.

6. Label the three points.
7. The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x=0$.

Graphing a Stretch or Compression of the Parent Function $y = \log_a(x)$

Sketch a graph of $f(x)=2\log_4(x)$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Since the function is $f(x)=2\log_4(x)$, we will notice $a=2$.

This means we will stretch the function $f(x)=\log_4(x)$ by a factor of 2.

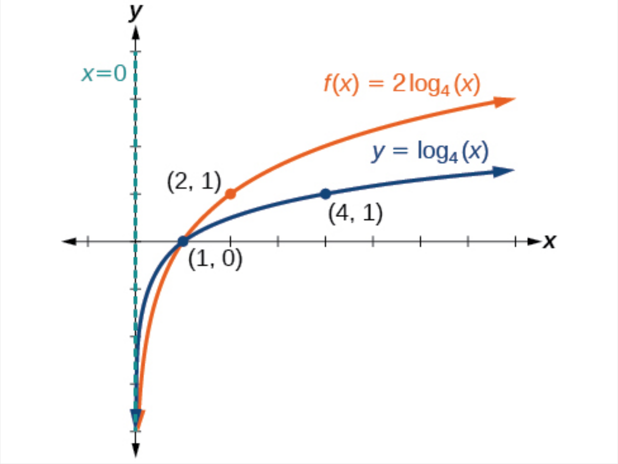
The vertical asymptote is $x=0$.

Consider the three key points from the parent function, $\left(\frac{1}{4},-1\right)$, $(1,0)$, and $(4,1)$.

The new coordinates are found by multiplying the y coordinates by 2.

Label the points $\left(\frac{1}{4},-2\right)$, $(1,0)$, and $(4,2)$.

The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x=0$. See [link].



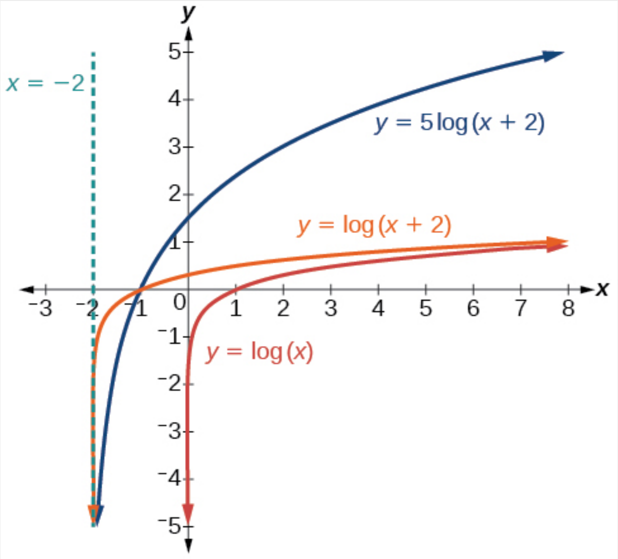
The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x=0$.

Combining a Shift and a Stretch

Sketch a graph of $f(x)=5\log(x+2)$. State the domain, range, and asymptote.

Remember: what happens inside parentheses happens first. First, we move the graph left 2 units, then stretch the function vertically by a factor of 5, as in [link]. The vertical asymptote

will be shifted to $x = -2$. The x-intercept will be $(-1, 0)$. The domain will be $(-2, \infty)$. Two points will help give the shape of the graph: $(-1, 0)$ and $(8, 5)$. We chose $x = 8$ as the x-coordinate of one point to graph because when $x = 8$, $x + 2 = 10$, the base of the common logarithm.

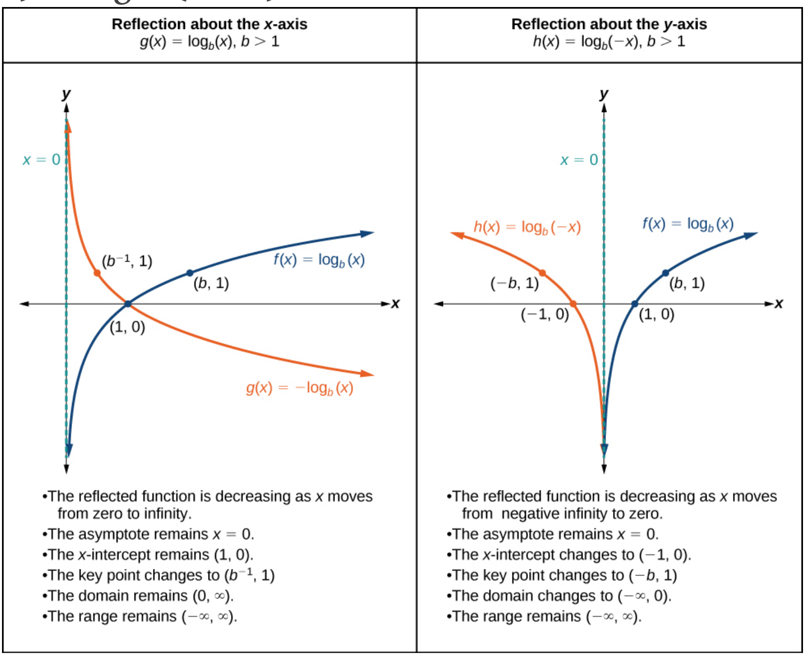


The domain is $(-2, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = -2$.

Graphing Reflections of $f(x) = \log_a(x)$

When the parent function $f(x) = \log_a(x)$ is

multiplied by -1, the result is a reflection about the *x*-axis. When the *input* is multiplied by -1, the result is a reflection about the *y*-axis. To visualize reflections, we restrict $a > 1$, and observe the general graph of the parent function $f(x) = \log a(x)$ alongside the reflection about the *x*-axis, $g(x) = -\log a(x)$ and the reflection about the *y*-axis, $h(x) = \log a(-x)$.



Reflections of the Parent Function $y = \log a(x)$

The function $f(x) = -\log a(x)$

- reflects the parent function $y = \log a(x)$ about the *x*-axis.
- has domain, $(0, \infty)$, range, $(-\infty, \infty)$, and

vertical asymptote, $x=0$, which are unchanged from the parent function.

The function $f(x)=\log_a(-x)$

- reflects the parent function $y=\log_a(x)$ about the *y*-axis.
- has domain $(-\infty,0)$.
- has range, $(-\infty,\infty)$, and vertical asymptote, $x=0$, which are unchanged from the parent function.

Given a logarithmic function with the parent function $f(x)=\log_a(x)$, graph a translation.

If $f(x)=-\log_a(x)$		**If $f(x)=\log_a(-x)$**

1. Draw the vertical asymptote, $x=0$.
1. Plot the *x* intercept, $(1,0)$.
1. Reflect the graph of the parent function $f(x)=\log_a(x)$ about the *x* axis.
1. Draw a smooth curve through the points.
1. State the domain, $(0,\infty)$, the range, $(-\infty,\infty)$, and the vertical asymptote

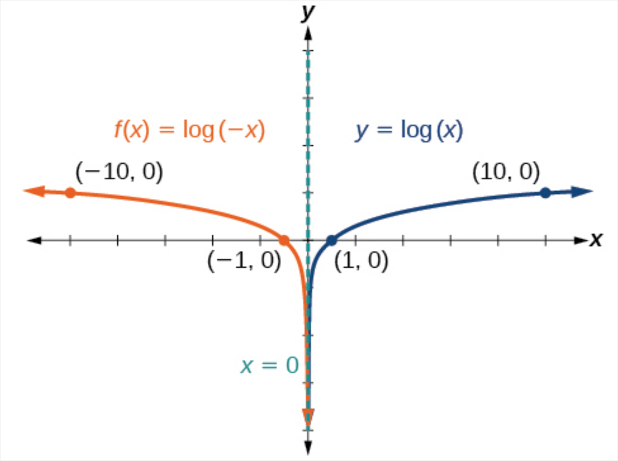
$x = 0.$

Graphing a Reflection of a Logarithmic Function

Sketch a graph of $f(x) = \log(-x)$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Before graphing $f(x) = \log(-x)$, identify the behavior and key points for the graph.

- Since $a = 10$ is greater than one, we know that the parent function is increasing. Since the *input* value is multiplied by -1, f is a reflection of the parent graph about the *y*-axis. Thus, $f(x) = \log(-x)$ will be decreasing as x moves from negative infinity to zero, and the right tail of the graph will approach the vertical asymptote $x = 0$.
- The *x*-intercept is $(-1, 0)$.
- We draw and label the asymptote, plot and label the points, and draw a smooth curve through the points.



The domain is $(-\infty,0)$, the range is $(-\infty,\infty)$, and the vertical asymptote is $x=0$.

Summarizing Translations of the Logarithmic Function

Now that we have worked with each type of translation for the logarithmic function, we can summarize each in [link] to arrive at the general equation for translating exponential functions.

Translations of the Parent Function $y=\log$

$a(x)$

Translation	Form
Shift • Horizontally c units to the left • Vertically d units up	$y=\log_a(x+c)+d$
Stretch and Compress • Stretch if $\|a\|>1$ • Compression if $\|a\|<1$	$y=a\log_a(x)$
Reflect about the *x* axis	$y=-\log_a(x)$
Reflect about the *y* axis	$y=\log_a(-x)$
General equation for all translations	$y=a\log_a(x+c)+d$

Translations of Logarithmic Functions

All translations of the parent logarithmic function, $y=\log_a(x)$, have the form

$f(x)=a\log_a(x+c)+d$

where the parent function, $y=\log_a(x), a>1$, is

- shifted vertically up d units.
- shifted horizontally to the left c units.
- stretched vertically by a factor of $|a|$ if $|a|>0$.
- compressed vertically by a factor of $|a|$ if $0<|a|<1$.
- reflected about the *x*-axis when $a<0$.

For $f(x)=\log(-x)$, the graph of the parent function is reflected about the *y*-axis.

Finding the Vertical Asymptote of a Logarithm Graph

What is the vertical asymptote of $f(x)=-2\log_3(x+4)+5$?

The vertical asymptote is at $x=-4$.

Analysis

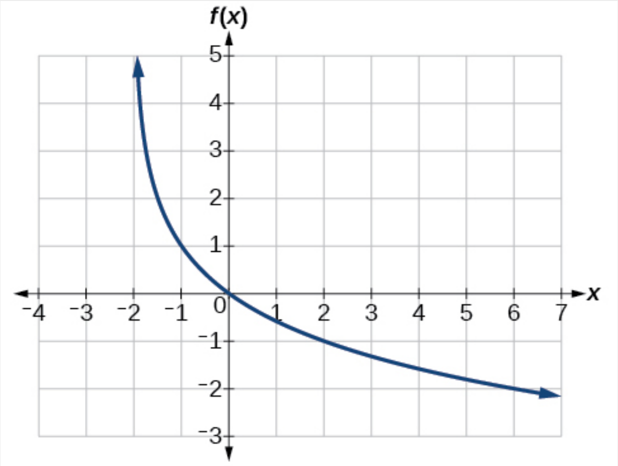
The coefficient, the base, and the upward translation do not affect the asymptote. The shift of the curve 4 units to the left shifts the vertical asymptote to $x=-4$.

What is the vertical asymptote of $f(x)=3+\ln(x-1)$?

$x=1$

Finding the Equation from a Graph

Find a possible equation for the common logarithmic function graphed in [link].



This graph has a vertical asymptote at $x = –2$ and has been vertically reflected. We do not know yet the vertical shift or the vertical stretch. We know so far that the equation will have form:

$f(x) = -a\log(x+2) + k$

It appears the graph passes through the points $(–1,1)$ and $(2,–1)$. Substituting $(–1,1)$,

$1 = -a\log(-1+2) + k$ Substitute $(-1,1)$.
$1 = -a\log(1) + k$ Arithmetic. $1 = k$ $\log(1) = 0$.

Next, substituting in $(2,–1)$,

$-1 = -a\log(2+2) + 1$ Plug in $(2,-1)$. $-2 =$

$-a\log(4)$ Arithmetic. $a = 2\log(4)$ Solve for a.

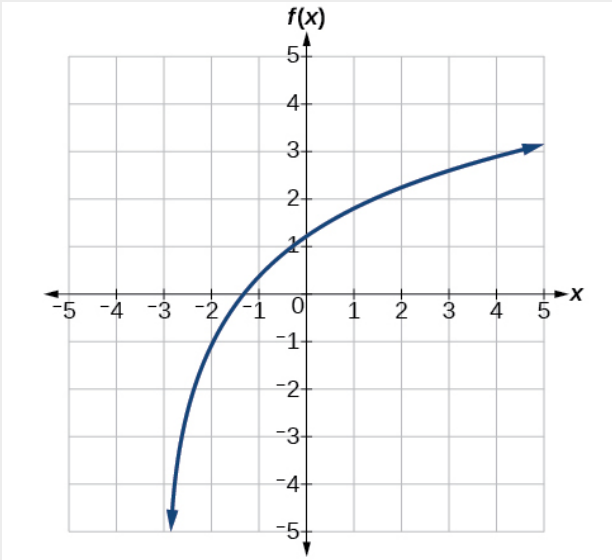
This gives us the equation $f(x) = -\frac{2}{\log(4)}\log(x+2)+1$.

Analysis

We can verify this answer by comparing the function values in [link] with the points on the graph in [link].

x	−1	0	1	2	3
f(x)	1	0	−0.58496	1	−1.3219
x	4	5	6	7	8
f(x)	−1.5850	−1.8074	−2	−2.1699	−2.3219

Give the equation of the natural logarithm graphed in [link].



$f(x) = 2\ln(x+3) - 1$

Natural and Common Logarithms

When we talked about exponential functions, we introduced the number *e*. Just as *e* was a base for an exponential function, it can be used a base for logarithmic functions too. The logarithmic function with base *e* is called the **natural logarithmic function**. The function $f(x) = \log_e x$ is generally written $f(x) = \ln x$ and we read it as “el en of x.”

Natural Logarithmic Function

The function $f(x) = \ln x$ is the **natural logarithmic function** with base e, where $x > 0$.

$y = \ln x$ is equivalent to $x = e^y$

We read $\ln(x)$ as, "the logarithm with base e of x" or "the natural logarithm of x."

The logarithm y is the exponent to which e must be raised to get x.

Since the functions $y = e^x$ and $y = \ln(x)$ are inverse functions, $\ln(e^x) = x$ for all x and $e^{\ln(x)} = x$ for $x > 0$.

Sometimes we may see a logarithm written without a base. In this case, we assume that the base is 10. In other words, the expression $\log(x)$ means $\log_{10}(x)$. We call a base-10 logarithm a **common logarithm function** and the base is not shown.

Common Logarithmic Function

The function $f(x) = \log x$ is the **common logarithmic function** with base10, where $x > 0$.

$y = \log(x)$ is equivalent to $10^y = x$

We read $\log(x)$ as, "the logarithm with base 10 of x" or "log base 10 of x."

It will be important for you to use your calculator to evaluate both common and natural logarithms.

Look for the log and ln keys on your calculator.

To solve logarithmic equations, one strategy is to change the equation to exponential form and then solve the exponential equation as we did before. As we solve logarithmic equations, $y=\log_a x$, we need to remember that for the base *a*, $a>0$ and $a\neq 1$. Also, the domain is $x>0$. Just as with radical equations, we must check our solutions to eliminate any extraneous solutions.

Given a common logarithm of the form $y=\log(x)$, evaluate it mentally.

1. Rewrite the argument x as a power of 10: $10^y=x$.
2. Use previous knowledge of powers of 10 to identify y by asking, "To what exponent must 10 be raised in order to get x? "

Solve: ⓐ $\log_a 49=2$ and ⓑ $\ln x=3$.

ⓐ

$\log_a 49=2$ Rewrite in exponential

form.$a^2=49$Solve the equation using the square root property.$a=\pm 7$The base cannot be negative, so we eliminate$a=-7$.$a=7, a=-7$Check.$a=7 \log_a 49=2 \log_7 49 \overset{?}{=} 2 \; 7^2 \overset{?}{=} 49 \; 49=49$✓

ⓑ

$\ln x=3$ Rewrite in exponential form.$e^3=x$
Check. $x=e^3$ $\ln x=3$ $\ln e^3 \overset{?}{=} 3$ $e^3=e^3$ ✓

Solve: ⓐ $\log_a 121=2$ ⓑ $\ln x=7$

ⓐ
$a=11$
ⓑ $x=e^7$

Solve: ⓐ $\log_a 64=3$ ⓑ $\ln x=9$

ⓐ
$a=4$

ⓑ $x=e9$

Solve: ⓐ $\log 2(3x-5)=4$ and ⓑ $\ln e2x=4$.

ⓐ

$\log 2(3x-5)=4$ Rewrite in exponential form.$24=3x-5$ Simplify.$16=3x-5$ Solve the equation.$21=3x$ $7=x$ Check. $x=7\log 2(3x-5)=4$ $\log 2(3\cdot 7-5)=?4$ $\log 2(16)=?4$ $24=?16$ $16=16$✓

ⓑ

$\ln e2x=4$ Rewrite in exponential form.$e4=e2x$ Since the bases are the same the exponents are equal.$4=2x$ Solve the equation.$2=x$ Check. $x=2\ln e2x=4$ $\ln e2\cdot 2=?4$ $\ln e4=?4$ $e4=e4$✓

Solve: ⓐ $\log 2(5x-1)=6$ ⓑ $\ln e3x=6$

ⓐ

$x=13$

ⓑ $x=2$

Solve: ⓐ $\log 3(4x+3)=3$ ⓑ $\ln e4x=4$

ⓐ
$x=6$
ⓑ $x=1$

Use Logarithmic Models in Applications

There are many applications that are modeled by logarithmic equations. We will first look at the logarithmic equation that gives the decibel (dB) level of sound. Decibels range from 0, which is barely audible to 160, which can rupture an eardrum. The $10-12$ in the formula represents the intensity of sound that is barely audible.

Decibel Level of Sound

The loudness level, *D*, measured in decibels, of a sound of intensity, *I*, measured in watts per square inch is

$D = 10\log(I10 - 12)$

Extended exposure to noise that measures 85 dB can cause permanent damage to the inner ear which will result in hearing loss. What is the decibel level of music coming through ear phones with intensity $10 - 2$ watts per square inch?

[redacted]

Substitute in the intensity level, *I*.

[redacted]

Simplify.

[redacted]

Since $\log 1010 = 10$.

[redacted]

Multiply.

[redacted]

The decibel level of music coming through earphones is 100 dB.

What is the decibel level of one of the new quiet dishwashers with intensity $10 - 7$ watts per square inch?

The quiet dishwashers have a decibel level of 50 dB.

What is the decibel level heavy city traffic with intensity $10 - 3$ watts per square inch?

The decibel level of heavy traffic is 90 dB.

The magnitude R of an earthquake is measured by a logarithmic scale called the Richter scale. The model is $R = \log I$, where I is the intensity of the shock wave. This model provides a way to measure earthquake intensity.

Earthquake Intensity

The magnitude R of an earthquake is measured by $R = \log I$, where I is the intensity of its shock wave.

In 1906, San Francisco experienced an intense earthquake with a magnitude of 7.8 on the Richter scale. Over 80% of the city was destroyed by the resulting fires. In 2014, Los Angeles experienced a moderate earthquake that measured 5.1 on the Richter scale and caused $108 million dollars of damage. Compare the intensities of the two earthquakes.

To compare the intensities, we first need to convert the magnitudes to intensities using the log formula. Then we will set up a ratio to compare the intensities.

Convert the magnitudes to intensities.R=logI 1906 earthquake7.8=logI Convert to exponential form.I=107.8 2014 earthquake5.1=logI Convert to exponential form.I=105.1 Form a ratio of the intensities.Intensityfor1906Intensityfor2014 Substitute in the values.107.8105.1 Divide by subtracting the exponents.102.7 Evaluate.501 The intensity of the 1906 earthquakewas about 501 times the intensity ofthe 2014 earthquake.

In 1906, San Francisco experienced an intense earthquake with a magnitude of 7.8 on the Richter scale. In 1989, the Loma Prieta earthquake also affected the San Francisco area, and measured 6.9 on the Richter scale. Compare the intensities of the two earthquakes.

The intensity of the 1906 earthquake was about 8 times the intensity of the 1989

earthquake.

Access these online resources for additional instruction and practice with evaluating and graphing logarithmic functions.

- Re-writing logarithmic equations in exponential form
- Simplifying Logarithmic Expressions
- Graphing logarithmic functions
- Using logarithms to calculate decibel levels

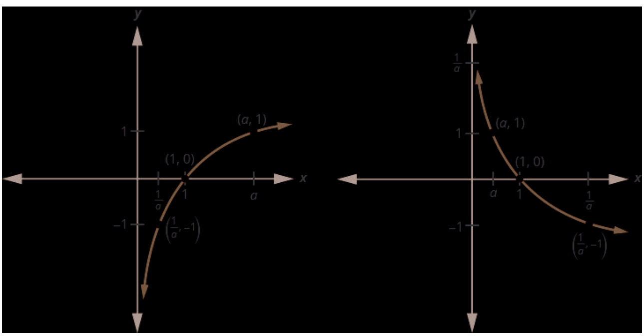
Access these online resources for additional instruction and practice with graphing logarithms.

- Graph an Exponential Function and Logarithmic Function
- Match Graphs with Exponential and Logarithmic Functions
- Find the Domain of Logarithmic Functions

Key Concepts

- **Properties of the Graph of** $y = \log_a x$:

	when $a > 1$		when $0 < a < 1$
Domain	$(0, \infty)$	Domain	$(0, \infty)$
Range	$(-\infty, \infty)$	Range	$(-\infty, \infty)$
x intercept	$(1, 0)$	x intercept	$(1, 0)$
y intercept	none	y intercept	none
Contains	$(a,1),(1a,-1)$	Contains	$(a,1),(1a,-1)$
Asymptote	y axis	Asymptote	y axis
Basic shape	increasing	Basic shape	decreasing



- **Decibel Level of Sound:** The loudness level, D, measured in decibels, of a sound of intensity, I, measured in watts per square inch is $D = 10\log(I10 - 12)$.
- **Earthquake Intensity:** The magnitude R of an earthquake is measured by $R = \log I$, where I is the intensity of its shock wave.

Definition of the logarithmic function	For $x>0, b>0, b\neq 1$, $y=\log_b(x)$ if and only if $b^y=x$.
Definition of the common logarithm	For $x>0$, $y=\log(x)$ if and only if $10^y=x$.
Definition of the natural logarithm	For $x>0$, $y=\ln(x)$ if and only if $e^y=x$.

Practice Makes Perfect

Convert Between Exponential and Logarithmic Form

In the following exercises, convert from exponential to logarithmic form.

$25=32$

$\log 232=5$

$53=125$

$\log 5125=3$

$10-2=1100$

$\log1100 = -2$

$x13 = 63$

$\log x63 = 13$

$17x = 175$

$\log17175 = x$

$(13)4 = 181$

$\log13181 = 4$

$4-3 = 164$

$\log4164 = -3$

$e3 = x$

$\ln x = 3$

In the following exercises, convert each logarithmic equation to exponential form.

$6 = \text{log}264$

$64 = 26$

$5 = \text{log}x32$

$32 = x5$

$0 = \text{log}71$

$1 = 70$

$1 = \text{log}99$

$9 = 91$

$3 = \text{log}101{,}000$

$1{,}000 = 103$

$x = \log_e 43$

$43 = e^x$

Evaluate Logarithmic Functions

In the following exercises, find the value of x in each logarithmic equation.

$\log_x 121 = 2$

$x = 11$

$\log_x 64 = 3$

$x = 4$

$\log_5 x = 3$

$x = 125$

$\log_3 x = -5$

$x = \frac{1}{243}$

$\log 1981 = x$

$x = -2$

In the following exercises, find the exact value of each logarithm without using a calculator.

$\log 636$

2

$\log 51$

0

$\log 273$

13

$\log 124$

-2

log3127

−3

For the following exercises, evaluate the natural logarithmic expression without using a calculator.

ln(1)

0

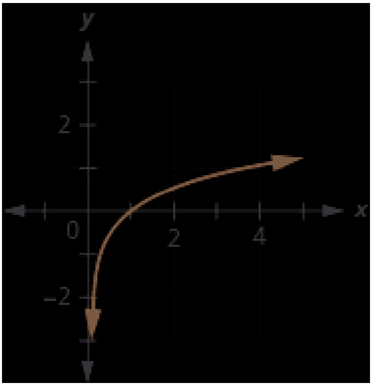
25ln(e 2 5)

10

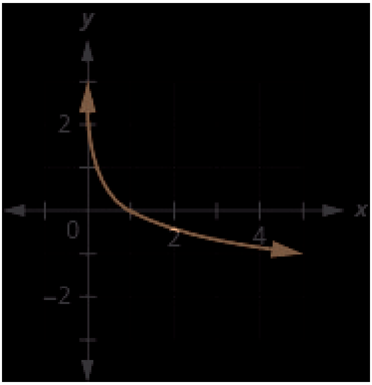
Graph Logarithmic Functions

In the following exercises, graph each logarithmic function.

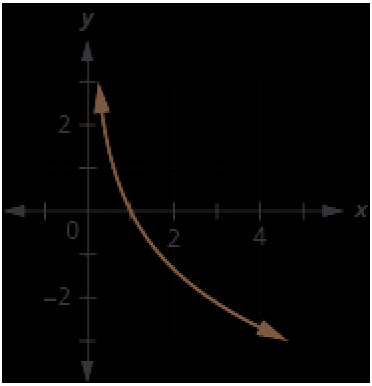
y=log4x



y = log15x



y = log0.6x



For the following exercises, state the domain and range of the function.

$h(x)=\ln\left(\frac{1}{2}-x\right)$

Domain: $\left(-\infty,\frac{1}{2}\right)$; Range: $\left(-\infty,\infty\right)$

$h(x)=\ln\left(4x+17\right)-5$

Domain: $\left(-\frac{17}{4},\infty\right)$; Range: $\left(-\infty,\infty\right)$

For the following exercises, state the domain and the vertical asymptote of the function.

$f(x)=\log_b(x-5)$

Domain: $(5,\infty)$; Vertical asymptote: $x=5$

$f(x)=\log(3x+1)$

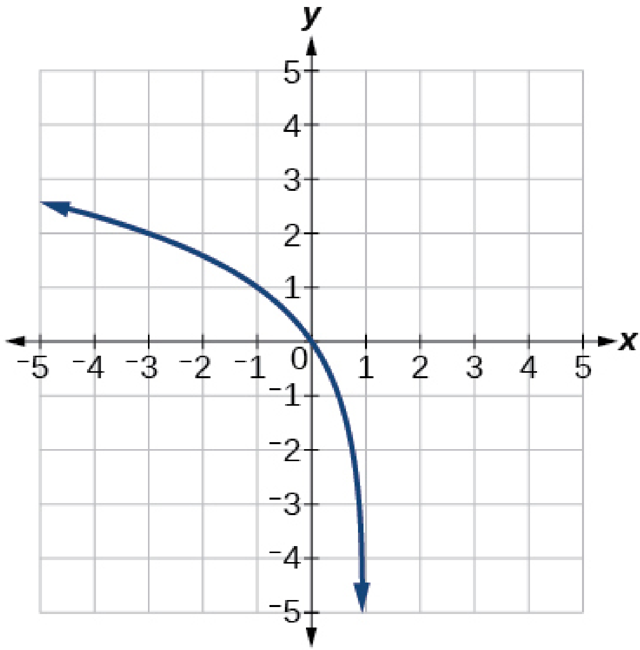
Domain: $\left(-\frac{1}{3},\infty\right)$; Vertical asymptote: $x=-\frac{1}{3}$

$g(x)=-\ln(3x+9)-7$

Domain: $(-3,\infty)$; Vertical asymptote: $x=-3$

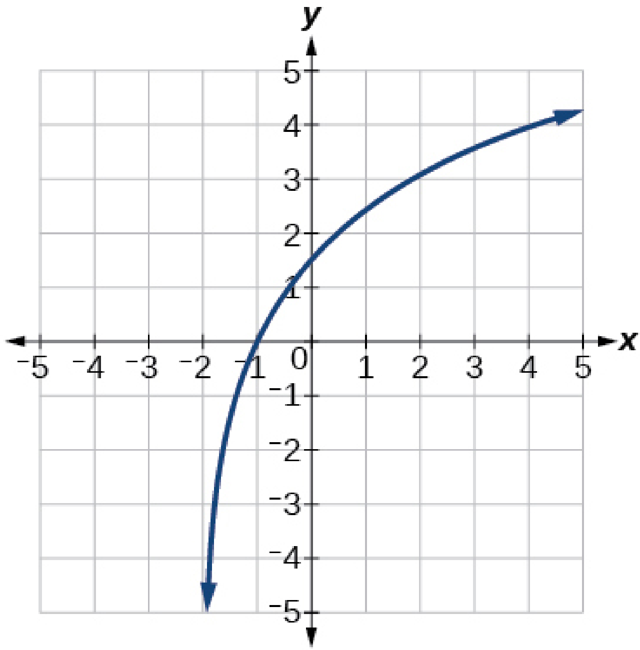
For the following exercises, write a logarithmic equation corresponding to the graph shown.

Use $y=\log_2(x)$ as the parent function.



$f(x) = \log 2\,(-(x-1))$

Use $f(x) = \log 4\,(x)$ as the parent function.



$f(x) = 3 \log 4 (x + 2)$

Solve Logarithmic Equations

In the following exercises, solve each logarithmic equation.

$\log a81 = 2$

$a = 9$

$\log a24 = 3$

$a=243$
$\ln x=4$
$x=e4$
$\log 2(6x+2)=5$
$x=5$
$\log 3(5x-4)=4$
$x=17$
$\log 4(3x-2)=2$
$x=6$
$\ln e2x=6$
$x=3$

$\log(x2-25)=2$

$x=-55, x=55$

$\log3(x2+2)=3$

$x=-5, x=5$

Use Logarithmic Models in Applications

In the following exercises, use a logarithmic model to solve.

What is the decibel level of a whisper with intensity $10-10$ watts per square inch?

A whisper has a decibel level of 20 dB.

What is the decibel level of the noise from a motorcycle with intensity $10-2$ watts per square inch?

What is the decibel level of the sound of a garbage disposal with intensity $10-2$ watts per square inch?

The sound of a garbage disposal has a decibel level of 100 dB.

The Los Angeles area experiences many earthquakes. In 1994, the Northridge earthquake measured magnitude of 6.7 on the Richter scale. In 2014, Los Angeles also experienced an earthquake which measured 5.1 on the Richter scale. Compare the intensities of the two earthquakes.

The intensity of the 1994 Northridge earthquake in the Los Angeles area was about 40 times the intensity of the 2014 earthquake.

Glossary

common logarithmic function
: The function $f(x) = \log x$ is the common logarithmic function with base10, where $x > 0$.
$y = \log x$is equivalent to$x = 10y$

logarithmic function
: The function $f(x) = \log_a x$ is the logarithmic function with base a, where $a > 0, x > 0$, and $a \neq 1$.
$y = \log_a x$is equivalent to$x = ay$

natural logarithmic function

The function $f(x) = \ln x$ is the natural logarithmic function with base e, where $x > 0$. $y = \ln x$is equivalent to$x = e^y$

Properties of Logarithms (4.3)
By the end of this section, you will be able to:

- Use the properties of logarithms
- Use the Change of Base Formula

This Module supports section **4.3 of Mat 1023**.
In case you missed something in class, or just want to review a specific topic covered in this Module, here is a list of topics covered:

1. Properties of Logarithms [link]
2. Expand and Condense Logarithms [link]
3. Change of Base Formula[link]
4. Key Concepts [link]

Now that we have learned about exponential and logarithmic functions, we can introduce some of the properties of logarithms. These will be very helpful as we continue to solve both exponential and logarithmic equations. In the previous section we already introduced Basic property and inverse property, which we will review below.

Properties of Logarithms

The first two properties derive from the definition of logarithms. Since $a0=1$, we can convert this to logarithmic form and get $loga1=0$. Also, since $a1=a$, we get $logaa=1$.

$loga1=0logaa=1$

Inverse Properties of Logarithms

These two properties are called inverse properties because, when we have the same base, raising to a power “undoes” the log and taking the log “undoes” raising to a power. These two properties show the composition of functions. Both ended up with the identity function which shows again that the exponential and logarithmic functions are inverse functions.

For $a>0, x>0$ and $a\neq 1$,

$alogax=xlogaax=x$

Use the Properties of Logarithms

Recall that the logarithmic and exponential functions “undo” each other. This means that logarithms have similar properties to exponents. Our

definition of logarithm shows us that a logarithm is the exponent of the equivalent exponential function. The properties of exponents have related properties for logarithms. There are three more properties of logarithms that will be useful in our work.

Before we can solve an equation like $\log_3(3x) + \log_3(2x+5) = 2$, we need a method for combining terms on the left side of the equation. Recall that we use the *Product Property of Exponents*, $a^m \cdot a^n = a^{m+n}$, we see that to multiply the same base, we add the exponents. We have a similar property for logarithms, called the **Product Property of Logarithms**, which tells us that to take the log of a product, we add the log of the factors. Because logs are exponents, and we multiply like bases, we can add the exponents.

Product Property of Logarithms

If $M > 0, N > 0, a > 0$ and $a \neq 1$, then,

$$\log_a(M \cdot N) = \log_a M + \log_a N$$

The logarithm of a product is the sum of the logarithms.

We use this property to write the log of a product as a sum of the logs of each factor.

Given the logarithm of a product, use the product rule of logarithms to write an equivalent sum of logarithms.

1. Factor the argument completely, expressing each whole number factor as a product of primes.
2. Write the equivalent expression by summing the logarithms of each factor.

Use the Product Property of Logarithms to write each logarithm as a sum of logarithms. Simplify, if possible: ⓐ log37x and ⓑ log464xy.

ⓐ

log37x Use the Product Property,loga(M·N) = logaM + logaN.log37 + log3x log37x = log37 + log3x

ⓑ

log464xy Use the Product Property,loga(M·N) = logaM + logaN.log464 + log4x + log4y Simplify by evaluatinglog464.3 + log4x + log4y log464xy = 3 + log4x + log4y

Use the Product Property of Logarithms to write each logarithm as a sum of logarithms. Simplify, if possible.

ⓐ $\log 99x$ ⓑ $\log 327xy$

ⓐ $1 + \log 9x$
ⓑ $3 + \log 3x + \log 3y$

Using the Product Rule for Logarithms

Expand $\log 3(30x(3x+4))$.

We begin by factoring the argument completely, expressing 30 as a product of primes.
$\log 3(30x(3x+4)) = \log 3(2\cdot 3\cdot 5\cdot x\cdot (3x+4))$

Next we write the equivalent equation by summing the logarithms of each factor.
$\log 3(30x(3x+4)) = \log 3(2) + \log 3(3) + \log 3(5) + \log 3(x) + \log 3(3x+4)$

Quotient Property

Similarly, in the Quotient Property of Exponents, $aman=am-n$, we see that to divide the same base, we subtract the exponents. The **Quotient Property of Logarithms**, $logaMN=logaM-logaN$ tells us to take the log of a quotient, we subtract the log of the numerator and denominator.

Quotient Property of Logarithms

If $M>0, N>0, a>0$ and $a\neq 1$, then,

$logaMN=logaM-logaN$

The logarithm of a quotient is the difference of the logarithms.

Note that $logaM-logaN\neq loga(M-N)$.

We use this property to write the log of a quotient as the difference of the logs of each factor.

Given the logarithm of a quotient, use the quotient rule of logarithms to write an equivalent difference of logarithms.

1. Express the argument in lowest terms by factoring the numerator and denominator and canceling common terms.

2. Write the equivalent expression by subtracting the logarithm of the denominator from the logarithm of the numerator.
3. Check to see that each term is fully expanded. If not, apply the product rule for logarithms to expand completely.

Use the Quotient Property of Logarithms to write each logarithm as a difference of logarithms. Simplify, if possible.
ⓐ log557 and ⓑ logx100

ⓐ
log557 Use the Quotient Property,logaMN = logaM − logaN.log55 − log57 Simplify.1 − log57
log557 = 1 − log57

ⓑ
logx100Use the Quotient Property,logaMN = logaM − logaN.logx − log100 Simplify.logx − 2 logx100 = logx − 2

Use the Quotient Property of Logarithms to write each logarithm as a difference of logarithms. Simplify, if possible.

ⓐ log434 ⓑ logx1000

ⓐ log43 − 1 ⓑ logx − 3

Use the Quotient Property of Logarithms to write each logarithm as a difference of logarithms. Simplify, if possible.

ⓐ log254 ⓑ log10y

ⓐ log25 − 2 ⓑ 1 − logy

Using the Quotient Rule for Logarithms

Expand log 2 (15x(x − 1) (3x + 4)(2 − x)).

First we note that the quotient is factored and in lowest terms, so we apply the quotient rule.

$\log_2\left(\frac{15x(x-1)}{(3x+4)(2-x)}\right) = \log_2\left(15x(x-1)\right) - \log_2\left((3x+4)(2-x)\right)$

Notice that the resulting terms are logarithms of products. To expand completely, we apply the product rule, noting that the prime factors of the factor 15 are 3 and 5.

$\log_2(15x(x-1)) - \log_2((3x+4)(2-x)) = [\log_2(3) + \log_2(5) + \log_2(x) + \log_2(x-1)] - [\log_2(3x+4) + \log_2(2-x)] = \log_2(3) + \log_2(5) + \log_2(x) + \log_2(x-1) - \log_2(3x+4) - \log_2(2-x)$

Analysis

There are exceptions to consider in this and later examples. First, because denominators must never be zero, this expression is not defined for $x = -\frac{4}{3}$ and $x = 2$. Also, since the argument of a logarithm must be positive, we note as we observe the expanded logarithm, that $x > 0$, $x > 1$, $x > -\frac{4}{3}$, and $x < 2$. Combining these conditions is beyond the scope of this section, and we will not consider them here or in subsequent exercises.

Power Property

We've explored the product rule and the quotient rule, but how can we take the logarithm of a power, such as x^2? The third property of logarithms is

related to the Power Property of Exponents, $(a^m)^n = a^{m \cdot n}$, we see that to raise a power to a power, we multiply the exponents.

$\log_b(x^2) = \log_b(x \cdot x) = \log_b x + \log_b x = 2 \log_b x$

Notice that we used the product property for logarithms to find a solution for the example above. By doing so, we have derived the **Power Property of Logarithms**, $\log_a M^p = p \log_a M$, which tells us to take the log of a number raised to a power, we multiply the power times the log of the number.

Power Property of Logarithms

If $M > 0, a > 0, a \neq 1$ and p is any real number then,

$\log_a M^p = p \log_a M$

We use this property to write the log of a number raised to a power as the product of the power times the log of the number. **We essentially take the exponent and throw it in front of the logarithm.**

Use the Power Property of Logarithms to write each logarithm as a product of logarithms.

Simplify, if possible.
ⓐ $\log_5 4^3$ and ⓑ $\log x^{10}$

ⓐ
$\log_5 4^3$ Use the Power Property, $\log_a M^p = p \log_a M$. $3 \log_5 4$
$\log_5 4^3 = 3 \log_5 4$

ⓑ
$\log x^{10}$ Use the Power Property, $\log_a M^p = p \log_a M$. $10 \log x$
$\log x^{10} = 10 \log x$

Use the Power Property of Logarithms to write each logarithm as a product of logarithms. Simplify, if possible.

ⓐ $\log_7 5^4$ ⓑ $\log x^{100}$

ⓐ $4 \log_7 5$ ⓑ $100 \cdot \log x$

Expand and Condense Logarithms

We summarize the Properties of Logarithms here for easy reference. While the natural logarithms are a special case of these properties, it is often helpful to also show the natural logarithm version of each property.

Properties of Logarithms

If $M>0, a>0, a\neq 1$ and p is any real number then,

Property	Base a	Base e
	$\log_a 1 = 0$	$\ln 1 = 0$
	$\log_a a = 1$	$\ln e = 1$
Inverse Properties	$a^{\log_a x} = x$ $\log_a a^x = x$	$e^{\ln x} = x$ $\ln e^x = x$
Product Property of Logarithms	$\log_a(M\cdot N) = \log_a M + \log_a N$	$\ln(M\cdot N) = \ln M + \ln N$
Quotient Property of Logarithms	$\log_a \frac{M}{N} = \log_a M - \log_a N$	$\ln \frac{M}{N} = \ln M - \ln N$
Power Property of Logarithms	$\log_a M^p = p\log_a M$	$\ln M^p = p\ln M$

Now that we have the properties we can use them to **“expand”** a logarithmic expression. This means to write the logarithm as a sum or difference and without any powers.

Taken together, the product rule, quotient rule, and power rule are often called “laws of logs.” Sometimes we apply more than one rule in order to simplify an expression. For example:
log b (6x y) = log b (6x) − log b y = log b 6 + log b x − log b y

We generally apply the Product and Quotient Properties before we apply the Power Property. Remember, however, that we can only do this with products, quotients, powers, and roots—never with addition or subtraction inside the argument of the logarithm.

Use the Properties of Logarithms to expand the logarithm log4(2x3y2). Simplify, if possible.
log4(2x3y2) Use the Product Property,logaM·N = logaM + logaN.log42 + log4x3 + log4y2 Use the Power Property,logaMp = plogaM,on the last two terms.log42 + 3log4x + 2log4y

Simplify.12 + 3log4x + 2log4y
log4(2x3y2) = 12 + 3log4x + 2log4y

Use the Properties of Logarithms to expand the logarithm log2(5x4y2). Simplify, if possible.

log25 + 4log2x + 2log2y

Expanding Logarithms Using Product, Quotient, and Power Rules

Rewrite ln(x 4 y 7) as a sum or difference of logs.

First, because we have a quotient of two expressions, we can use the quotient rule:
ln(x 4 y 7) = ln(x 4 y) − ln(7)

Then seeing the product in the first term, we use the product rule:
ln(x 4 y) − ln(7) = ln(x 4) + ln(y) − ln(7)

Finally, we use the power rule on the first

term:
$\ln(x^4) + \ln(y) - \ln(7) = 4\ln(x) + \ln(y) - \ln(7)$

Expand $\log(x^2 y^3 z^4)$.

$2\log x + 3\log y - 4\log z$

Expanding Complex Logarithmic Expressions

Expand $\log_6\left(\frac{64x^3(4x+1)}{(2x-1)}\right)$.

We can expand by applying the Product and Quotient Rules.
$\log_6\left(\frac{64x^3(4x+1)}{(2x-1)}\right) = \log_6 64 + \log_6 x^3 + \log_6(4x+1) - \log_6(2x-1)$ Apply the Quotient Rule. $= \log_6 2^6 + \log_6 x^3 + \log_6(4x+1) - \log_6(2x-1)$ Simplify by writing 64 as 2^6. $= 6\log_6 2 + 3\log_6 x + \log_6(4x+1) - \log_6(2x-1)$ Apply the Power Rule.

When we have a radical in the logarithmic expression, it is helpful to first write its radicand as a rational exponent.

Using the Power Rule for Logarithms to Simplify the Logarithm of a Radical Expression

Expand log(x).

log(x) =log x (1 2) = 1 2 logx

Use the Properties of Logarithms to expand the logarithm log2x33y2z4. Simplify, if possible.

log2x33y2z4 Rewrite the radical with a rational exponent.log2(x33y2z)14 Use the Power Property,logaMp=plogaM.14log2(x33y2z) Use the Quotient Property,logaM·N=logaM −logaN.14(log2(x3)−log2(3y2z)) Use the Product Property,logaM·N=logaM+logaN,in the second term.14(log2(x3)−(log23+log2y2+log2z))

Use the Power Property, $\log_a M^p = p\log_a M$, inside the parentheses. $\frac{1}{4}(3\log_2 x - (\log_2 3 + 2\log_2 y + \log_2 z))$ Simplify by distributing. $\frac{1}{4}(3\log_2 x - \log_2 3 - 2\log_2 y - \log_2 z)$ $\log_2 \sqrt[4]{\frac{x^3}{3y^2z}} = \frac{1}{4}(3\log_2 x - \log_2 3 - 2\log_2 y - \log_2 z)$

Use the Properties of Logarithms to expand the logarithm $\log_4 \sqrt[5]{\frac{x^4}{2y^3z^2}}$. Simplify, if possible.

$\frac{1}{5}(4\log_4 x - \frac{1}{2} - 3\log_4 y - 2\log_4 z)$

Expand $\ln(x^{\frac{2}{3}})$.

$\frac{2}{3}\ln x$

Can we expand $\ln(x^2 + y^2)$?
No. There is no way to expand the logarithm of a sum

or difference inside the argument of the logarithm.

Condense Logarithms

The opposite of expanding a logarithm is to **condense** a sum or difference of logarithms that have the same base into a single logarithm. We again use the properties of logarithms to help us, but in reverse.

To condense logarithmic expressions **with the same base** into one logarithm, we start by using the Power Property to get the coefficients of the log terms to be one and then the Product and Quotient Properties as needed.

Given a sum, difference, or product of logarithms with the same base, write an equivalent expression as a single logarithm.

1. Apply the power property first. Identify terms that are products of factors and a logarithm, and rewrite each as the logarithm of a power.
2. Next apply the product property. Rewrite sums of logarithms as the logarithm of a product.
3. Apply the quotient property last. Rewrite differences of logarithms as the logarithm of a

quotient.

Use the Properties of Logarithms to condense the logarithm $\log_4 3+\log_4 x-\log_4 y$. Simplify, if possible.

The log expressions all have the same base, 4. $\log_4 3+\log_4 x-\log_4 y$ The first two terms are added, so we use the Product Property, $\log_a M+\log_a N=\log_a M\cdot N$. $\log_4 3x-\log_4 y$ Since the logs are subtracted, we use the Quotient Property, $\log_a M-\log_a N=\log_a \frac{M}{N}$. $\log_4 \frac{3x}{y}$ $\log_4 3+\log_4 x-\log_4 y=\log_4 \frac{3x}{y}$

Use the Properties of Logarithms to condense the logarithm $\log_2 5+\log_2 x-\log_2 y$. Simplify, if possible.

$\log_2 \frac{5x}{y}$

Use the Properties of Logarithms to condense the logarithm log36 − log3x − log3y. Simplify, if possible.

log36xy

Use the Properties of Logarithms to condense the logarithm 2log3x + 4log3(x + 1). Simplify, if possible.

The log expressions have the same base, 3.2log3x + 4log3(x + 1) Use the Power Property,logaM + logaN = logaM·N.log3x2 + log3(x + 1)4 The terms are added, so we use the ProductProperty,logaM + logaN = logaM·N.log3x2(x + 1)4 2log3x + 4log3(x + 1) = log3x2(x + 1)4

Use the Properties of Logarithms to condense

the logarithm $3\log_2 x + 2\log_2(x-1)$. Simplify, if possible.

$\log_2 x^3(x-1)^2$

Condensing Complex Logarithmic Expressions

Condense $\log_2(x^2) + \frac{1}{2}\log_2(x-1) - 3\log_2((x+3)^2)$.

We apply the power rule first:

$$\log_2(x^2) + \frac{1}{2}\log_2(x-1) - 3\log_2((x+3)^2) = \log_2(x^2) + \log_2(x-1) - \log_2((x+3)^6)$$

Next we apply the product rule to the sum:

$$\log_2(x^2) + \log_2(x-1) - \log_2((x+3)^6) = \log_2(x^2 x-1) - \log_2((x+3)^6)$$

Finally, we apply the quotient rule to the difference:

$$\log_2(x^2 x-1) - \log_2((x+3)^6) = \log_2 \frac{x^2 x-1}{(x+3)^6}$$

Use the Change-of-Base Formula

Most calculators can evaluate only common and natural logs. In order to evaluate logarithms with a base other than 10 or e, we use the **change-of-base formula** to rewrite the logarithm as the quotient of logarithms of any other base; when using a calculator, we would change them to common or natural logs. We will show how this is derived by using the one-to-one property and **power rule for logarithms**.

Suppose we want to evaluatelogaM.logaM Lety = logaM.y = logaM Rewrite the expression in exponential form.ay = M Take thelogbof each side by applying the one-to-one property.logbay = logbM Use the Power Property.ylogba = logbM Solve fory.y = logbMlogba Substitutey = logaM.logaM = logbMlogba

The Change-of-Base Formula introduces a new base b. This can be any base *b* we want where $b > 0, b \neq 1$. Because our calculators have keys for logarithms base 10 and base *e*, we will rewrite the Change-of-Base Formula with the new base as 10 or *e*.

Change-of-Base Formula

The **change-of-base formula** can be used to evaluate a logarithm with any base.
For any logarithmic bases a,b and $M>0$,
logaM=logbMlogbalogaM=logMlogalogaM=lnMlna
new basebnew base 10new basee

Given a logarithm with the form $\log_b M$, use the change-of-base formula to rewrite it as a quotient of logs with any positive base n, where $n \neq 1$.

1. Determine the new base n, remembering that the common log, $\log(x)$, has base 10, and the natural log, $\ln(x)$, has base e.
2. Rewrite the log as a quotient using the change-of-base formula
    1. The numerator of the quotient will be a logarithm with base n and argument M.
    2. The denominator of the quotient will be a logarithm with base n and argument b.

Changing Logarithmic Expressions to Expressions Involving Only Natural Logs

Change $\log_5 3$ to a quotient of natural logarithms.

Because we will be expressing $\log_5 3$ as a quotient of natural logarithms, the new base, $n = e$.

We rewrite the log as a quotient using the change-of-base formula. The numerator of the quotient will be the natural log with argument 3. The denominator of the quotient will be the natural log with argument 5.

$\log_b M = \frac{\ln M}{\ln b} \quad \log_5 3 = \frac{\ln 3}{\ln 5}$

When we use a calculator to find the logarithm value, we usually round to three decimal places. This gives us an approximate value and so we use the approximately equal symbol ($\approx$).

Rounding to three decimal places, approximate $\log_4 35$.

$\log_4 35$

Use the Change-of-Base Formula.

$\log_a M = \dfrac{\log_b M}{\log_b a}$

Identify *a* and *M*. Choose 10 for *b*.

$\log_4 35 = \dfrac{\log 35}{\log 4}$

Enter the expression log35log4 in the calculator using the log button for base 10. Round to three decimal places.

$\log_4 35 \approx 2.565$

Using the Change-of-Base Formula with a Calculator

Evaluate log 2 (10) using the change-of-base formula with a calculator.

According to the change-of-base formula, we can rewrite the log base 2 as a logarithm of any other base. Since our calculators can

evaluate the natural log, we might choose to use the natural logarithm, which is the log base e.

$\log_2 10 = \frac{\ln 10}{\ln 2}$ Apply the change of base formula using base e. ≈ 3.3219 Use a calculator to evaluate to 4 decimal places.

Rounding to three decimal places, approximate $\log_3 42$.

3.402

Access these online resources for additional instruction and practice with using the properties of logarithms.

- The Properties of Logarithms
- Using Properties of Logarithms to Expand Logs
- Using Properties of Logarithms to Condense Logs
- Change of Base

Key Concepts

- **Properties of Logarithms**
  $loga1=0logaa=1$
- **Inverse Properties of Logarithms**

  - For $a>0,x>0$ and $a\neq1$
    $alogax=xlogaax=x$

- **Product Property of Logarithms**

  - If $M>0,N>0,a>0$ and $a\neq1$, then,
    $logaM·N=logaM+logaN$
    The logarithm of a product is the sum of the logarithms.

- **Quotient Property of Logarithms**

  - If $M>0,N>0,a>0$ and $a\neq1$, then,
    $logaMN=logaM-logaN$
    The logarithm of a quotient is the difference of the logarithms.

- **Power Property of Logarithms**

  - If $M>0,a>0,a\neq1$ and p is any real number then,
    $logaMp=plogaM$
    The log of a number raised to a power is

the product of the power times the log of the number.

- **Properties of Logarithms Summary**
  If $M>0, a>0, a\neq 1$ and p is any real number then,

Property	Base a	Base e
	$\log_a 1=0$	$\ln 1=0$
	$\log_a a=1$	$\ln e=1$
Inverse Properties	$a^{\log_a x}=x$ $\log_a a^x=x$	$e^{\ln x}=x$ $\ln e^x=x$
Product Property of Logarithms	$\log_a(M\cdot N)=\log_a M+\log_a N$	$\ln(M\cdot N)=\ln M+\ln N$
Quotient Property of Logarithms	$\log_a \frac{M}{N}=\log_a M-\log_a N$	$\ln\frac{M}{N}=\ln M-\ln N$
Power Property of Logarithms	$\log_a M^p=p\log_a M$	$\ln M^p=p\ln M$

- **Change-of-Base Formula**
  For any logarithmic bases *a* and *b*, and $M>0$,
  $\log_a M=\frac{\log_b M}{\log_b a}$ (new base b) $\quad \log_a M=\frac{\log M}{\log a}$ (new base 10) $\quad \log_a M=\frac{\ln M}{\ln a}$ (new base e)

Practice Makes Perfect

Use the Properties of Logarithms

In the following exercises, use the properties of logarithms to evaluate.

ⓐ log121 ⓑ lne

ⓐ 0 ⓑ 1

ⓐ 5log510 ⓑ log4410

ⓐ 10 ⓑ 10

ⓐ 6log615 ⓑ log88−4

ⓐ 15 ⓑ −4

ⓐ 10log3 ⓑ log10−1

ⓐ 3 ⓑ −1

ⓐ eln3 ⓑ lne7

ⓐ 3 ⓑ 7

In the following exercises, use the Product Property of Logarithms to write each logarithm as a sum of logarithms. Simplify if possible.

log58y

---

log58 + log5y

log381xy

---

4 + log3x + log3y

log1000y

---

3 + logy

In the following exercises, use the Quotient Property of Logarithms to write each logarithm as a sum of logarithms. Simplify if possible.

log656

---

log65 − 1

log5125x

---

3 − log5x

log10,000y

---

4 − logy

lne416

---

4 − ln16

In the following exercises, use the Power Property of Logarithms to expand each. Simplify if possible.

log2x5

5log2x

logx − 3

−3logx

log5x3

13log5x

lnx43

43lnx

In the following exercises, use the Properties of Logarithms to expand the logarithm. Simplify if possible.

log2(3x5y3)

log23 + 5log2x + 3log2y

log5(214y3)

14log521 + 3log5y

log54ab3c4d2

log54 + log5a + 3log5b
 + 4log5c − 2log5d

log3x2327y4

23log3x − 3 − 4log3y

log33x + 2y25z2

12log3(3x + 2y2) − log35 − 2log3z

log53x24y3z3

13(log53 + 2log5x − log54
− 3log5y − log5z)

In the following exercises, use the Properties of Logarithms to condense the logarithm. Simplify if possible.

log4 + log25

2

log25 − log2(x − 1)

$\log 25x-1$

$\log 52-\log 5x-\log 5y$

$\log 52xy$

$6\log 3x+9\log 3y$

$\log 3x6y9$

$\log(x2+2x+1)-2\log(x+1)$

0

$3\ln x+4\ln y-2\ln z$

$\ln x3y4z2$

$2\log(2x+3)+12\log(x+1)$

$\log(2x+3)2\cdot x+1$

Use the Change-of-Base Formula

In the following exercises, use the Change-of-Base Formula, rounding to three decimal places, to approximate each logarithm.

log546

2.379

log1593

1.674

log321

5.542

For the following exercises, rewrite each expression as an equivalent ratio of logs using the indicated base.

log 7 (15) to base e

---

log 7 (15)= ln(15) ln(7)

Glossary

change-of-base formula
: a formula for converting a logarithm with any base to a quotient of logarithms with any other base.

power property for logarithms
: a rule of logarithms that states that the log of a power is equal to the product of the exponent and the log of its base

product property for logarithms
: a rule of logarithms that states that the log of a product is equal to a sum of logarithms

quotient property for logarithms
: a rule of logarithms that states that the log of a quotient is equal to a difference of logarithms

Systems of Linear Equations with Two Variables (5.1)

By the end of this section, you will be able to:

- Determine whether an ordered pair is a solution of a system of equations
- Solve a system of linear equations by graphing
- Solve a system of equations by substitution
- Solve a system of equations by elimination
- Choose the most convenient method to solve a system of linear equations

This Module supports section **5.1 of Mat 1023**.

In case you missed something in class, or just want to review a specific topic covered in this Module, here is a list of topics covered:

1. Ordered Pairs and Systems of Equations [link]
2. Solve by Graphing [link]
3. Solve by Substitution Method [link]
4. Solve by Addition/Elimination Method [link]
5. Choose the Method to Solve Systems of Equations [link]
6. Key Concepts [link]

Determine Whether an Ordered Pair is a Solution

In Solving Linear Equations, we learned how to solve linear equations with one variable. Now we will work with two or more linear equations grouped together, which is known as a **system of linear equations**.

System of Linear Equations
When two or more linear equations are grouped together, they form a **system of linear equations**.

An example of a system of two linear equations is shown below. We use a brace to show the two equations are grouped together to form a system of equations.

$$\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$$

A linear equation in two variables, such as $2x+y=7$, has an infinite number of solutions. Its graph is a line. Remember, every point on the line is a solution to the equation and every solution to the equation is a point on the line.

To solve a system of two linear equations, we want to find the values of the variables that are solutions

to *both* equations. In other words, we are looking for the ordered pairs (x,y) that make both equations true. These are called the **solutions of a system of equations**.

Solutions of a System of Equations

The **solutions of a system of equations** are the values of the variables that make *all* the equations true. A solution of a system of two linear equations is represented by an ordered pair (x,y).

To determine if an ordered pair is a solution to a system of two equations, we substitute the values of the variables into each equation. If the ordered pair makes both equations true, it is a solution to the system.

Determine whether the ordered pair is a solution to the system $\{x-y=-12x-y=-5$.

ⓐ $(-2,-1)$ ⓑ $(-4,-3)$

ⓐ

$$x - y = -1$$
$$2x - y = -5$$

We substitute $x = -2$ and $y =$ into both equations.

$x - y = -1$	$2x - y = -5$
$-2 - (\quad) \stackrel{?}{=} -1$	$2 \cdot -2 - (\quad) \stackrel{?}{=} -5$
$-1 = -1$ ✓	$5 \neq -5$
(−2, −1) does not make both equations true.	(−2, −1) is not a solution.

ⓑ

We substitute $x = -4$ and $y =$ into both equations.

$x - y = -1$	$2x - y = -5$
$-4 - (\quad) \stackrel{?}{=} -1$	$2 \cdot (-4) - (\quad) \stackrel{?}{=} -5$
$-1 = -1$ ✓	$-5 = -5$ ✓
(−4, −3) does make both equations true.	(−4, −3) is a solution.

Determine whether the ordered pair is a solution to the system $\{3x + y = 0x + 2y = -5$.

ⓐ$(1, -3)$ ⓑ $(0,0)$

ⓐ yes ⓑ no

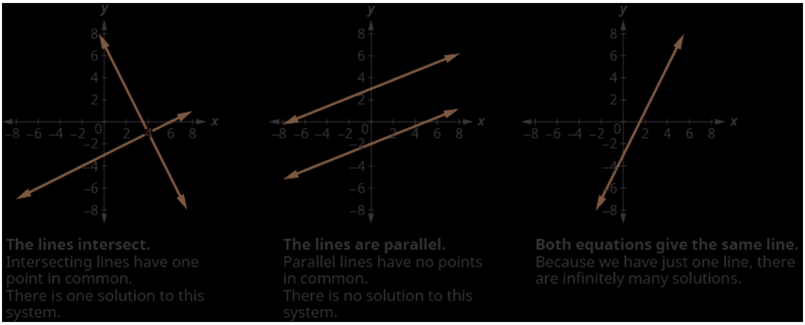
Solve a System of Linear Equations by Graphing

In this section, we will use three methods to solve a system of linear equations. The first method we'll use is graphing.

The graph of a linear equation is a line. Each point on the line is a solution to the equation. For a system of two equations, we will graph two lines. Then we can see all the points that are solutions to each equation. And, by finding what the lines have in common, we'll find the solution to the system.

Most linear equations in one variable have one solution, but we saw that some equations, called contradictions, have no solutions and for other equations, called identities, all numbers are solutions.

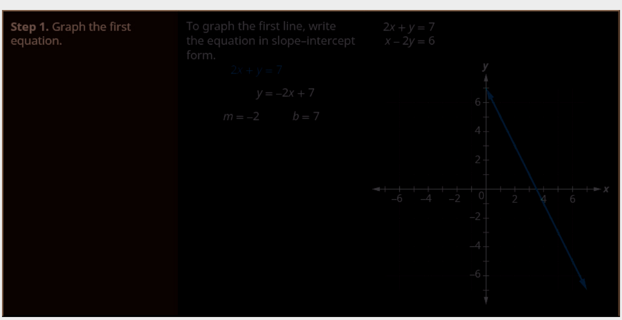
Similarly, when we solve a system of two linear equations represented by a graph of two lines in the same plane, there are three possible cases, as shown.

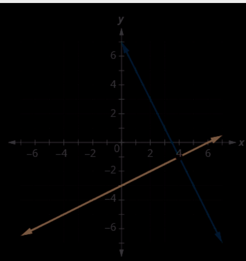


Each time we demonstrate a new method, we will use it on the same system of linear equations. At the end of the section you'll decide which method was the most convenient way to solve this system.

How to Solve a System of Equations by Graphing

Solve the system by graphing $\{2x+y=7x-2y=6$.



Step 2. Graph the second equation on the same rectangular coordinate system.	To graph the second line, use intercepts. $(0, -3)$ $(6, 0)$	

Step 3. Determine whether the lines intersect, are parallel, or are the same line.	Look at the graph of the lines.	The lines intersect.

Step 4. Identify the solution to the system. • If the lines intersect, identify the point of intersection. Check to make sure it is a solution to both equations. This is the solution to the system. • If the lines are parallel, the system has no solution. • If the lines are the same, the system has an infinite number of solutions.	Since the lines intersect, find the point of intersection. Check the point in both equations.	The lines intersect at $(4, -1)$. $2x + y = 7$ $2(4) + (-1) \stackrel{?}{=} 7$ $8 - 1 \stackrel{?}{=} 7$ $7 = 7 \checkmark$ $x - 2y = 6$ $4 - 2(-1) \stackrel{?}{=} 6$ $6 = 6 \checkmark$ The solution is $(4, -1)$.

Solve a system of linear equations by graphing.

Graph the first equation. Graph the second equation on the same rectangular coordinate system. Determine whether the lines intersect, are parallel, or are the same line. Identify the solution to the system.

- If the lines intersect, identify the point of intersection. This is the solution to the system.
- If the lines are parallel, the system has no solution.
- If the lines are the same, the system has an infinite number of solutions.

Check the solution in both equations.

In the next example, we'll first re-write the equations into slope–intercept form as this will make it easy for us to quickly graph the lines.

Solve the system by graphing: $\{3x+y=-12x+y=0$.

We'll solve both of these equations for y so that we can easily graph them using their slopes and *y*-intercepts.

[redacted]

Solve the first equation for y.

[redacted]

Find the slope and y-intercept.

[redacted]

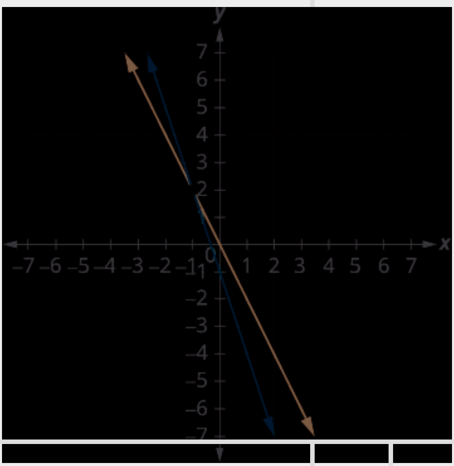
Solve the second equation for y.

[redacted]

Find the slope and y-intercept.

[redacted]

Graph the lines.



[redacted]

Determine the point of intersection. The lines intersect at $(-1, 2)$.

Check the solution in both equations.

$$\begin{array}{ll} 3x+y=-1 & 2x+y=0 \\ 3(-1)+2 \stackrel{?}{=} -1 & 2(-1)+2 \stackrel{?}{=} 0 \\ -1=-1 \checkmark & 0=0 \checkmark \end{array}$$

The solution is $(-1,2)$.

In all the systems of linear equations so far, the lines intersected and the solution was one point. In the next two examples, we'll look at a system of equations that has no solution and at a system of equations that has an infinite number of solutions.

Solve the system by graphing: $\{y=12x-3x$
$-2y=4$.

$$y=\frac{1}{2}x-3$$

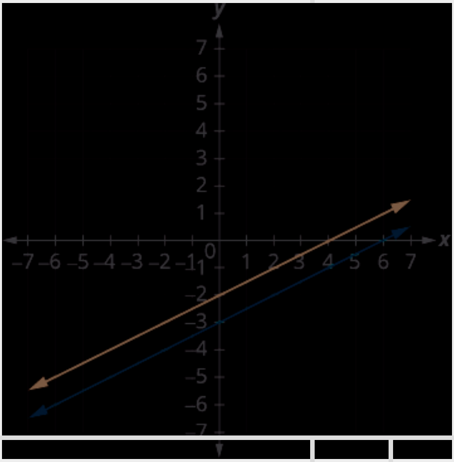
$$x-2y=4$$

To graph the first equation, we will use its [redacted] slo[redacted]

To graph the second equation, we will use th[redacted]

[redacted]

Graph the lines.



Determine the points of intersection.

The lines are parallel. Since no point is on both lines, there is no ordered pair that makes both equations true. There is no solution to this system.

Solve the system by graphing: $\{y = -14x + 2x + 4y = -8$.

no solution

Sometimes the equations in a system represent the same line. Since every point on the line makes both equations true, there are infinitely many ordered pairs that make both equations true. There are infinitely many solutions to the system.

Solve the system by graphing: $\{y = 2x - 3 - 6x + 3y = -9$.

[redacted]

Find the slope and *y*-intercept of the first eq[redacted]

Find the intercepts of the second equation.

[redacted]

[redacted]

Graph the lines.

[redacted]

The lines are the same! Since every point on the line makes both equations true, there are infinitely many

ordered pairs that make both equations true.
There are infinitely many solutions to this system.

If you write the second equation in slope-intercept form, you may recognize that the equations have the same slope and same *y*-intercept.

When we graphed the second line in the last example, we drew it right over the first line. We say the two lines are **coincident**. Coincident lines have the same slope and same *y*-intercept.

Coincident Lines
Coincident lines have the same slope and same *y*-intercept.

The systems of equations in [link] and [link] each had two intersecting lines. Each system had one solution.

In [link], the equations gave coincident lines, and so the system had infinitely many solutions.

The systems in those three examples had at least one solution. A system of equations that has at least one solution is called a *consistent* system.

A system with parallel lines, like [link], has no solution. We call a system of equations like this *inconsistent.* It has no solution.

Consistent and Inconsistent Systems

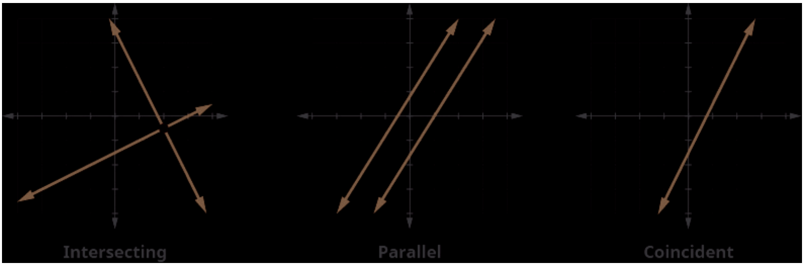
A **consistent system of equations** is a system of equations with at least one solution.

An **inconsistent system of equations** is a system of equations with no solution.

We also categorize the equations in a system of equations by calling the equations *independent* or *dependent*. If two equations are independent, they each have their own set of solutions. Intersecting lines and parallel lines are independent.

If two equations are dependent, all the solutions of one equation are also solutions of the other equation. When we graph two dependent equations, we get coincident lines.

Let's sum this up by looking at the graphs of the three types of systems. See below and [link].



Lines	Intersecting	Parallel	Coincident
Number of solutions	1 point	No solution	Infinitely many
Consistent/ inconsistent	Consistent	Inconsistent	Consistent
Dependent/ independent	Independent	Independent	Dependent

Without graphing, determine the number of solutions and then classify the system of equations.

ⓐ $\{y=3x-16x-2y=12$ ⓑ $\{2x+y=-3x-5y=5$

ⓐ We will compare the slopes and intercepts of the two lines.

The first equation is already in slope-intercept form.$\{y=3x-16x-2y=12y=3x-1$

Write the second equation in slope-intercept form.Find the slope and intercept of each line.$6x-2y=12-2y=-6x+12-2y-2=-6x+12-2y=3x-6y=3x-1y=3x-6m=3m=3b=-1b=-6$Since the slopes are the same andy-intercepts aredifferent, the lines are parallel.

A system of equations whose graphs are parallel lines has no solution and is inconsistent and independent.

ⓑ We will compare the slope and intercepts of the two lines.

$\{2x+y=-3x-5y=5$Write both equations in slope–intercept form.$2x+y=-3x-5y=5y=-2x-3-5y=-x+5-5y-5=-x+5-5y=15x-1$Find the slope and intercept of each line.$y=-2x-3y=15x-1m=-2m=15b=-3b=-1$Since the slopes are different, the lines intersect.

A system of equations whose graphs are intersect has 1 solution and is consistent and

independent.

Without graphing, determine the number of solutions and then classify the system of equations.

ⓐ $\{y=-2x-44x+2y=9$ ⓑ $\{3x+2y=22x+y=1$

ⓐ no solution, inconsistent, independent ⓑ one solution, consistent, independent

Solving systems of linear equations by graphing is a good way to visualize the types of solutions that may result. However, there are many cases where solving a system by graphing is inconvenient or imprecise. If the graphs extend beyond the small grid with x and y both between -10 and 10, graphing the lines may be cumbersome. And if the solutions to the system are not integers, it can be hard to read their values precisely from a graph.

Solve a System of Equations by Substitution

We will now solve systems of linear equations by the substitution method.

We will use the same system we used first for graphing.
$\{2x+y=7x-2y=6$

We will first solve one of the equations for either *x* or *y*. We can choose either equation and solve for either variable—but we'll try to make a choice that will keep the work easy.

Then we substitute that expression into the other equation. The result is an equation with just one variable—and we know how to solve those!

After we find the value of one variable, we will substitute that value into one of the original equations and solve for the other variable. Finally, we check our solution and make sure it makes both equations true.

How to Solve a System of Equations by Substitution

Solve the system by substitution: $\{2x+y=7x$

$-2y = 6.$

Step 1. Solve one of the equations for either variable.	We'll solve the first equation for *y*.	$2x + y = 7$ $x - 2y = 6$ $2x + y = 7$ $y = 7 - 2x$
Step 2. Substitute the expression from Step 1 into the other equation.	We replace *y* in the second equation with the expression $7 - 2x$.	$x - 2y = 6$ $x - 2(7 - 2x) = 6$
Step 3. Solve the resulting equation.	Now we have an equation with just 1 variable. We know how to solve this!	$x - 2(7 - 2x) = 6$ $x - 14 + 4x = 6$ $5x = 20$ $x = 4$
Step 4. Substitute the solution from Step 3 into one of the original equations to find the other variable.	We'll use the first equation and replace *x* with 4.	$2x + y = 7$ $2(4) + y = 7$ $8 + y = 7$ $y = -1$
Step 5. Write the solution as an ordered pair.	The ordered pair is (x, y).	$(4, -1)$
Step 6. Check that the ordered pair is a solution to **both** original equations.	Substitute $x = 4$, $y = -1$ into both equations and make sure they are both true.	$2x + y = 7$ $x - 2y = 6$ $2(4) + (-1) \stackrel{?}{=} 7$ $4 - 2(-1) \stackrel{?}{=} 6$ $7 = 7$ ✓ $6 = 6$ ✓ Both equations are true. $(4, -1)$ is the solution to the system.

Solve the system by substitution: $\{2x+y=$
$-14x+3y=3.$

$(-3,5)$

Solve a system of equations by substitution.

Solve one of the equations for either variable. Substitute the expression from Step 1 into the other equation. Solve the resulting equation. Substitute the solution in Step 3 into either of the original equations to find the other variable. Write the solution as an ordered pair. Check that the ordered pair is a solution to **both** original equations.

Be very careful with the signs in the next example.

Solve the system by substitution: $\{4x$

$+2y=46x-y=8.$

We need to solve one equation for one variable. We will solve the first equation for *y*.

[redacted]

Solve the first equation for *y*.

Su[redacted]

y i

eq

Replace the *y* with $-2x+2.$

[redacted]

Solve the equation for *x*.

[redacted]

Substitute $x=54$ into

4x

$$
\begin{aligned}
4x + 2y &= 4 \\
4\left(\frac{5}{4}\right) + 2y &= 4 \\
5 + 2y &= 4 \\
2y &= -1 \\
y &= -\frac{1}{2}
\end{aligned}
$$

The ordered pair is (54, −12).

Check the ordered pair in both equations.

$$
\begin{array}{cc}
4x + 2y = 4 & 6x - y = 8 \\
4\left(\frac{5}{4}\right) + 2\left(-\frac{1}{2}\right) \stackrel{?}{=} 4 & 6\left(\frac{5}{4}\right) - \left(-\frac{1}{2}\right) \stackrel{?}{=} 8 \\
5 - 1 \stackrel{?}{=} 4 & \frac{15}{4} - \left(-\frac{1}{2}\right) \stackrel{?}{=} 8 \\
4 = 4 \checkmark & \frac{16}{2} \stackrel{?}{=} 8 \\
 & 8 = 8 \checkmark
\end{array}
$$

The solution is (54, −12).

Solve the system by substitution: {x − 4y = −4 − 3x + 4y = 0.

(2,32)

Solve a System of Equations by Addition (also called Elimination)

We have solved systems of linear equations by graphing and by substitution. Graphing works well when the variable coefficients are small and the solution has integer values. Substitution works well when we can easily solve one equation for one of the variables and not have too many fractions in the resulting expression.

The third method of solving systems of linear equations is called the Addition (Elimination) Method. When we solved a system by substitution, we started with two equations and two variables and reduced it to one equation with one variable. This is what we'll do with the elimination method, too, but we'll have a different way to get there.

The Elimination Method is based on the Addition Property of Equality. The Addition Property of Equality says that when you add the same quantity to both sides of an equation, you still have equality. We will extend the Addition Property of Equality to say that when you add equal quantities to both sides of an equation, the results are equal.

For any expressions *a, b, c,* and *d.*
$\text{if}\, a = b\, \text{and}\, c = d\, \text{then}\, a + c = b + d.$

To solve a system of equations by elimination, we start with both equations in standard form. Then we decide which variable will be easiest to eliminate. How do we decide? We want to have the coefficients of one variable be opposites, so that we can add the equations together and eliminate that variable.

Notice how that works when we add these two equations together:
$\begin{cases} 3x+y=5 \\ 2x-y=0 \end{cases}$ ———————— $5x=5$

The y's add to zero and we have one equation with one variable.

Let's try another one:
$\begin{cases} x+4y=2 \\ 2x+5y=-2 \end{cases}$

This time we don't see a variable that can be immediately eliminated if we add the equations.

But if we multiply the first equation by -2, we will make the coefficients of x opposites. We must multiply every term on both sides of the equation by -2.

$$-2(x+4y)=-2(2)$$
$$2x+5y=-2$$

Then rewrite the system of equations.

$$\begin{aligned} -2x - 8y &= -4 \\ 2x + 5y &= -2 \end{aligned}$$

Now we see that the coefficients of the *x* terms are opposites, so *x* will be eliminated when we add these two equations.

$$\begin{array}{r} -2x - 8y = -4 \\ 2x + 5y = -2 \\ \hline -3y = -6 \end{array}$$

Once we get an equation with just one variable, we solve it. Then we substitute that value into one of the original equations to solve for the remaining variable. And, as always, we check our answer to make sure it is a solution to both of the original equations.

Now we'll see how to use elimination to solve the same system of equations we solved by graphing and by substitution.

How to Solve a System of Equations by Elimination

Solve the system by elimination: $\{2x+y=7x$
$-2y=6.$

Step 1. Write both equations in standard form. • If any coefficients are fractions, clear them.	Both equations are in standard form, $Ax + By = C$. There are no fractions.	$2x + y = 7$ $x - 2y = 6$
Step 2. Make the coefficients of one variable opposites. • Decide which variable you will eliminate. • Multiply one or both equations so that the coefficients of that variable are opposites.	We can eliminate the y's by multiplying the first equation by 2. Multiply both sides of $2x + y = 7$ by 2.	$2x + y = 7$ $x - 2y = 6$ $2(2x + y) = 2(7)$ $x - 2y = 6$
Step 3. Add the equations resulting from Step 2 to eliminate one variable.	We add the x's, y's, and constants.	$4x + 2y = 14$ $x - 2y = 6$ $5x \quad = 20$
Step 4. Solve for the remaining variable.	Solve for x.	$x = 4$
Step 5. Substitute the solution from Step 4 into one of the original equations. Then solve for the other variable.	Substitute $x = 4$ into the second equation, $x - 2y = 6$. Then solve for y.	$x - 2y = 6$ $4 - 2y = 6$ $-2y = 2$ $y = -1$
Step 6. Write the solution as an ordered pair.	Write it as (x, y).	$(4, -1)$
Step 7. Check that the ordered pair is a solution to **both** original equations.	Substitute $x = 4$, $y = -1$ into $2x + y = 7$ and $x - 2y = 6$. Do they make both equations true? Yes!	$2x + y = 7$; $x - 2y = 6$ $2(4) + (-1) \stackrel{?}{=} 7$; $4 - 2(-1) \stackrel{?}{=} 6$ $7 = 7$ ✓ ; $6 = 6$ ✓ The solution is $(4, -1)$.

The steps are listed here for easy reference.

Solve a system of equations by elimination.

Write both equations in standard form. If any coefficients are fractions, clear them. Make the coefficients of one variable opposites.

- Decide which variable you will eliminate.
- Multiply one or both equations so that the coefficients of that variable are opposites.

Add the equations resulting from Step 2 to eliminate one variable. Solve for the remaining variable. Substitute the solution from Step 4 into one of the original equations. Then solve for the other variable. Write the solution as an ordered pair. Check that the ordered pair is a solution to **both** original equations.

Now we'll do an example where we need to multiply both equations by constants in order to make the coefficients of one variable opposites.

Solve the system by elimination: $\begin{cases} 4x - 3y = 9 \\ 7x + 2y = -6 \end{cases}$.

In this example, we cannot multiply just one equation by any constant to get opposite coefficients. So we will strategically multiply both equations by different constants to get the opposites.

$$\begin{cases} 4x - 3y = 9 \\ 7x + 2y = -6 \end{cases}$$

Both equations are in standard form.

To get opposite coefficients of y, we will multiply the first equation by 2 and the second equation by 3.

$$\begin{cases} 2(4x - 3y) = 2(9) \\ 3(7x + 2y) = 3(-6) \end{cases}$$

Simplify.

$$\begin{cases} 8x - 6y = 18 \\ 21x + 6y = -18 \end{cases}$$

Add the two equations to eliminate y.

[redacted]

Solve for x.

[redacted]

Substitute $x = 0$ into one of the original equations.

[redacted]

Solve for y.

[redacted]

Write the solution as an ordered pair. | The ordered pair is $(0, -3)$.

Check that the ordered pair is a solution to **both** original equations.

[redacted]

The solution is $(0, -3)$.

Solve each system by elimination: { 7x +8y=43x−5y=27.
(4,−3)

When the system of equations contains fractions, we will first clear the fractions by multiplying each equation by the LCD of all the fractions in the equation.

Solve the system by elimination: {x +12y=632x+23y=172.
In this example, both equations have fractions. Our first step will be to multiply each equation by the LCD of all the fractions in the equation to clear the fractions.

$$x + \frac{1}{2}y = 6$$
$$\frac{3}{2}x + \frac{2}{3}y = \frac{17}{2}$$

To clear the fractions, multiply each eq

$$2\left(x + \frac{1}{2}y\right) = 2(6)$$
$$6\left(\frac{3}{2}x + \frac{2}{3}y\right) = 6\left(\frac{17}{2}\right)$$

Simplify.

$$2x + y = 12$$
$$9x + 4y = 51$$

Now we are ready to eliminate one of the variables. Notice that both equations are in standard form.

We can eliminate y by multiplying the top eq

$$-4(2x + y) = -4(12)$$
$$9x + 4y = 51$$

Simplify and add.

Su
on
eq

$$-8x - 4y = -48$$
$$9x + 4y = 51$$
$$x = 3$$

$$x + \frac{1}{2}y = 6$$

Solve for y.

$$3 + \frac{1}{2}y = 6$$

[redacted]

[redacted]

Write the solution as an ordered pair.	The ordered pair is (3,6).
Check that the ordered pair is a solution to both original equations.	[redacted]
	The solution is (3,6).

Solve each system by elimination: { 13x
$-12y=134x-y=52$.

(6,2)

When we solved the system by graphing, we saw that not all systems of linear equations have a single ordered pair as a solution. When the two equations were really the same line, there were infinitely many solutions. We called that a consistent system. When the two equations described parallel lines, there was no solution. We called that an inconsistent system.

The same is true using substitution or elimination. If the equation at the end of substitution or elimination is a true statement, we have a consistent but dependent system and the system of equations has infinitely many solutions. If the equation at the end of substitution or elimination is a false statement, we have an inconsistent system and the system of equations has no solution.

Solve the system by elimination: {3x
$+4y=12y=3-34x$.

{3x + 4y = 12y = 3 − 34x Write the second equation in standard form.{3x + 4y = 1234x + y = 3 Clear the fractions by multiplying thesecond equation by 4.{3x + 4y = 124(34x + y) = 4(3) Simplify.{3x + 4y = 123x + 4y = 12 To eliminate a variable, we multiply thesecond equation by − 1.Simplify and add.{3x + 4y = 12 − 3x − 4y = − 12__________0 = 0

This is a true statement. The equations are consistent but dependent. Their graphs would be the same line. The system has infinitely many solutions.

After we cleared the fractions in the second equation, did you notice that the two equations were the same? That means we have coincident lines.

Solve the system by elimination: { x + 2y = 6y = − 12x + 3.

infinitely many solutions

Choose the Most Convenient Method to Solve a System of Linear Equations

When you solve a system of linear equations in in an application, you will not be told which method to use. You will need to make that decision yourself. So you'll want to choose the method that is easiest to do and minimizes your chance of making mistakes.

Choose the Most Convenient Method to Solve a System of Linear EquationsGraphing———— Substitution—————Elimination—————Use when you need aUse when one equation isUse when the equations are picture of the situation.already solved or can bein standard form. easily solved for onevariable.

For each system of linear equations, decide whether it would be more convenient to solve it by substitution or elimination. Explain your answer.

ⓐ $\{3x+8y=407x-4y=-32$ ⓑ $\{5x+6y=12y=23x-1$

ⓐ

$\{3x+8y=407x-4y=-32$

Since both equations are in standard form, using elimination will be most convenient.

ⓑ

$\{5x+6y=12y=23x-1$

Since one equation is already solved for *y*, using substitution will be most convenient.

For each system of linear equations decide whether it would be more convenient to solve it by substitution or elimination. Explain your answer.

ⓐ $\{4x-5y=-323x+2y=-1$ ⓑ $\{x=2y-13x-5y=-7$

ⓐ Since both equations are in standard form, using elimination will be most convenient. ⓑ Since one equation is already solved for *x*, using substitution will be most convenient.

Key Concepts

- **How to solve a system of linear equations by graphing.**

  Graph the first equation. Graph the second equation on the same rectangular coordinate system. Determine whether the lines intersect, are parallel, or are the same line. Identify the solution to the system.
  If the lines intersect, identify the point of intersection. This is the solution to the system.
  If the lines are parallel, the system has no solution.
  If the lines are the same, the system has an infinite number of solutions. Check the solution in both equations.

- **How to solve a system of equations by substitution.**

  Solve one of the equations for either variable. Substitute the expression from Step 1 into the other equation. Solve the resulting equation. Substitute the solution in Step 3 into either of the original equations to find the other variable. Write the solution as an ordered pair. Check that the ordered pair is a solution to **both** original equations.

- **How to solve a system of equations by Addition(elimination).**

  Write both equations in standard form. If any coefficients are fractions, clear them. Make the coefficients of one variable opposites.
  Decide which variable you will eliminate.
  Multiply one or both equations so that the coefficients of that variable are opposites. Add the equations resulting from Step 2 to eliminate one variable. Solve for the remaining variable. Substitute the solution from Step 4 into one of the original equations. Then solve for the other variable. Write the solution as an ordered pair. Check that the ordered pair is a solution to **both** original equations.
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Practice Makes Perfect

Determine Whether an Ordered Pair is a Solution of a System of Equations

In the following exercises, determine if the following points are solutions to the given system of equations.

$\{2x-6y=03x-4y=5$

ⓐ $(3,1)$
ⓑ $(-3,4)$

ⓐ yes ⓑ no

$\{x+y=2y=34x$

ⓐ $(87,67)$
ⓑ $(1,34)$

ⓐ yes ⓑ no

Solve a System of Linear Equations by Graphing

In the following exercises, solve the following systems of equations by graphing.

$\{3x+y=-32x+3y=5$

$(-2,3)$

$\{y=x+2y=-2x+2$

(0,2)

$\{y=32x+1y=-12x+5$

(2,4)

$\{-2x+3y=3x+3y=12$

(3,3)

$\{-2x+4y=4y=12x$

---

no solution

$\{x=-3y+42x+6y=8$

---

infinite solutions

Without graphing, determine the number of solutions and then classify the system of equations.

$\{y = 23x + 1 - 2x + 3y = 5$

No solutions, inconsistent, independent

$\{5x + 3y = 42x - 3y = 5$

---

1 point, consistent and independent

$\{5x - 2y = 10y = 52x - 5$

---

infinite solutions, consistent, dependent

Solve a System of Equations by Substitution

In the following exercises, solve the systems of equations by substitution.

$\{2x + y = -23x - y = 7$

$(1, -4)$

$\{x - 3y = -92x + 5y = 4$

$(-3,2)$

$\{-2x+2y=6y=-3x+1$

$(-1/2,5/2)$

$\{3x+4y=1y=-25x+2$

$(-5,4)$

$\{y=x-6y=-32x+4$

$(4,-2)$

$\{y=-23x+52x+3y=11$

none

Solve a System of Equations by Addition/Elimination

In the following exercises, solve the systems of equations by elimination.

$\{6x-5y=-12x+y=13$

$(4,5)$
$\{5x-3y=-12x-y=2$
$(7,12)$
$\{11x+9y=-57x+5y=-1$
$(2,-3)$
$\{x+12y=3215x-15y=3$
$(6/-9,24/7)$
$\{13x-y=-323x+52y=3$
$(-3,2)$
$\{x-4y=-1-3x+12y=3$
infinitely many

$\{4x+3y=220x+15y=10$

---

infinitely many

Choose the Most Convenient Method to Solve a System of Linear Equations

In the following exercises, decide whether it would be more convenient to solve the system of equations by substitution or elimination.

ⓐ$\{y=7x-53x-2y=16$
ⓑ$\{12x-5y=-423x+7y=-15$

ⓐ substitution ⓑ elimination

ⓐ$\{14x-15y=-307x+2y=10$
ⓑ$\{x=9y-112x-7y=-27$

ⓐ elimination ⓑ substituion

Glossary

coincident lines

Coincident lines have the same slope and same *y*-intercept.

consistent and inconsistent systems

Consistent system of equations is a system of equations with at least one solution; inconsistent system of equations is a system of equations with no solution.

solutions of a system of equations

Solutions of a system of equations are the values of the variables that make *all* the equations true; solution is represented by an ordered pair (x,y).

system of linear equations

When two or more linear equations are grouped together, they form a system of linear equations.